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Existence of viscosity multi-valued solutions with asymptotic behavior for Hessian equations

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Abstract
The Perron method is used to establish the existence of viscosity multi-valued solutions for a class of Hessian-type equations with prescribed behavior at infinity.

Keywords: Hessian equation; multi-valued solution; asymptotic behavior

1 Introduction

In [1, 2], the multi-valued solutions of the eikonal equation were studied. Later, in [3, 4] Jin *et al.* provided a level set method for the computation of multi-valued geometric solutions to general quasilinear partial differential equations and multi-valued physical observables to the semiclassical limit of the Schrödinger equations. In [5], Caffarelli and Li investigated the multi-valued solutions of the Monge-Ampère equation where they first introduced the geometric situation of the multi-valued solutions and obtained the existence, regularity and the asymptotic behavior at infinity of the multi-valued viscosity solutions. In [6] Ferrer *et al.* used complex variable methods to study the multi-valued solutions for the Dirichlet problems of Monge-Ampère equations on exterior planar domains. Recently, Bao and Dai discussed the multi-valued solutions of Hessian equations, see [7, 8]. Motivated by the above works, in this paper we study the viscosity multi-valued solutions of the Hessian equation

$$F\big(\lambda\big(D^2u\big)\big) = \sigma > 0, \tag{1.1}$$

where σ is a constant and $\lambda(D^2u) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ are eigenvalues of the Hessian matrix D^2u. F is assumed to be defined in the symmetric open convex cone Γ, with vertex at the origin, containing

$$\Gamma^+ = \big\{\lambda \in \mathbf{R}^n : \text{each component of } \lambda, \lambda_i > 0, i = 1, 2, \ldots, n\big\},$$

satisfies the fundamental structure conditions

$$F_i(\lambda) = \frac{\partial F}{\partial \lambda_i} > 0 \quad \text{in } \Gamma, 1 \le i \le n, \tag{1.2}$$

and F is a continuous concave function. In addition, F will be assumed to satisfy some more technical assumptions such as

$$F > 0 \quad \text{in } \Gamma, \qquad F = 0 \quad \text{on } \partial\Gamma, \tag{1.3}$$

and for any $r \geq 1$, $R > 0$,

$$F\left(R\left(\frac{1}{r^{n-1}}, r, \ldots, r\right)\right) \geq F\big(R(1, 1, \ldots, 1)\big). \tag{1.4}$$

For every $C > 0$ and every compact set K in Γ, there is $\Lambda = \Lambda(C, K)$ such that

$$F(\Lambda\lambda) \geq C \quad \text{for all } \lambda \in K. \tag{1.5}$$

There exists a number Λ sufficiently large such that at every point $x \in \partial\Omega$, if $x_1, \ldots, x_{n-1}$ represent the principal curvatures of $\partial\Omega$, then

$$(x_1, \ldots, x_{n-1}, \Lambda) \in \Gamma. \tag{1.6}$$

Inequality (1.4) is satisfied by each kth root of an elementary symmetric function ($1 \leq k \leq n$) and the $(k-l)$th root of each quotient of the kth elementary symmetric function and the lth elementary symmetric function ($1 \leq l < k \leq n$).

2 Preliminaries

The geometric situation of the multi-valued function is given in [5]. Let $n \geq 2$, $D \subset \mathbf{R}^n$ be a bounded domain with smooth boundary ∂D, and let $\Sigma \subset D$ be homeomorphic in $\mathbf{R}^n$ to an $n-1$ dimensional closed disc. $\partial\Sigma$ is homeomorphic to an $n-2$ dimensional sphere for $n \geq 3$.

Let $\mathbf{Z}$ be the set of integers and $M = (D \setminus \partial\Sigma) \times \mathbf{Z}$ denote a covering of $D \setminus \partial\Sigma$ with the following standard parametrization: fixing $x^* \in D \setminus \partial\Sigma$ and connecting x^* by a smooth curve in $D \setminus \partial\Sigma$ to a point x in $D \setminus \partial\Sigma$. If the curve goes through Σ $m \geq 0$ times in the positive direction (fixing such a direction), then we arrive at (x, m) in M. If the curve goes through Σ $m \geq 0$ times in the negative direction, then we arrive at $(x, -m)$ in M.

For $k = 2, 3, \ldots$, we introduce an equivalence relation '$\sim k$' on M as follows: (x, m) and (y, j) in M are '$\sim k$' equivalent if $x = y$ and $m - j$ is an integer multiple of k. We let $M_k = M/\sim k$ denote the k-sheet cover of $D \setminus \partial\Sigma$, and let $\partial' M_k = \bigcup_{m=1}^{k} (\partial D \times \{m\})$.

We define a distance in M_k as follows: for any $(x, m), (y, j) \in M_k$, let $l((x, m), (y, j))$ denote a smooth curve in M_k which connects (x, m) and (y, j), and let $|l((x, m), (y, j))|$ denote its length. Define

$$d\big((x, m), (y, j)\big) = \inf_{l} \big| l\big((x, m), (y, j)\big)\big|,$$

where the infimum is taken over all smooth curves connecting (x, m) and (y, j). Then $d((x, m), (y, j))$ is a distance.

Definition 2.1 We say that a function u is continuous at (x, m) in M_k if

$$\lim_{d((x,m),(y,j)) \to 0} u(y, j) = u(x, m),$$

and $u \in C^0(M_k)$ if for any (x, m), u is continuous at (x, m).

Similarly, we can define $u \in C^\alpha(M_k)$, $C^{0,1}(M_k)$ and $C^2(M_k)$.

Definition 2.2 A function $u \in C^2(M_k)$ is called admissible if $\lambda \in \overline{\Gamma}$, where $\lambda = \lambda(D^2u(x,m)) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ are the eigenvalues of the Hessian matrix $D^2u(x,m)$.

Definition 2.3 A function $u \in C^0(M_k)$ is called a viscosity subsolution (resp. supersolution) to (1.1) if for any $(y,m) \in M_k$ and $\xi \in C^2(M_k)$ satisfying

$$u(x,m) \le (\text{resp.} \ge)\ \xi(x,m), \quad (x,m) \in M_k \quad \text{and} \quad u(y,m) = \xi(y,m),$$

we have

$$F\big(\lambda\big(D^2\xi(y,m)\big)\big) \ge (\text{resp.} \le)\ \sigma.$$

Definition 2.4 A function $u \in C^0(M_k)$ is called a viscosity solution to (1.1) if it is both a viscosity subsolution and a viscosity supersolution to (1.1).

Definition 2.5 A function $u \in C^0(M_k)$ is called admissible if for any $(y,m) \in M_k$ and any function $\xi \in C^2(M_k)$ satisfying $u(x,m) \le (\ge)\ \xi(x,m)$, $x \in M_k$, $u(y,m) = \xi(y,m)$, we have $\lambda(D^2\xi(y,m)) \in F$.

Remark It is obvious that if u is a viscosity subsolution, then u is admissible.

Lemma 2.1 *Let Ω be a bounded strictly convex domain in $\mathbf{R}^n$, $\partial\Omega \in C^2$, $\varphi \in C^2(\overline{\Omega})$. Then there exists a constant C only dependent on n, φ and Ω such that for any $\xi \in \partial\Omega$, there exists $\overline{x}(\xi) \in \mathbf{R}^n$ such that*

$$\big|\overline{x}(\xi)\big| \le C, \qquad w_\xi(x) < \varphi(x) \quad \text{for } x \in \overline{\Omega} \setminus \{\xi\},$$

where $w_\xi(x) = \varphi(\xi) + \frac{\overline{R}}{2}(|x - \overline{x}(\xi)|^2 - |\xi - \overline{x}(\xi)|^2)$ for $x \in \mathbf{R}^n$ and $\overline{R}$ is a constant satisfying $F(\overline{R}, \overline{R}, \ldots, \overline{R}) = \sigma$.

This is a modification of Lemma 5.1 in [5].

Lemma 2.2 *Let Ω be a domain in $\mathbf{R}^n$ and $f \in C^0(\mathbf{R}^n)$ be nonnegative. Assume that the admissible functions $v \in C^0(\overline{\Omega})$, $u \in C^0(\mathbf{R}^n)$ satisfy, respectively,*

$$F\big(\lambda\big(D^2v\big)\big) \ge f(x), \quad x \in \Omega,$$
$$F\big(\lambda\big(D^2u\big)\big) \ge f(x), \quad x \in \mathbf{R}^n.$$

Moreover,

$$u \le v, \quad x \in \overline{\Omega},$$
$$u = v, \quad x \in \partial\Omega.$$

Set

$$w(x) = \begin{cases} v(x), & x \in \Omega, \\ u(x), & x \in \mathbf{R}^n \setminus \Omega. \end{cases}$$

Then $w \in C^0(\mathbf{R}^n)$ is an admissible function and satisfies in the viscosity sense

$$F(\lambda(D^2 w(x))) \geq f(x), \quad x \in \mathbf{R}^n.$$

Lemma 2.3 *Let B be a ball in $\mathbf{R}^n$ and let $f \in C^{0,\alpha}(\overline{B})$ be positive. Suppose that $\underline{u} \in C^0(\overline{B})$ satisfies in the viscosity sense*

$$F(\lambda(D^2 u)) \geq f(x), \quad x \in B.$$

Then the Dirichlet problem

$$F(\lambda(D^2 u)) = f(x), \quad x \in B,$$
$$u = \underline{u}(x), \quad x \in \partial B$$

admits a unique admissible viscosity solution $u \in C^0(\overline{B})$.

We refer to [9] for the proof of Lemmas 2.2 and 2.3.

3 Existence of viscosity multi-valued solutions with asymptotic behavior

In this section, we establish the existence of viscosity multi-valued solutions with prescribed asymptotic behavior at infinity of (1.1). Let Ω be a bounded strictly convex domain with smooth boundary $\partial\Omega$. Let Σ, diffeomorphic to an $(n-1)$-disc, be the intersection of Ω any hyperplane in $\mathbf{R}^n$. Let $M = (\mathbf{R}^n \setminus \partial\Sigma) \times \mathbf{Z}$, $M_k = M/\sim k$ be covering spaces of $\mathbf{R}^n \setminus \partial\Sigma$ as in Section 2. Σ divides Ω into two open parts, denoted as Ω^+ and Ω^-. Fixing $x^* \in \Omega^-$, we use the convention that going through Σ from Ω^- to Ω^+ denotes the positive direction through Σ. Our main result is the following theorem.

Theorem 3.1 *Let $k \geq 3$. Then, for any $C_m \in \mathbf{R}$, there exists an admissible viscosity solution $u \in C^0(M_k)$ of*

$$F(\lambda(D^2 u)) = \sigma, \quad (x,m) \in M_k \tag{3.1}$$

satisfying

$$\limsup_{|x|\to\infty} |x|^{n-2} \left| u(x,m) - \left(\frac{\overline{R}}{2}|x|^2 + C_m\right)\right| < +\infty, \tag{3.2}$$

where $\overline{R}$ is a constant satisfying $F(\overline{R}, \overline{R}, \ldots, \overline{R}) = \sigma$.

When

$$F(\lambda(D^2 u)) = \sigma_k(\lambda(D^2 u)), \qquad \Gamma = \Gamma_k = \{\lambda \in \mathbf{R}^n : \sigma_j > 0, j = 1, 2, \ldots, k\},$$

where the kth elementary symmetric function

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

for $\lambda = (\lambda_1, \ldots, \lambda_n)$, in [8] Dai obtained the following result.

Theorem 3.2 *Let $k \geq 3$. Then, for any $C_m \in \mathbf{R}$, there exists a k-convex viscosity solution $u \in C^0(M_k)$ of*

$$\sigma_k\big(\lambda\big(D^2 u\big)\big) = 1, \quad (x,m) \in M_k$$

satisfying

$$\limsup_{|x|\to\infty}\left(|x|^{k-2}\left|u(x,m) - \left(\frac{C_*}{2}|x|^2 + C_m\right)\right|\right) < \infty,$$

where $C_* = (\frac{1}{C_n^k})^{\frac{1}{k}}$.

Proof of Theorem 3.1 We divide the proof of Theorem 3.1 into two steps.

Step 1. By [10], there is an admissible solution $\Phi \in C^\infty(\overline{\Omega})$ of the Dirichlet problem:

$$F\big(\lambda\big(D^2\Phi\big)\big) = C_0 > \sigma, \quad x \in \Omega,$$
$$\Phi = 0, \quad x \in \partial\Omega.$$

By the comparison principles in [11], $\Phi \leq 0$ in Ω. Further, by Lemma 2.1, for each $\xi \in \partial\Omega$, there exists $\overline{x}(\xi) \in \mathbf{R}^n$ such that

$$W_\xi(x) < \Phi(x), \quad x \in \overline{\Omega} \setminus \{\xi\},$$

where

$$W_\xi(x) = \frac{\overline{R}}{2}\big(\big|x - \overline{x}(\xi)\big|^2 - \big|\xi - \overline{x}(\xi)\big|^2\big), \quad \xi \in \mathbf{R}^n,$$

and $\sup_{\xi\in\partial\Omega}|\overline{x}(\xi)| < \infty$. Therefore

$$W_\xi(\xi) = 0, \qquad W_\xi(x) \leq \Phi(x) \leq 0, \quad x \in \overline{\Omega},$$
$$F\big(\lambda\big(D^2 W_\xi(x)\big)\big) = F(\overline{R}, \overline{R}, \ldots, \overline{R}) = \sigma, \quad \xi \in \mathbf{R}^n.$$

Denote

$$W(x) = \sup_{\xi\in\partial\Omega} W_\xi(x).$$

Then

$$W(x) \leq \Phi(x), \quad x \in \Omega,$$

and by [12]

$$F\big(\lambda\big(D^2 W\big)\big) \geq \sigma, \quad x \in \mathbf{R}^n.$$

Define

$$V(x) = \begin{cases} \Phi(x), & x \in \Omega, \\ W(x), & x \in \mathbf{R}^n \setminus \Omega. \end{cases}$$

Then $V \in C^0(\mathbf{R}^n)$ is an admissible viscosity solution of

$$F\big(\lambda\big(D^2 V\big)\big) \geq \sigma, \quad x \in \mathbf{R}^n.$$

Fix some $R_1 > 0$ such that $\overline{\Omega} \subset B_{R_1}(0)$, where $B_{R_1}(0)$ is the ball centered at the origin with radius R_1.

Let $R_2 = 2R_1\overline{R}^{\frac{1}{2}}$. For $a > 1$, defuse

$$W_a(x) = \inf_{B_{R_1}} V + \int_{2R_2}^{|\overline{R}^{\frac{1}{2}} x|} \big(s^n + a\big)^{\frac{1}{n}}\, ds, \quad x \in \mathbf{R}^n.$$

Then

$$D_{ij} W_a = \big(|y|^n + a\big)^{\frac{1}{n}-1} \left[\left(|y|^{n-1} + \frac{a}{|y|}\right)\overline{R}\delta_{ij} - \frac{a\overline{R}^2 x_i x_j}{|y|^3}\right], \quad |x| > 0,$$

where $y = \overline{R}^{\frac{1}{2}} x$. By rotating the coordinates, we may set $x = (r, 0, \ldots, 0)$. Therefore

$$D^2 W_a = \big(R^n + a\big)^{\frac{1}{n}-1} \overline{R}\,\mathrm{diag}\left(R^{n-1}, R^{n-1} + \frac{a}{R}, \ldots, R^{n-1} + \frac{a}{R}\right),$$

where $R = |y|$. Consequently, $\lambda(D^2 W_a) \in \Gamma$ for $|x| > 0$ and by (1.4)

$$F\big(\lambda\big(D^2 W_a\big)\big) \geq F(\overline{R}, \overline{R}, \ldots, \overline{R}) = \sigma, \quad |x| > 0.$$

Moreover,

$$W_a(x) \leq V(x), \quad |x| \leq R_1. \tag{3.3}$$

Fix some $R_3 > 3R_2$ satisfying

$$R_3 \overline{R}^{\frac{1}{2}} > 3R_2.$$

We choose $a_1 > 1$ such that for $a \geq a_1$,

$$W_a(x) > \inf_{B_{R_1}} V + \int_{2R_2}^{3R_2} \big(s^n + a\big)^{\frac{1}{n}}\, ds \geq V(x), \quad |x| = R_3.$$

Then by (3.3) $R_3 \geq R_1$. According to the definition of W_a,

$$\begin{aligned} W_a(x) &= \inf_{B_{R_1}} V + \int_{2R_2}^{|\overline{R}^{\frac{1}{2}} x|} s\left(\left(1 + \frac{a}{s^n}\right)^{\frac{1}{n}} - 1\right) ds + \int_{2R_2}^{|\overline{R}^{\frac{1}{2}} x|} s\, ds \\ &= \frac{\overline{R}}{2}|x|^2 + C_m + \inf_{B_{R_1}} V + \int_{2R_2}^{+\infty} s\left(\left(1 + \frac{a}{s^n}\right)^{\frac{1}{n}} - 1\right) ds - C_m \\ &\quad - 2R_2^2 - \int_{|\overline{R}^{\frac{1}{2}} x|}^{+\infty} s\left(\left(1 + \frac{a}{s^n}\right)^{\frac{1}{n}} - 1\right) ds, \quad x \in \mathbf{R}^n. \end{aligned}$$

Let

$$\mu(m,a) = \inf_{B_{R_1}} V + \int_{2R_2}^{+\infty} s\left(\left(1+\frac{a}{s^n}\right)^{\frac{1}{n}} - 1\right) ds - C_m - 2R_2^2.$$

Then $\mu(m,a)$ is continuous and monotonic increasing for a and when $a \to \infty$, $\mu(m,a) \to \infty$, $1 \le m \le k$. Moreover,

$$W_a(x) = \frac{\overline{R}}{2}|x|^2 + C_m + \mu(m,a) - O\big(|x|^{2-n}\big), \quad \text{when } |x| \to \infty. \tag{3.4}$$

Define, for $a \ge a_1$ and $1 \le m \le k$,

$$\underline{u}_{m,a}(x) = \begin{cases} \max\{V(x), W_a(x)\} - \mu(m,a), & |x| \le R_3, \\ W_a - \mu(m,a), & |x| \ge R_3. \end{cases}$$

Then by (3.4), for $1 \le m \le k$,

$$\underline{u}_{m,a}(x) = \frac{\overline{R}}{2}|x|^2 + C_m - O\big(|x|^{2-n}\big), \quad \text{when } |x| \to \infty,$$

and by the definition of V,

$$\underline{u}_{m,a}(x) = -\mu(m,a), \quad x \in \partial\Sigma.$$

Choose $a_2 \ge a_1$ large enough such that when $a \ge a_2$,

$$\begin{aligned} V(x) - \mu(m,a) &= V(x) - \inf_{B_{R_1}} V - \int_{2R_2}^{+\infty} s\left(\left(1+\frac{a}{s^n}\right)^{\frac{1}{n}} - 1\right) ds + C_m + 2R_2^2 \\ &\le C_m \\ &\le \frac{\overline{R}}{2}|x|^2 + C_m, \quad |x| \le R_3. \end{aligned}$$

Therefore

$$\underline{u}_{m,a}(x) \le \frac{\overline{R}}{2}|x|^2 + C_m, \quad a \ge a_2, x \in \mathbf{R}^n.$$

By Lemma 2.2, $\underline{u}_{m,a} \in C^0(\mathbf{R}^n)$ is admissible and satisfies in the viscosity sense

$$F\big(\lambda\big(D^2\underline{u}_{m,a}\big)\big) \ge \sigma, \quad x \in \mathbf{R}^n.$$

It is easy to see that there exists a continuous function $a^{(m)}(a)$ such that $\lim_{a\to\infty} a^{(m)}(a) = \infty$ and $\mu(m, a^{(m)}(a)) = \mu(1,a)$ for $2 \le m \le k$. So there exists $a_3 \ge a_2$ such that $a^{(m)}(a) > a_2$ whenever $a \ge a_3$ and $2 \le m \le k$. Let $a^{(1)}(a) = a$ and define

$$\underline{u}_a(x,m) = \underline{u}_{m,a^{(m)}(a)}(x), \quad (x,m) \in M_k.$$

Then, by the definition of $\underline{u}_{m,a}$, when $a \geq a_3$, $\underline{u}_a \in C^0(M_k)$ is a locally admissible function satisfying

$$\underline{u}_a(x,m) = \frac{\overline{R}}{2}|x|^2 + C_m - O\bigl(|x|^{2-n}\bigr), \quad \text{when } |x| \to \infty,$$
$$\underline{u}_a(x,m) \leq \frac{\overline{R}}{2}|x|^2 + C_m, \quad x \in \mathbf{R}^n, 1 \leq m \leq k,$$
$$\lim_{x \to \overline{x}} \underline{u}_a(x,m) = -\mu(1,a), \quad \overline{x} \in \partial\Sigma, 1 \leq m \leq k,$$

and in the viscosity sense

$$F\bigl(\lambda\bigl(D^2\underline{u}_a\bigr)\bigr) \geq \sigma, \quad (x,m) \in M_k.$$

Step 2. We define the solution of (3.1) by the Perron method.

For $a \geq a_3$, let S_a denote the set of admissible functions $V \in C^0(M_k)$ which can be extended to $\partial\Sigma$ and satisfies

$$F\bigl(\lambda\bigl(D^2V\bigr)\bigr) \geq \sigma, \quad (x,m) \in M_k,$$
$$\lim_{x \to \overline{x}} V(x,m) \leq -\mu(1,a), \quad \overline{x} \in \Gamma,$$
$$V(x,m) \leq \frac{\overline{R}}{2}|x|^2 + C_m, \quad x \in \mathbf{R}^n, 1 \leq m \leq k.$$

It is obvious that $\underline{u}_a \in S_a$. Hence $S_a \neq \emptyset$. Define

$$u_a(x,m) = \sup\bigl\{V(x,m) : V \in S_a\bigr\}, \quad (x,m) \in M_k.$$

Next we prove that u_a is a viscosity solution of (3.1). From the definition of u_a, it is a viscosity subsolution of (3.1) and satisfies

$$u_a(x,m) \leq \frac{\overline{R}}{2}|x|^2 + C_m, \quad x \in \mathbf{R}^n.$$

So we need only to prove that u_a is a viscosity supersolution of (3.1) satisfying (3.2).

For any $x_0 \in \mathbf{R}^n \setminus \partial\Sigma$, fix $\varepsilon > 0$ such that $\overline{B} = \overline{B_\varepsilon(x_0)} \subset \mathbf{R}^n \setminus \partial\Sigma$. Then the lifting of B into M_k is the k disjoint balls denoted as $\{B^{(i)}\}_{i=1}^k$. For any $(x,m) \in B^{(i)}$, by Lemma 2.3, there exists an admissible viscosity solution $\widetilde{u} \in C^0(\overline{B^{(i)}})$ to the Dirichlet problem

$$F\bigl(\lambda\bigl(D^2\widetilde{u}\bigr)\bigr) = \sigma, \quad (x,m) \in B^{(i)},$$
$$\widetilde{u} = u_a, \quad (x,m) \in \partial B^{(i)}.$$

By the comparison principle in [11],

$$u_a \leq \widetilde{u}, \quad (x,m) \in B^{(i)}. \tag{3.5}$$

Define

$$\psi(x,m) = \begin{cases} \widetilde{u}(x,m), & (x,m) \in B^{(i)}, \\ u_a(x,m), & (x,m) \in M_k \setminus \{B^{(i)}\}_{i=1}^k. \end{cases}$$

By Lemma 2.2,

$$F\big(\lambda\big(D^2\psi(x,m)\big)\big) \geq \sigma, \quad x \in \mathbf{R}^n.$$

As

$$F\big(\lambda\big(D^2\widetilde{u}\big)\big) = \sigma = F\big(\lambda\big(D^2 g\big)\big), \quad (x,m) \in B^{(i)},$$
$$\widetilde{u} = u_a \leq g, \quad (x,m) \in \partial B^{(i)},$$

where $g(x,m) = \frac{\overline{R}}{2}|x|^2 + C_m$, we have

$$\widetilde{u} \leq g, \quad (x,m) \in \overline{B^{(i)}}$$

by the comparison principle in [11]. Therefore $\psi \in S_a$.

By the definition of $u_a, u_a \geq \psi$ in M_k. Consequently, $\widetilde{u} \leq u_a$ in $B^{(i)}$ and further $\widetilde{u} = u_a$, $(x,m) \in B^{(i)}$ in view of (3.5). Since x_0 is arbitrary, we conclude that u_a is an admissible viscosity solution of (3.1).

By the definition of u_a,

$$\underline{u}_a \leq u_a \leq g, \quad (x,m) \in M_k,$$

so u_a satisfies (3.2) and we complete the proof of Theorem 3.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally and read and approved the final version of the manuscript.

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