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Global existence of a coupled Euler-Bernoulli plate system with variable coefficients

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Abstract

In this paper, we study the initial boundary value problem of a coupled Euler-Bernoulli plate system with spatially varying coefficients of viscosity, damping and source term in a bounded domain. We prove that under some conditions on the initial value, the growth orders of the damping terms and the source terms the solution to the problem exists globally.

MSC: 35L70; 35L75; 93D20

Keywords: coupled plate system; variable coefficients; global existence

1 Introduction

The purpose of this paper is to present the global existence result of the solution to a coupled Euler-Bernoulli plate system with variable coefficients.

Let Ω be a bounded domain in R^n with smooth boundary and $\Gamma = \partial\Omega$. We shall consider the following initial-boundary value problem for a coupled Euler-Bernoulli plate system with variable coefficients:

$$\begin{cases} \rho_1(x)u_{tt} + \mathcal{A}^2u - \gamma\mathcal{A}v + |u_t|^{p-1}u_t = f_1(u,v), & x\in\Omega, t\in(0,T), \\ \rho_2(x)v_{tt} + \mathcal{A}^2v + \gamma\mathcal{A}u + |v_t|^{q-1}v_t = f_2(u,v), & x\in\Omega, t\in(0,T), \\ u = v = \frac{\partial u}{\partial v_{\mathcal{A}}} = \frac{\partial u}{\partial v_{\mathcal{A}}} = 0, & x\in\Gamma, t\in(0,T), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), & x\in\bar{\Omega}, \\ (\sqrt{\rho_1}u_t(x,0), \sqrt{\rho_2}v_t(x,0)) = (\sqrt{\rho_1}u_1(x), \sqrt{\rho_2}v_1(x)), & x\in\bar{\Omega}, \end{cases} \tag{1.1}$$

in which the constants $\gamma \ge 0$, $p \ge 1$, $q \ge 1$, and

$$\mathcal{A}u = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right), \quad x\in R^n,$$

where $a_{ij}(x) = a_{ji}(x)$ are C^∞ functions in R^n satisfying

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \lambda\sum_{i=1}^n \xi_i^2 > 0, \quad \forall x\in R^n, 0 \ne (\xi_1,\xi_2,\ldots,\xi_n)^T \in R^n, \tag{1.2}$$

λ is a positive constant. $\frac{\partial}{\partial \nu_{\mathcal{A}}} = \sum_{i,j=1}^{n} a_{ij}(x)\nu_i \frac{\partial}{\partial x_j}$ is the so-called co-normal derivative and $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is the unit normal of Γ pointing towards the exterior of Ω. In physical terms the entries $a_{ij}(x)$ are related to the coefficients of elasticity.

We suppose that $\rho_i \in C^1(\overline{\Omega})$ $(i = 1, 2)$ is the density (the mass per unit length) of the plates respectively, and it satisfies the following hypothesis:

$$\rho_i \geq \rho_0 > 0, \quad i = 1, 2, \tag{1.3}$$

where ρ_0 is a constant. We also assume that there exists a function $F(\xi, \eta)$ such that

$$f_1(\xi, \eta) = \frac{\partial F(\xi, \eta)}{\partial \xi}, \qquad f_2(\xi, \eta) = \frac{\partial F(\xi, \eta)}{\partial \eta} \quad \text{for } (\xi, \eta) \in R^2, \tag{1.4}$$

and suppose that $F(\xi, \eta)$ satisfies

$$0 \leq F(\xi, \eta) \leq d_1\bigl(|\xi|^{m+1} + |\eta|^{m+1}\bigr) \quad \text{for } (\xi, \eta) \in R^2, \tag{1.5}$$

in which d_1 is a positive constant, $1 < m \leq \frac{6}{n-4}$ for $n > 4$, and $m > 1$ for $n \leq 4$.

The systems of plate equation have been studied by numerous authors, see [1–9]. Messaoudi [7] considered the semilinear Petrovsky equation with the damping and the source term, showed that the solution blows up in finite time if the growth order of damping term is larger than the growth order of source term and the energy is negative, whereas the solution is global if the growth order of damping term is not larger than the growth order of source term. Hoffmann and Rybka [5] studied the analyticity of the nonlinear term forces convergence of solutions for two equations of continuum mechanics. They showed that any solution with appropriate boundary and initial conditions has a limit as t goes to infinity. Yao [9] discussed the initial-boundary value problem for Euler-Bernoulli plate with variable coefficients and gave the observability inequality. The authors of [1] were interested in the case of thermoelastic plate and they established stability of the rest state. The authors of [2] discussed the exponential attractors for an extensible beam equation. [5] differs from [1, 2] because [5] included a viscous term Δu_t. This term played an important role in their consideration. Guesmia [3] considered the system of plate equations with damping, proved that the solution decays exponentially if the damping term behaves like a linear function, whereas the decay is of a polynomial order otherwise. Li and Wu [6] discussed the plate stabilization problem with infinite damping and showed that the energy of the problem decays exponentially provided that the negative damping is sufficiently small. For the Cauchy problem of multidimensional generalized double dispersion equation, Xu and Liu [8] proved the existence and nonexistence of global weak solution by the potential well method. Komornik and Zuazua [10] considered a wave equation with mixed boundary condition. They showed the stabilization without a geometrical hypothesis on the domain. This is an original work that used the multiplicative technique in the study of asymptotic behavior of a dissipative system in PDE using the energy method.

Our purpose in this paper is to give the global existence of the solution to the initial boundary value problem for a coupled Euler-Bernoulli plate system with variable coefficients.

The present work is organized as follows. In Section 2, we give the local existence result of problem (1.1). Section 3 is devoted to the global existence of the solution.

We shall write $\|\cdot\|$ denoting the usual $L^2(\Omega)$ norm $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_p$ denoting the usual $L^p(\Omega)$ norm $\|\cdot\|_{L^p(\Omega)}$, $\langle\cdot,\cdot\rangle$ denoting the usual inner product in $L^2(\Omega)$.

Let

$$W = \left\{ w \in H^2(\Omega) \middle| w = \frac{\partial w}{\partial \nu_{\mathcal{A}}} = 0 \text{ on } \Gamma \right\}.$$

Let $B > 0$ be the best constant of the embedding inequality such that

$$\|w\|_{m+1} \leq B\|\mathcal{A}w\|, \quad \forall w \in W. \tag{1.6}$$

2 Local existence result

In order to state our main results, we introduce the definition of weak solution for system (1.1).

Definition By a weak solution (u, v) of system (1.1) on $[0, T]$, we mean functions $u, v \in C([0, T], W) \cap C^1([0, T], H_0^1(\Omega))$, $u_0, v_0 \in W$, $u_1, v_1 \in H_0^1(\Omega)$ satisfying

$$\begin{cases} \langle \rho_1 u_t, \varphi\rangle - \langle \rho_1 u_1, \varphi\rangle - \int_0^t \langle \rho_1 u_\tau, \varphi_\tau\rangle \, d\tau + \int_0^t \langle \mathcal{A}u, \mathcal{A}\varphi\rangle \, d\tau \\ \quad + \gamma \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \, dx + \int_0^t \langle |u_t|^{p-1} u_t, \varphi\rangle \, d\tau = \int_0^t \langle f_1, \varphi\rangle \, d\tau, \\ \langle \rho_2 v_t, \psi\rangle - \langle \rho_2 v_1, \psi\rangle - \int_0^t \langle \rho_2 v_\tau, \psi_\tau\rangle \, d\tau + \int_0^t \langle \mathcal{A}v, \mathcal{A}\psi\rangle \, d\tau \\ \quad - \gamma \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \psi}{\partial x_j} \, dx + \int_0^t \langle |v_t|^{q-1} v_t, \psi\rangle \, d\tau = \int_0^t \langle f_2, \psi\rangle \, d\tau, \\ (u(0), v(0)) = (u_0, v_0), \qquad (\sqrt{\rho_1} u_t(0), \sqrt{\rho_2} v_t(0)) = (\sqrt{\rho_1} u_1, \sqrt{\rho_2} v_1), \end{cases}$$

for all text functions $\varphi, \psi \in W$ and almost all $t \in [0, T]$.

Theorem 2.1 (Local weak solution) *Suppose the validity of assumptions* (1.2)-(1.5). *Then there exists a unique local weak solution* (u, v) *to system* (1.1) *defined on* $[0, T_0]$ *for some* $T_0 > 0$.

Proof Similar to [11].

We define the energy of system (1.1) by

$$E(t) := \frac{1}{2}\|\sqrt{\rho_1} u_t\|^2 + \frac{1}{2}\|\sqrt{\rho_2} v_t\|^2 + \frac{1}{2}\|\mathcal{A}u\|^2 + \frac{1}{2}\|\mathcal{A}v\|^2 - \int_\Omega F(u, v) \, dx. \tag{2.1}$$

□

Lemma 2.2 *Suppose the validity of assumptions* (1.2)-(1.5). *Then the energy function of system* (1.1) *satisfies the energy identity,*

$$E(t) + \int_0^t \|u_\tau\|_{p+1}^{p+1} \, d\tau + \int_0^t \|v_\tau\|_{q+1}^{q+1} \, d\tau = E(0). \tag{2.2}$$

Proof Multiplying the first equation of (1.1) by u_t and the second equation of (1.1) by v_t, integrating over $\Omega \times [0, t]$ and summing them, by a straightforward calculation we obtain (2.2). □

3 Global existence result

In this section, we give the global existence of system (1.1).

Define

$$J(t) = \|\sqrt{\rho_1}u_t\|^2 + \|\sqrt{\rho_2}v_t\|^2 + \|\mathcal{A}u\|^2 + \|\mathcal{A}v\|^2 - (m+1)\int_\Omega F(u,v)\,dx. \tag{3.1}$$

Theorem 3.1 *Suppose that* (u,v) *is the unique weak solution of system* (1.1), *the initial data* $(u_0,v_0,u_1,v_1) \in W \times W \times L^2(\Omega) \times L^2(\Omega)$ *satisfy*

$$J(0) > 0, \qquad d_1 B^{m+1}(m+1) < \left[\frac{2(m+1)}{m-1}E(0)\right]^{\frac{1-m}{2}}.$$

Then the solution (u,v) *is global.*

Proof From the continuity of $E(t)$ and $F(u,v)$, we have the continuity of $J(t)$ on $[0,T]$.

Denote

$$t_0 := \sup\bigl\{t \in [0,T]; J(\tau) > 0, \tau \in [0,t)\bigr\}.$$

From the assumption $J(0) > 0$, it is not difficult to see that $0 < t_0 \le T$. Next we show that $t_0 = T$ and $T > 0$ is arbitrary. From the definition of $E(t)$ and $J(t)$, we can rewrite $E(t)$ for $t \in [0,t_0)$ as

$$\begin{aligned} E(t) &= \frac{m-1}{2(m+1)}\bigl[\|\sqrt{\rho_1}u_t\|^2 + \|\sqrt{\rho_2}v_t\|^2 + \|\mathcal{A}u\|^2 + \|\mathcal{A}v\|^2\bigr] + \frac{1}{m+1}J(t) \\ &> \frac{m-1}{2(m+1)}\bigl[\|\sqrt{\rho_1}u_t\|^2 + \|\sqrt{\rho_2}v_t\|^2 + \|\mathcal{A}u\|^2 + \|\mathcal{A}v\|^2\bigr]. \end{aligned} \tag{3.2}$$

Thus from the nonincreasing property of $E(t)$ (2.2), it follows that for $t \in [0,t_0)$,

$$\|\sqrt{\rho_1}u_t\|^2 + \|\sqrt{\rho_2}v_t\|^2 + \|\mathcal{A}u\|^2 + \|\mathcal{A}v\|^2 < \frac{2(m+1)}{m-1}E(t) \le \frac{2(m+1)}{m-1}E(0). \tag{3.3}$$

From assumptions (1.5)-(1.6) and estimate (3.3), we get that for $t \in [0,t_0)$,

$$\begin{aligned} \int_\Omega F(u,v)\,dx &\le d_1\bigl(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\bigr) \\ &\le d_1 B^{m+1}\bigl(\|\mathcal{A}u\|^{m+1} + \|\mathcal{A}v\|^{m+1}\bigr) \\ &\le d_1 B^{m+1}\bigl(\|\mathcal{A}u\|^2 + \|\mathcal{A}v\|^2\bigr)^{\frac{m+1}{2}} \\ &< d_1 B^{m+1}\bigl(\|\mathcal{A}u\|^2 + \|\mathcal{A}v\|^2\bigr)\left[\frac{2(m+1)}{m-1}E(0)\right]^{\frac{m-1}{2}}. \end{aligned} \tag{3.4}$$

Then it follows that for $\tau \in (0,1)$ small enough,

$$J(t) > \tau\left(\|\mathcal{A}u\|^2 + \|\mathcal{A}v\|^2\right) + (m+1)\left\{(1-\tau)\left[d_1 B^{m+1}(m+1)\left(\frac{2(m+1)}{m-1}E(0)\right)^{\frac{m-1}{2}}\right]^{-1} - 1\right\} \times \int_\Omega F(u,v)\,dx. \tag{3.5}$$

Thanks to the assumption of $E(0)$, we have

$$d_1 B^{m+1}(m+1)\left[\frac{2(m+1)}{m-1}E(0)\right]^{\frac{m-1}{2}} < 1.$$

Thus, by selecting $1-\tau = d_1 B^{m+1}(m+1)[\frac{2(m+1)}{m-1}E(0)]^{\frac{m-1}{2}}$, we get

$$J(t) > \tau\left(\|\mathcal{A}u\|^2 + \|\mathcal{A}v\|^2\right)$$

for $t \in [0,t_0)$. This implies $J(t_0) \geq 0$. If $J(t_0) = 0$, then it follows that $u(x,t_0) = v(x,t_0) = u_t(x,t_0) = v_t(x,t_0) = 0$ for $x \in \bar{\Omega}$. By the uniqueness of the solution to system (1.1), we obtain that the solution is zero solution for $t \in [t_0,+\infty)$. Then the solution is global.

For the case $J(t_0) > 0$, from the definition of t_0, we have $t_0 = T$ and $J(t) > 0$ for $t \in [0,T]$, where $T > 0$ is arbitrary and the following estimate holds:

$$\frac{m-1}{2(m+1)}\left[\|\sqrt{\rho_1}u_t\|^2 + \|\sqrt{\rho_2}v_t\|^2 + \|\mathcal{A}u\|^2 + \|\mathcal{A}v\|^2\right] < E(t) \leq E(0). \tag{3.6}$$

Then the global existence of the solution follows. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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