Jeong et al. *Boundary Value Problems* 2014, **2014**:167
http://www.boundaryvalueproblems.com/content/2014/1/167

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Solvability for nonlocal boundary value problems on a half line with $\dim(\ker L) = 2$

Jeongmi Jeong[1], Chan-Gyun Kim[2] and Eun Kyoung Lee[2*]

*Correspondence: eunkyoung165@gmail.com
[2]Department of Mathematics Education, Pusan National University, Busan, 609-735, South Korea
Full list of author information is available at the end of the article

Abstract
The existence of at least one solution to the second-order nonlocal boundary value problems on a half line is investigated by using Mawhin's continuation theorem.
MSC: Primary 34B10; 34B40; secondary 34B15

Keywords: resonance; nonlocal boundary condition; semilinear; a half line

1 Introduction

Boundary value problems on an infinite interval arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and in various applications such as an unsteady flow of gas through a semi-infinite porous media, theory of drain flows and plasma physics. For an extensive collection of results as regards boundary value problems on unbounded domains, we refer the reader to a monograph by Agarwal and O'Regan [1]. For more recent results on unbounded domains, we refer the reader to [2–15] and the references therein.

A boundary value problem is called to be a resonance one if the corresponding homogeneous boundary value problem has a non-trivial solution. Resonance problems can be expressed as an abstract equation $Lx = Nx$, where L is a noninvertible operator. When L is linear, Mawhin's continuation theorem [16] is an efficient tool in finding solutions for these problems. Recently, there have been many works concerning the existence of solutions for multi-point boundary value problems at resonance. For example, see [7, 17–21] in the case that $\dim(\ker L) = 1$, and see [4, 22–25] in the case that $\dim(\ker L) = 2$.

In this paper, we consider the existence of solutions to the following second-order nonlinear differential equation with nonlocal boundary conditions that contain integral and multi-point boundary conditions:

$$\begin{cases} (cu')'(t) = f(t, u(t), u'(t)), & \text{a.e. } t \in (0,\infty), \\ (cu')(0) = \int_0^\infty g(s)(cu')(s)\,ds, & \lim_{t\to\infty}(cu')(t) = \sum_{i=1}^m \alpha_i (cu')(\xi_i), \end{cases} \tag{1}$$

where $0 \le \xi_1 < \cdots < \xi_m < \infty$, $\alpha_i \in (-\infty,\infty)$, $c \in C[0,\infty)$, $g \in L^1(0,\infty)$, and $f : [0,\infty) \times (-\infty,\infty) \times (-\infty,\infty) \to (-\infty,\infty)$ is a Carathéodory function, *i.e.*, $f = f(t,u,v)$ is Lebesgue measurable in t for all $(u,v) \in (-\infty,\infty) \times (-\infty,\infty)$ and continuous in (u,v) for almost all $t \in [0,\infty)$. Throughout this paper, we assume that the following conditions hold:

(H1) $\sum_{i=1}^m \alpha_i = 1$, $\int_0^\infty g(s)\,ds = 1$, $c(t) > 0$ for $t \in (0,\infty)$, and $\frac{1}{c} \in L^1_{\text{loc}}[0,\infty)$;

(H2) let $w(t) := \int_0^t \frac{1}{c(s)}\,ds$, and there exist non-negative measurable functions α, β, and γ such that $(1+w)\alpha, \beta/c, \gamma \in L^1(0,\infty)$, and $|f(t,u,v)| \le \alpha(t)|u| + \beta(t)|v| + \gamma(t)$, a.e. $(t,u,v) \in [0,\infty) \times (-\infty,\infty) \times (-\infty,\infty)$;

(H3) let $k(t)$ be the function such that $(1+w(\cdot))e^{-k(\cdot)} \in L^1(0,\infty)$. Then

$$\triangle = \begin{vmatrix} Q_2(w(\cdot)e^{-k(\cdot)}) & -Q_1(w(\cdot)e^{-k(\cdot)}) \\ -Q_2(e^{-k(\cdot)}) & Q_1(e^{-k(\cdot)}) \end{vmatrix} =: \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0,$$

where the linear operators $Q_1, Q_2 : L^1(0,\infty) \to (-\infty,\infty)$ will be defined later in Section 3.

If $\frac{1}{c} \in L^1_{\mathrm{loc}}[0,\infty)$, then w is continuous in $[0,\infty)$, and in (H3), there exists a function k satisfying $(1+w(\cdot))e^{-k(\cdot)} \in L^1(0,\infty)$ (see, *e.g.*, Remark 3.1(1)). The boundary conditions in problem (1) are crucial since the differential operator $Lu = (cu')'$ under the boundary conditions in (1) satisfies $\dim(\ker L) = 2$. The purpose of this paper is to establish the sufficient conditions for the existence of solutions to problem (1) on a half line at resonance with $\dim(\ker L) = 2$ by using Mawhin's continuation theorem [16].

The remainder of this paper is organized as follows: some preliminaries are provided in Section 2, the main result is presented in Section 3, and finally an example to illustrate the main result is given in Section 4.

2 Preliminaries

In this section, we recall some notations and two theorems which will be used later. Let X and Y be two Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $L : \mathrm{dom}(L) \subset X \to Y$ be a Fredholm operator with index zero, and let $P : X \to X$, $Q : Y \to Y$ be projectors such that $\mathrm{Im}\,P = \ker L$ and $\ker Q = \mathrm{Im}\,L$. Then $X = \ker L \oplus \ker P$ and $Y = \mathrm{Im}\,L \oplus \mathrm{Im}\,Q$. It follows that $L|_{\mathrm{dom}\,L \cap \ker P} : \mathrm{dom}\,L \cap \ker P \to \mathrm{Im}\,L$ is invertible. We denote the inverse of it by K_P. If Ω is an open bounded subset of X with $\mathrm{dom}\,L \cap \overline{\Omega} \neq \emptyset$, then the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded in Y and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.1 ([16]) *Let $L : \mathrm{dom}(L) \subset X \to Y$ be a Fredholm operator with index zero and $N : X \to Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied*:

(1) *$Lu \neq \lambda Nu$ for every $(u,\lambda) \in [(\mathrm{dom}\,L \setminus \ker L) \cap \partial\Omega] \times (0,1)$;*

(2) *$Nu \notin \mathrm{Im}\,L$ for every $u \in \ker L \cap \partial\Omega$;*

(3) *$\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, where $Q : Y \to Y$ is a projector such that $\mathrm{Im}\,L = \ker Q$.*

Then the equation $Lu = Nu$ has at least one solution in $\mathrm{dom}\,L \cap \overline{\Omega}$.

Since the Arzelá-Ascoli theorem fails in the noncompact interval case, we use the following result in order to show that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.2 ([1]) *Let Z be the space of all bounded continuous vector-valued functions on $[0,\infty)$ and $S \subset Z$. Then S is relatively compact in Z if the following conditions hold*:

(i) *S is bounded in Z;*

(ii) *S is equicontinuous on any compact interval of $[0,\infty)$;*

(iii) *S is equiconvergent at ∞, that is, given $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that $\|\phi(t) - \phi(\infty)\|_{(-\infty,\infty)^n} < \epsilon$ for all $t > T$ and all $\phi \in S$.*

3 Main results

Let X be the set of the functions $u \in C[0,\infty) \cap C^1(0,\infty)$ such that $\frac{u(t)}{1+w(t)}$ and $(cu')(t)$ are uniformly bounded in $[0,\infty)$. Here, w is the function in the assumption (H2). Then X is a Banach space equipped with a norm $\|u\|_X = \|u\|_1 + \|u\|_2$, where

$$\|u\|_1 = \sup_{t\in[0,\infty)} \frac{|u(t)|}{1+w(t)} \quad \text{and} \quad \|u\|_2 = \sup_{t\in[0,\infty)} \left|\left(cu'\right)(t)\right|.$$

Let Y denote the Banach space $L^1(0,\infty)$ equipped with the usual norm, $\|h\|_Y = \int_0^\infty |h(s)|\,ds$.

Remark 3.1 (1) For any non-negative continuous function $w(t)$, we can choose a function $k(t)$ which satisfies $(1+w(\cdot))e^{-k(\cdot)} \in Y$. For example, put

$$k(t) = \int_0^t \left(1 + w(s)\right) ds.$$

Then $(1+w(\cdot))e^{-k(\cdot)} \in Y$.

(2) If $\frac{1}{c} \in Y$, then $w \in L^\infty(0,\infty)$, and the norm $\|\cdot\|_X$ is equivalent to the norm $\|u\| = \|u\|_\infty + \|u\|_2$. Here,

$$\|u\|_\infty = \sup_{t\in[0,\infty)} \left|u(t)\right|.$$

Define $L : \operatorname{dom} L \to Y$ by $Lu := (cu')'$, where

$$\operatorname{dom} L := \left\{ u \in X : \left(cu'\right)' \in Y, \left(cu'\right)(0) = \int_0^\infty g(s)\left(cu'\right)(s)\,ds, \right.$$
$$\left. \lim_{t\to\infty} \left(cu'\right)(t) = \sum_{i=1}^m \alpha_i \left(cu'\right)(\xi_i) \right\}.$$

Clearly, $\ker L := \{a + bw : a, b \in (-\infty,\infty)\}$. Now we define the linear operators $Q_1, Q_2 : Y \to (-\infty,\infty)$ under the hypothesis (H3) as follows:

$$Q_1(y) := \sum_{i=1}^m \alpha_i \int_{\xi_i}^\infty y(s)\,ds \quad \text{and} \quad Q_2(y) := \int_0^\infty g(s) \int_0^s y(\tau)\,d\tau\,ds.$$

Lemma 3.2 *Assume that* (H1) *holds. Then*

$$\operatorname{Im} L = \left\{ y \in Y : Q_1(y) = Q_2(y) = 0 \right\}.$$

Proof Let $y \in \operatorname{Im} L$. Then there exists $x \in \operatorname{dom} L$ such that $(cx')' = y$. For $t \in [0,\infty)$,

$$\left(cx'\right)(t) = \left(cx'\right)(0) + \int_0^t y(s)\,ds$$

and

$$\left(cx'\right)(\xi_i) = \left(cx'\right)(0) + \int_0^{\xi_i} y(s)\,ds,$$

which imply that

$$\lim_{t\to\infty}\left(cx'\right)(t) = \left(cx'\right)(0) + \int_0^\infty y(s)\,ds = \left(cx'\right)(\xi_i) + \int_{\xi_i}^\infty y(s)\,ds,$$

and thus $Q_1(y) = 0$ by the fact that $\sum_{i=1}^m \alpha_i = 1$. In a similar manner, $Q_2(y) = 0$. On the other hand, let $y \in Y$ satisfying $Q_1(y) = Q_2(y) = 0$. Take

$$x(t) = \int_0^t \frac{1}{c(s)} \int_0^s y(\tau)\,d\tau\,ds.$$

Then $x \in \operatorname{dom} L$, and $(cx')' = y \in \operatorname{Im} L$. Thus the proof is complete. □

By Lemma 3.2, $\operatorname{codim}(\operatorname{Im} L) = 2$. Since $\dim(\ker L) = 2$, L is a Fredholm operator with index 0. Let $T_1, T_2 : Y \to Y$ be linear operators which are defined as follows:

$$T_1 y := \frac{1}{\triangle}\left(a_{11}Q_1(y) + a_{12}Q_2(y)\right)e^{-k(\cdot)},$$
$$T_2 y := \frac{1}{\triangle}\left(a_{21}Q_1(y) + a_{22}Q_2(y)\right)e^{-k(\cdot)}.$$

Then, by simple calculations, $T_1(T_1y) = T_1y$, $T_1((T_2y)w) = 0$, $T_2(T_1y) = 0$, and $T_2((T_2y)w) = T_2y$. Define a bounded linear operator $Q : Y \to Y$ by

$$(Qy)(t) := (T_1y)(t) + (T_2y)(t)w(t), \quad t \in (0,\infty). \tag{2}$$

Then $Q^2y = Qy$, *i.e.*, $Q : Y \to Y$ is a linear projector. Since $\triangle \neq 0$, $\ker Q = \operatorname{Im} L$, and $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$.

Define a continuous projector $P : X \to X$ by

$$Px := x(0) + \left(cx'\right)(0)w(\cdot).$$

Clearly, $\operatorname{Im} P = \ker L$, and consequently $X = \ker L \oplus \ker P$. Define an operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ by

$$(K_Py)(t) := \int_0^t \frac{1}{c(s)} \int_0^s y(\tau)\,d\tau\,ds, \quad t \in [0,\infty).$$

Then K_P is the inverse operator $L|_{\operatorname{dom} L\cap\ker P}$, and it satisfies

$$\|K_Py\|_X \le 2\|y\|_Y. \tag{3}$$

Let a nonlinear operator $N : X \to Y$ be defined by $(Nx)(t) := f(t, x(t), x'(t))$, $t \in [0,\infty)$. Then problem (1) is equivalent to $Lx = Nx$, $x \in \operatorname{dom} L$.

From now on, we consider the case $\frac{1}{c} \notin Y$. The case $\frac{1}{c} \in Y$ can be dealt in a similar manner.

Lemma 3.3 *Let* $\frac{1}{c} \notin Y$, *and assume that* (H1)-(H3) *hold. Assume that* Ω *is a bounded open subset of* X *such that* $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$. *Then* N *is* L*-compact on* $\overline{\Omega}$.

Proof Since Ω is a bounded open subset of X, there exists a constant $r > 0$ such that $\|x\|_X \leq r$ for any $x \in \overline{\Omega}$. For any $x \in \overline{\Omega}$ and for almost all $t \in [0,\infty)$, by (H2),

$$\left|(Nx)(t)\right| = \left|f\left(t, x(t), x'(t)\right)\right| \leq \left(\left(1 + w(t)\right)\alpha(t) + \frac{\beta}{c}(t)\right)r + \gamma(t) =: \mathcal{N}_r(t).$$

Then $\mathcal{N}_r \in Y$ and $\|Nx\|_Y \leq \|\mathcal{N}_r\|_Y$ for all $x \in \overline{\Omega}$. Thus $N(\overline{\Omega})$ is bounded in Y.

For any $x \in \overline{\Omega}$,

$$\left|Q_1(Nx)\right| \leq \left(\sum_{i=1}^{m} |\alpha_i|\right)\|Nx\|_Y \leq \left(\sum_{i=1}^{m} |\alpha_i|\right)\|\mathcal{N}_r\|_Y, \qquad \left|Q_2(Nx)\right| \leq \|\mathcal{N}_r\|_Y. \tag{4}$$

It follows from (2) and (4) that, for almost all $t \in [0,\infty)$,

$$\begin{aligned}\left|(QNx)(t)\right| &\leq \left|(T_1Nx)(t)\right| + \left|(T_2Nx)(t)\right|w(t) \\ &\leq \frac{1}{|\triangle|}\left(\left(|a_{11}|\sum_{i=1}^{m}|\alpha_i| + |a_{12}|\right) + \left(|a_{21}|\sum_{i=1}^{m}|\alpha_i| + |a_{22}|\right)w(t)\right)e^{-k(t)}\|\mathcal{N}_r\|_Y \\ &=: \mathcal{Q}_r(t).\end{aligned}$$

Then $\mathcal{Q}_r \in Y$ and $\|QNx\|_Y \leq \|\mathcal{Q}_r\|_Y$ for all $x \in \overline{\Omega}$. Thus, $QN(\overline{\Omega})$ is bounded in Y.

Next we will prove that $K_P(I-Q)N(\overline{\Omega})$ is a relatively compact set in X. For $x \in \overline{\Omega}$, by (3),

$$\left\|K_P(I-Q)Nx\right\|_X \leq 2\left(\|Nx\|_Y + \|QNx\|_Y\right) \leq 2\left(\|\mathcal{N}_r\|_Y + \|\mathcal{Q}_r\|_Y\right).$$

Thus $K_P(I-Q)N(\overline{\Omega})$ is bounded in X.

Let $T > 0$ be given. For any $x \in \overline{\Omega}$ and let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$,

$$\begin{aligned}&\left|\frac{(K_P(I-Q)Nx)(t_1)}{1+w(t_1)} - \frac{(K_P(I-Q)Nx)(t_2)}{1+w(t_2)}\right| \\ &\quad\leq \left|\frac{w(t_2)-w(t_1)}{(1+w(t_1))(1+w(t_2))}\right|\left|\int_0^{t_1}\frac{1}{c(s)}\int_0^s (I-Q)Nx(\tau)\,d\tau\,ds\right| \\ &\qquad+ \left|\frac{\int_{t_1}^{t_2}\frac{1}{c(s)}\int_0^s(I-Q)Nx(\tau)\,d\tau\,ds}{1+w(t_2)}\right| \\ &\quad\leq 2\left|w(t_2)-w(t_1)\right|\left(\|\mathcal{N}_r\|_Y + \|\mathcal{Q}_r\|_Y\right)\end{aligned}$$

and

$$\left|\left(c\left(K_P(I-Q)Nx\right)'\right)(t_1) - \left(c\left(K_P(I-Q)Nx\right)'\right)(t_2)\right| \leq \int_{t_1}^{t_2}\left(\mathcal{N}_r(\tau) + \mathcal{Q}_r(\tau)\right)d\tau,$$

which imply that

$$\left\{\frac{K_P(I-Q)Nx}{1+w} : x \in \overline{\Omega}\right\} \quad \text{and} \quad \left\{c\left(K_P(I-Q)Nx\right)' : x \in \overline{\Omega}\right\}$$

are equicontinuous on $[0,T]$.

For any $x \in \overline{\Omega}$, by L'Hôspital's rule,

$$\begin{aligned}
&\lim_{t\to\infty} \frac{(K_P(I-Q)Nx)(t)}{1+w(t)} \\
&\quad= \lim_{t\to\infty} \int_0^t \bigl((I-Q)Nx\bigr)(s)\,ds \\
&\quad= \int_0^\infty \bigl((I-Q)Nx\bigr)(s)\,ds - \lim_{t\to\infty} \int_t^\infty \bigl((I-Q)Nx\bigr)(s)\,ds
\end{aligned}$$

and

$$\left| \bigl(c\bigl(K_P(I-Q)Nx\bigr)'\bigr)(t) - \int_0^\infty \bigl((I-Q)Nx\bigr)(\tau)\,d\tau \right| \le \int_t^\infty \bigl|\bigl((I-Q)Nx\bigr)(\tau)\bigr|\,d\tau.$$

Since $|(I-Q)Nx| \le \mathcal{N}_r + \mathcal{Q}_r$ for all $x \in \overline{\Omega}$,

$$\frac{(K_P(I-Q)Nx)(t)}{1+w(t)} \to \int_0^\infty \bigl((I-Q)Nx\bigr)(s)\,ds$$

and

$$\bigl(c\bigl(K_P(I-Q)Nx\bigr)'\bigr)(t) \to \int_0^\infty \bigl((I-Q)Nx\bigr)(\tau)\,d\tau$$

uniformly on $\overline{\Omega}$ as $t \to \infty$. In view of Theorem 2.2, $K_P(I-Q)N(\overline{\Omega})$ is a relatively compact set in X, and thus N is L-compact on $\overline{\Omega}$. □

The following theorem is the main result in this paper.

Theorem 3.4 *Let $\frac{1}{c} \notin Y$, and assume that* (H1)-(H3) *hold. Assume also that the following hold*:

(H4) *there exist positive constants A and B such that if $|x(t)| > A$ for every $t \in [0,B]$ or $|(cx')(t)| > A$ for every $t \in [0,\infty)$, then $QNx \neq 0$;*

(H5) *there exists a positive constant C such that if $|a| > C$ or $|b| > C$, then $QN(a+bw) \neq 0$, and only one of the following inequalities is satisfied*:

$$\begin{aligned}
&\text{(i)} \quad aQ_1\bigl(N(a+bw)\bigr) + bQ_2\bigl(N(a+bw)\bigr) \ge 0, \\
&\text{(ii)} \quad aQ_1\bigl(N(a+bw)\bigr) + bQ_2\bigl(N(a+bw)\bigr) \le 0.
\end{aligned}$$

If α and β satisfy

$$\bigl\|(1+w)\alpha\bigr\|_Y + \left\|\frac{\beta}{c}\right\|_Y < \frac{1}{4+w(B)}, \tag{5}$$

then problem (1) *has at least one solution in X.*

Proof We divide the proof into four steps.

Step 1. Let

$$\Omega_1 = \bigl\{x \in \operatorname{dom} L \setminus \ker L : Lx = \lambda Nx \text{ for some } \lambda \in (0,1)\bigr\}.$$

Then Ω_1 is bounded. In fact, $x \in \Omega_1$ means $\lambda \in (0,1)$ and $Nx \in \operatorname{Im} L$. Thus, by Lemma 3.2, $Q_1(Nx) = Q_2(Nx) = 0$. By (H4), there exist $t_0 \in [0,B]$ and $t_1 \in [0,\infty)$ such that $|x(t_0)| \le A$, $|(cx')(t_1)| \le A$. Then

$$\left|\left(cx'\right)(0)\right| = \left|\left(cx'\right)(t_1) - \int_0^{t_1} \left(cx'\right)'(s)\,ds\right| \le A + \|Nx\|_Y.$$

On the other hand,

$$\begin{aligned}\left|x(0)\right| &= \left|x(t_0) - \int_0^{t_0} \frac{1}{c(s)}\left[\left(cx'\right)(t_1) + \int_{t_1}^{s} \left(cx'\right)'(\tau)\,d\tau\right]ds\right| \\ &\le \left|x(t_0)\right| + \left(\left|\left(cx'\right)(t_1)\right| + \|Nx\|_Y\right)w(B) \le A\left(1 + w(B)\right) + w(B)\|Nx\|_Y.\end{aligned}$$

Thus,

$$\|Px\|_X \le \left|x(0)\right| + 2\left|\left(cx'\right)(0)\right| \le A\left(3 + w(B)\right) + \left(2 + w(B)\right)\|Nx\|_Y.$$

Since $\ker L = \operatorname{Im} P$, $LPx = 0$ for all $x \in X$, it follows from (3) and (H2) that

$$\begin{aligned}\|x\|_X &= \left\|Px + (I-P)x\right\|_X = \left\|Px + K_P L(I-P)x\right\|_X \le \|Px\|_X + 2\|Lx\|_Y \\ &\le \|Px\|_X + 2\|Nx\|_Y \le A\left(3 + w(B)\right) + \left(4 + w(B)\right)\|Nx\|_Y \\ &\le A\left(3 + w(B)\right) + \left(4 + w(B)\right)\left(\left(\left\|(1+w)\alpha\right\|_Y + \left\|\frac{\beta}{c}\right\|_Y\right)\|x\|_X + \|\gamma\|_Y\right),\end{aligned}$$

and, by (5),

$$\|x\|_X \le \frac{A(3 + w(B)) + (4 + w(B))\|\gamma\|_Y}{1 - (4 + w(B))(\|(1+w)\alpha\|_Y + \|\frac{\beta}{c}\|_Y)},$$

which implies that Ω_1 is bounded.

Step 2. Set

$$\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}.$$

Then Ω_2 is bounded. In fact, $x \in \Omega_2$ implies $x = a + bw$ and $Q_1(Nx) = Q_2(Nx) = 0$. By (H5), we obtain $|a| \le C$ and $|b| \le C$. Thus Ω_2 is bounded.

Step 3. Define an isomorphism $J : \ker L \to \operatorname{Im} Q$ by

$$J\left(a + bw(\cdot)\right) = \frac{1}{\triangle}\left(a_{11}a + a_{12}b + (a_{21}a + a_{22}b)w(\cdot)\right)e^{-k(\cdot)}.$$

Assume first that (i) in (H5) holds. Let

$$\Omega_3 = \left\{x \in \ker L : \lambda Jx + (1-\lambda)QNx = 0 \text{ for some } \lambda \in [0,1]\right\}.$$

Then Ω_3 is bounded. Indeed, $x \in \Omega_3$ means that there exist constants $a, b \in (-\infty,\infty)$ and $\lambda \in [0,1]$ such that $x = a + bw$ and $\lambda Jx + (1-\lambda)QNx = 0$. If $\lambda = 0$, then $QNx = 0$. It follows from $\triangle \neq 0$ that $Q_1(Nx) = Q_2(Nx) = 0$. By (H5), we obtain $|a| \leq C$ and $|b| \leq C$. If $\lambda = 1$, clearly $a = b = 0$. For $\lambda \in (0,1)$, by the facts that $-\lambda Jx = (1-\lambda)QNx$ and $\triangle \neq 0$, it follows that $-\lambda a = (1-\lambda)Q_1(Nx)$ and $-\lambda b = (1-\lambda)Q_2(Nx)$. If $|a| > C$ or $|b| > C$, by (i) in (H5),

$$-\lambda\bigl(a^2+b^2\bigr) = (1-\lambda)\bigl(aQ_1N(a+bw) + bQ_2N(a+bw)\bigr) \geq 0,$$

which is a contradiction. Thus Ω_3 is bounded in X. On the other hand, if (ii) in (H5) holds, taking

$$\Omega_3 = \bigl\{x \in \ker L : -\lambda Jx + (1-\lambda)QNx = 0 \text{ for some } \lambda \in [0,1]\bigr\},$$

one can show that Ω_3 is bounded in a similar manner.

Step 4. Take an open bounded set $\Omega \supset \bigcup_{i=1}^{3} \overline{\Omega}_i \cup \{0\}$. By Step 1 and Step 2, in view of Theorem 2.1, we only need to prove that $\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$ in order to show that $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. Let

$$H(x,\lambda) = \pm\lambda Jx + (1-\lambda)QNx.$$

By Step 3, $H(x,\lambda) \neq 0$ for all $(x,\lambda) \in (\ker L \cap \partial\Omega) \times [0,1]$. Thus, by the homotopy invariance property of degree,

$$\begin{aligned}\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) &= \deg\bigl(H(\cdot,0), \Omega \cap \ker L, 0\bigr)\\ &= \deg\bigl(H(\cdot,1), \Omega \cap \ker L, 0\bigr) = \deg(\pm J, \Omega \cap \ker L, 0) \neq 0.\end{aligned}$$

By Theorem 2.1, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, *i.e.*, problem (1) has at least one solution in X. □

In the case that $\frac{1}{c} \in Y$, $w \in L^\infty(0,\infty)$, and using the norm $\|\cdot\|$ on X in Remark 3.1(2), we have a similar result to Theorem 3.4. We omit the proof.

Theorem 3.5 *Let $\frac{1}{c} \in Y$, and assume that* (H1)-(H5) *hold. If α and β satisfy*

$$\|\alpha\|_Y + \left\|\frac{\beta}{c}\right\|_Y < \frac{1}{2+3\|w\|_\infty},$$

then problem (1) *has at least one solution in X.*

4 Example

Consider the following second-order nonlinear differential equation:

$$\begin{cases} (cu')'(t) = f(t,u(t),u'(t)), & \text{a.e. } t \in (0,\infty), \\ (cu')(0) = \int_0^\infty g(s)(cu')(s)\,ds, & \lim_{t\to\infty}(cu')(t) = (cu')(1), \end{cases} \tag{6}$$

where $f(t,u,v) = \alpha_1(t)u + \beta_1(t)v + \gamma_1(t)$ for $(t,u,v) \in [0,\infty) \times (-\infty,\infty) \times (-\infty,\infty)$, $\gamma_1 \in Y$ satisfies $\int_0^1 (1-t^2)\gamma_1(t)\,dt = \int_1^\infty \gamma_1(t)\,dt = 0$,

$$\alpha_1(t) = \begin{cases} Ke^{-t}, & t \in [0,1], \\ 0, & t \in (1,\infty), \end{cases} \qquad \beta_1(t) = \begin{cases} 0, & t \in [0,1], \\ K(e-4)e^{-t}, & t \in (1,\infty), \end{cases}$$

$$c(t) = \begin{cases} t^\rho, & t \in [0,1], \\ 1, & t \in (1,\infty), \end{cases} \qquad g(t) = \begin{cases} 2t, & t \in [0,1], \\ 0, & t \in (1,\infty), \end{cases}$$

$$w(t) = \begin{cases} \frac{1}{1-\rho} t^{1-\rho}, & t \in [0,1], \\ t + \frac{\rho}{1-\rho}, & t \in (1,\infty). \end{cases}$$

We also assume that the constants K and ρ satisfy

$$0 < K < K_\rho := \frac{1-\rho}{(5-4\rho)(3e^{-1} + \frac{1}{1-\rho}\int_0^1 t^{1-\rho} e^{-t}\, dt)}$$

and $\rho \in [0,1)$. Then $1/c \notin Y$, and (H1)-(H2) hold for $m = \alpha_1 = \xi_1 = 1$, $\alpha = \alpha_1$, $\beta = |\beta_1|$, and $\gamma = |\gamma_1|$. For $y \in Y$,

$$Q_1(y) = \int_1^\infty y(s)\, ds \quad \text{and} \quad Q_2(y) = \int_0^1 (1-s^2) y(s)\, ds.$$

Taking $k(t) = t$, then

$$a_{11} = \frac{1}{1-\rho}\int_0^1 (1-t^2) t^{1-\rho} e^{-t}\, dt, \qquad a_{12} = \frac{\rho - 2}{1-\rho} e^{-1},$$
$$a_{21} = 1 - 4e^{-1}, \qquad a_{22} = e^{-1}$$

and

$$\begin{aligned} \triangle = a_{11}a_{22} - a_{12}a_{21} &= \frac{1}{(1-\rho)e}\int_0^1 (1-t^2) t^{1-\rho} e^{-t}\, dt + \frac{2-\rho}{1-\rho} e^{-1}(1 - 4e^{-1}) \\ &= \frac{e-4}{e^2} + \frac{1}{(1-\rho)e}\left(\frac{e-4}{e} + \int_0^1 (1-t^2) t^{1-\rho} e^{-t}\, dt\right). \end{aligned}$$

Since, for any $\rho \in [0,1)$,

$$\frac{e-4}{e} + \int_0^1 (1-t^2) t^{1-\rho} e^{-t}\, dt \le \frac{e-4}{e} + \int_0^1 (1-t^2) e^{-t}\, dt = 0,$$

and $\triangle < 0$. Thus (H3) holds for any $\rho \in [0,1)$.

Take $B = 1$ and let $A > 0$ be arbitrary. Then (H4) holds. In fact, if $|x(t)| > A$ for $t \in [0,1]$, then

$$Q_2(Nx) = K\int_0^1 (1-t^2) e^{-t} x(t)\, dt \ne 0.$$

If $|(cx')(t)| > A$ for $t \in [0,\infty)$, then $|x'(t)| > A$ for $t \in [1,\infty)$, and

$$Q_1(Nx) = K(e-4)\int_1^\infty e^{-t} x'(t)\, dt \ne 0.$$

Since

$$Q_1\big(N(a+bw)\big) = K(1 - 4e^{-1})b$$

and

$$Q_2\big(N(a+bw)\big) = K\left(\big(4e^{-1}-1\big)a + \frac{b}{1-\rho}\int_0^1 \big(1-t^2\big)e^{-t}t^{1-\rho}\,dt\right)$$

for $(a,b) \neq (0,0)$,

$$\big|Q_1\big(N(a+bw)\big)\big| + \big|Q_2\big(N(a+bw)\big)\big| \neq 0,$$

i.e., $QN(a+bw) \neq 0$ for $(a,b) \neq (0,0)$, and

$$aQ_1\big(N(a+bw)\big) + bQ_2\big(N(a+bw)\big) = b^2\frac{K}{1-\rho}\int_0^1 \big(1-t^2\big)e^{-t}t^{1-\rho}\,dt \geq 0$$

for all $(a,b) \in (-\infty,\infty) \times (-\infty,\infty)$. Therefore (H5) holds for any $C > 0$. Since $0 < K < K_\rho$, (5) is satisfied, and thus there exists at least one solution to problem (6) in view of Theorem 3.4.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final draft.

Author details
[1]Department of Mathematics, Pusan National University, Busan, 609-735, South Korea. [2]Department of Mathematics Education, Pusan National University, Busan, 609-735, South Korea.

Acknowledgements
This study was supported by the Research Fund Program of Research Institute for Basic Sciences, Pusan National University, Korea, 2014, Project No. RIBS-PNU-2014-101.

Received: 3 March 2014 Accepted: 25 June 2014 Published online: 26 September 2014

References

1. Agarwal, RP, O'Regan, D: Infinite Interval Problems for Differential, Difference and Integral Equations. Kluwer Academic, Dordrecht (2001). doi:10.1007/978-94-010-0718-4
2. Agarwal, RP, O'Regan, D: Infinite interval problems modeling phenomena which arise in the theory of plasma and electrical potential theory. Stud. Appl. Math. **111**(3), 339-358 (2003). doi:10.1111/1467-9590.t01-1-00237
3. Cupini, G, Marcelli, C, Papalini, F: On the solvability of a boundary value problem on the real line. Bound. Value Probl. **2011**, 26 (2011)
4. Jiang, W, Wang, B, Wang, Z: Solvability of a second-order multi-point boundary-value problems at resonance on a half-line with $\dim\ker L = 2$. Electron. J. Differ. Equ. **2011**, 120 (2011)
5. Kim, C-G: Solvability of multi-point boundary value problems on the half-line. J. Nonlinear Sci. Appl. **5**(1), 27-33 (2012) (Special issue)
6. Kim, C-G: Existence and iteration of positive solutions for multi-point boundary value problems on a half-line. Comput. Math. Appl. **61**(7), 1898-1905 (2011). doi:10.1016/j.camwa.2011.02.023
7. Kosmatov, N: Multi-point boundary value problems on an unbounded domain at resonance. Nonlinear Anal. **68**(8), 2158-2171 (2008). doi:10.1016/j.na.2007.01.038
8. Kosmatov, N: Second order boundary value problems on an unbounded domain. Nonlinear Anal. **68**(4), 875-882 (2008). doi:10.1016/j.na.2006.11.043
9. Lian, H, Ge, W: Solvability for second-order three-point boundary value problems on a half-line. Appl. Math. Lett. **19**(10), 1000-1006 (2006). doi:10.1016/j.aml.2005.10.018
10. Liang, S, Zhang, J: Positive solutions for singular third-order boundary value problem with dependence on the first order derivative on the half-line. Acta Appl. Math. **111**(1), 27-43 (2010). doi:10.1007/s10440-009-9528-z
11. Liu, L, Wang, Y, Hao, X, Wu, Y: Positive solutions for nonlinear singular differential systems involving parameter on the half-line. Abstr. Appl. Anal. **2012**, Article ID 161925 (2012)
12. Ma, R: Existence of positive solutions for second-order boundary value problems on infinity intervals. Appl. Math. Lett. **16**(1), 33-39 (2003). doi:10.1016/S0893-9659(02)00141-6
13. Marcelli, C: The role of boundary data on the solvability of some equations involving non-autonomous nonlinear differential operators. Bound. Value Probl. **2013**, 252 (2013)
14. O'Regan, D, Yan, B, Agarwal, RP: Nonlinear boundary value problems on semi-infinite intervals using weighted spaces: an upper and lower solution approach. Positivity **11**(1), 171-189 (2007). doi:10.1007/s11117-006-0050-5

15. O'Regan, D, Yan, B, Agarwal, RP: Solutions in weighted spaces of singular boundary value problems on the half-line. J. Comput. Appl. Math. **205**(2), 751-763 (2007). doi:10.1016/j.cam.2006.02.055
16. Mawhin, J: Topological Degree Methods in Nonlinear Boundary Value Problems. CBMS Regional Conference Series in Mathematics, vol. 40. Am. Math. Soc., Providence (1979) (Expository lectures from the CBMS Regional Conference held at Harvey Mudd College, Claremont, Calif., June 9-15, 1977)
17. Cui, Y: Solvability of second-order boundary-value problems at resonance involving integral conditions. Electron. J. Differ. Equ. **2012**, 45 (2012)
18. Feng, W, Webb, JRL: Solvability of three point boundary value problems at resonance. In: Proceedings of the Second World Congress of Nonlinear Analysts: Part 6 (Athens, 1996), vol. 30, pp. 3227-3238 (1997). doi:10.1016/S0362-546X(96)00118-6
19. Franco, D, Infante, G, Zima, M: Second order nonlocal boundary value problems at resonance. Math. Nachr. **284**(7), 875-884 (2011). doi:10.1002/mana.200810841
20. Wang, F, Zhang, F: Existence of *n* positive solutions to second-order multi-point boundary value problem at resonance. Bull. Korean Math. Soc. **49**(4), 815-827 (2012). doi:10.4134/BKMS.2012.49.4.815
21. Zima, M, Drygaś, P: Existence of positive solutions for a kind of periodic boundary value problem at resonance. Bound. Value Probl. **2013**, 19 (2013). doi:10.1186/1687-2770-2013-19
22. Du, Z, Meng, F: Solutions to a second-order multi-point boundary value problem at resonance. Acta Math. Sci. **30**(5), 1567-1576 (2010). doi:10.1016/S0252-9602(10)60150-6
23. Kosmatov, N: A multi-point boundary value problem with two critical conditions. Nonlinear Anal. **65**(3), 622-633 (2006). doi:10.1016/j.na.2005.09.042
24. Meng, F, Du, Z: Solvability of a second-order multi-point boundary value problem at resonance. Appl. Math. Comput. **208**(1), 23-30 (2009). doi:10.1016/j.amc.2008.11.026
25. Zhang, X, Feng, M, Ge, W: Existence result of second-order differential equations with integral boundary conditions at resonance. J. Math. Anal. Appl. **353**(1), 311-319 (2009). doi:10.1016/j.jmaa.2008.11.082

doi:10.1186/s13661-014-0167-6
Cite this article as: Jeong et al.: **Solvability for nonlocal boundary value problems on a half line with** $\dim(\ker L) = 2$**.** *Boundary Value Problems* 2014 **2014**:167.