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The local well-posedness of solutions for a nonlinear pseudo-parabolic equation

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Abstract
The local existence and uniqueness of solutions for a nonlinear pseudo-parabolic equation are established in the Sobolev space $C([0,T);H^s(R^n)) \cap C^1([0,T);H^{s-1}(R^n))$ with $s > \frac{n}{2}$. In addition, we prove the global existence of solutions for two special cases of the equation.

MSC: 35Q35; 35Q51

Keywords: local strong solution; well-posedness; nonlinear pseudo-parabolic equation

1 Introduction

The pseudo-parabolic equation possesses the form

$$\frac{\partial u}{\partial t} - k\frac{\partial \triangle u}{\partial t} = \triangle u + u^p, \quad x \in R^n, t > 0, \tag{1}$$

where constant $k > 0$, $p > 0$, and $\triangle = \sum_1^n \frac{\partial^2}{\partial x_i^2}$. If $k = 0$, Eq. (1) becomes the heat equation with sources. If $k > 0$, we call Eq. (1) as the pseudo-parabolic model (see Ting [1], Showalter and Ting [2]). The pseudo-parabolic equation has many important physical backgrounds such as the seepage of homogeneous fluids through a fissured rock [3], the unidirectional propagation of nonlinear dispersive long waves [4, 5] and the aggregation of populations [6] (where u is the population density). Equation (1) is employed in the analysis of nonstationary processes in the area of semiconductors [7, 8], where the term $k\frac{\partial \triangle u}{\partial t} - \frac{\partial u}{\partial t}$ is regarded as the free electron density rate, term $\triangle u$ is regarded as the linear dissipation of the free charge current and u^p is a source of free electron current. Equation (1) is also named a Sobolev type model or a Sobolev-Galpern type model [9].

The initial-boundary value problem and the initial problem for the linear pseudo-parabolic equation were investigated in [1, 2, 10] where the existence and uniqueness of solutions for the equation were established. Various dynamic properties of solutions for nonlinear pseudo-parabolic equations, including singular pseudo-parabolic equations and degenerate pseudo-parabolic equations can be found in [11–18]. It is worth to mention that Kaikina *et al.* [19] considered the superlinear case of the Cauchy problem for Eq. (1) with $p > 1$ and showed the existence and uniqueness of the solutions. Furthermore, it was shown that the Cauchy problem for Eq. (1) has a unique global solution under the assumptions $p > 1 + \frac{2}{n}$ and sufficiently small initial value u_0. The existence, uniqueness, and comparison principle for mild solutions of Eq. (1) were established in Cao *et al.* [20] by whom the large time behavior of the solutions and the critical global existence exponent and the critical Fujita exponent for Eq. (1) were obtained.

In this work, we study the following nonlinear pseudo-parabolic equation:

$$\frac{\partial u}{\partial t} - k\frac{\partial \triangle u}{\partial t} = \triangle u + \alpha u^q + \beta Df(u), \quad x \in R^n, t > 0, \tag{2}$$

where $q \geq 1$ is an integer, α and β are constants, $f(u)$ is a polynomial with order m, $f(0) = 0$, and $D = \sum_1^n \frac{\partial}{\partial x_i}$. When $\beta = 0$, Eq. (2) reduces to Eq. (1). The existence and uniqueness of local solutions for Eq. (2) are established in the Sobolev space $C([0,T);H^s(R^n)) \cap C^1([0,T);H^{s-1}(R^n))$ with $s > \frac{n}{2}$. We find that the local solution in the space $H^s(R^n)$ blows up if and only if $\lim_{t\to T}\|u(t,\cdot)\|_{L^\infty(R^n)} = \infty$. For the space dimension $n = 1$, assuming that the initial value $u_0 \in H^1(R^1)$, $\alpha < 0$, and p is an odd number, we find the global existence of solutions for Eq. (2). For the other case $n = 1$, $p = 1$, and initial value $u_0 \in H^1(R)$, we also acquire the global existence result of solutions for Eq. (2).

The rest of this paper is organized as follows. The main results are stated in Section 2. Several lemmas and the proofs of main results are given in Section 3.

2 Main results

Firstly, we state some notations.

Let $L^p = L^p(R^n)$ $(1 \leq p < +\infty)$ be the space of all measurable functions h such that $\|h\|_{L^p}^p = \int_{R^n} |h(t,x)|^p\,dx < \infty$. We define $L^\infty = L^\infty(R^n)$ with the standard norm $\|h\|_{L^\infty} = \inf_{m(e)=0}\sup_{x\in R^n\setminus e}|h(t,x)|$. For any real number s, $H^s = H^s(R^n)$ denotes the Sobolev space with the norm defined by

$$\|h\|_{H^s} = \left(\int_{R^n} \left(1 + |\xi|^2\right)^s \left|\hat{h}(t,\xi)\right|^2 d\xi\right)^{\frac{1}{2}} < \infty,$$

where $\hat{h}(t,\xi) = \int_{R^n} e^{-ix\xi} h(t,x)\,dx$.

For $T > 0$ and nonnegative number s, $C([0,T);H^s(R^n))$ denotes the Frechet space of all continuous H^s-valued functions on $[0,T)$. We set $\Lambda = (1 - \sum_1^n \frac{\partial^2}{\partial x_i^2})^{\frac{1}{2}}$ and $\Theta = (1 - k\triangle)^{\frac{1}{2}}$. For simplicity, throughout this article, we let c denote any positive constant.

We consider the Cauchy problem for Eq. (2)

$$\begin{cases} \frac{\partial u}{\partial t} - k\frac{\partial\triangle u}{\partial t} = \triangle u + \alpha u^q + \beta Df(u), & x \in R^n, t > 0, \\ u(0,x) = u_0(x), & x \in R^n, \end{cases} \tag{3}$$

which is equivalent to

$$\begin{cases} \frac{\partial u}{\partial t} = -\frac{1}{k}u + \Theta^{-2}[\frac{u}{k} + \alpha u^q + \beta Df(u)], & x \in R^n, t > 0, \\ u(0,x) = u_0(x), & x \in R^n, \end{cases} \tag{4}$$

where Θ^{-2} is the inverse operator of $\Theta^2 = 1 - k\triangle$.

Now, we give our main results for problem (3).

Theorem 2.1 *Let $u_0(x) \in H^s(R^n)$ with $s > \frac{n}{2}$. Then the Cauchy problem* (3) *has a unique solution $u(t,x) \in C([0,T);H^s(R^n)) \cap C^1([0,T);H^{s-1}(R^n))$ where T is the maximum existence* *time. Moreover,*

$$\lim_{t\to T}\|u(t,\cdot)\|_{H^s(R^n)} = \infty$$

if and only if

$$\lim_{t\to T}\|u(t,\cdot)\|_{L^\infty(R^n)} = \infty.$$

For the case of space dimension $n = 1$, we have the result.

Theorem 2.2 *Let $n = 1$, $u_0 \in H^1(R)$ in system* (3), *and assume that q is an odd number and $\alpha \le 0$. Then problem* (3) *has a unique global solution $u(t,x)$ satisfying*

$$u(t,x) \in C\big([0,\infty); H^s(R)\big) \cap C^1\big([0,\infty); H^{s-1}(R)\big), \quad s > \frac{1}{2}.$$

Theorem 2.3 *Let $n = 1$, $q = 1$, and $u_0 \in H^1(R)$ in system* (3). *For any constants α and β, then problem* (3) *has a unique global solution $u(t,x)$ satisfying*

$$u(t,x) \in C\big([0,\infty); H^s(R)\big) \cap C^1\big([0,\infty); H^{s-1}(R)\big), \quad s > \frac{1}{2}.$$

3 Several lemmas

Lemma 3.1 *Let r and ρ be real numbers such that $-r < \rho \le r$. Then*

$$\|uv\|_{H^\rho(R^n)} \le c\|u\|_{H^r(R^n)}\|v\|_{H^\rho(R^n)}, \quad \text{if } r > \frac{n}{2},$$
$$\|uv\|_{H^{r+\rho-1/2}(R^n)} \le c\|u\|_{H^r(R^n)}\|v\|_{H^\rho(R^n)}, \quad \text{if } r < \frac{n}{2}.$$

This lemma can be found in [21] or [22].

Lemma 3.2 (Kato and Ponce [23]) *If $r \ge 0$, then $H^r \cap L^\infty$ is an algebra. Moreover,*

$$\|uv\|_{H^r(R^n)} \le c\big(\|u\|_{L^\infty(R^n)}\|v\|_{H^r(R^n)} + \|u\|_{H^r(R^n)}\|v\|_{L^\infty(R^n)}\big),$$

where c is a constant depending only on r.

Lemma 3.3 *Assume $u_0 \in H^s(R^n)$ with $s > \frac{n}{2}$. Then problem* (3) *admits a unique local solution*

$$u(t,x) \in C\big([0,T); H^s(R)\big) \cap C^1\big([0,T); H^{s-1}(R)\big).$$

Proof For the first equation of problem (4), we have

$$u = u_0 + \int_0^t \left(-\frac{u}{k} + \Theta^{-2}\left[\frac{u}{k} + \alpha u^q + \beta Df(u)\right]\right) dt. \tag{5}$$

Letting functions u and v be in the closed ball $B_{M_0}(0)$ of radius $M_0 > 1$ about the zero function in $C([0,T]; H^s(R^n))$ and letting Γ be the operator on the right-hand side of (5), for fixed $t \in [0, T]$, we get

$$\begin{aligned}
&\left\| \int_0^t \left(-\frac{u}{k} + \Theta^{-2}\left[\frac{u}{k} + \alpha u^q + \beta Df(u) \right] \right) dt \right. \\
&\qquad \left. - \int_0^t \left(-\frac{v}{k} + \Theta^{-2}\left[\frac{v}{k} + \alpha v^q + \beta Df(v) \right] \right) dt \right\|_{H^s} \\
&\quad \le T \Big(\sup_{0 \le t \le T} \|u - v\|_{H^s(R^n)} + \sup_{0 \le t \le T} \left\| u^q - v^q \right\|_{H^s(R^n)} \\
&\qquad + \sup_{0 \le t \le T} \left\| f(u) - f(v) \right\|_{H^s(R^n)} \Big). \qquad (6)
\end{aligned}$$

Using Lemma 3.1 derives

$$\begin{aligned}
&\left\| u^q - v^q \right\|_{H^s(R^n)} \\
&\quad = \left\| (u - v)\left(u^{q-1} + u^{q-2}v + \cdots + uv^{q-2} + v^{q-1} \right) \right\|_{H^s(R^n)} \\
&\quad \le \|u - v\|_{H^s(R^n)} \left\| \left(u^{q-1} + u^{q-2}v + \cdots + uv^{q-2} + v^{q-1} \right) \right\|_{H^s(R^n)} \\
&\quad \le c M_0^{q-1} \|u - v\|_{H^s(R^n)} \qquad (7)
\end{aligned}$$

and

$$\left\| f(u) - f(v) \right\|_{H^s(R^n)} \le c M_0^{m-1} \|u - v\|_{H^s(R^n)}. \qquad (8)$$

From (5)-(8), we obtain

$$\|\Gamma u - \Gamma v\|_{H^s} \le \theta \|u - v\|_{H^s(R^n)}, \qquad (9)$$

where $\theta = \max(cTM_0, cTM_0^{q-1}, cTM_0^{m-1})$ and c is independent of T. Choosing T sufficiently small such that $\theta < 1$, we know that operator Γ is a contractive mapping. Applying the above inequality and (5) yields

$$\|\Gamma u\|_{H^s(R^n)} \le \|u_0\|_{H^s(R^n)} + \theta \|u\|_{H^s(R^n)}. \qquad (10)$$

Choosing T sufficiently small such that $\theta M_0 + \|u_0\|_{H^s} < M_0$, we know that Γ maps $B_{M_0}(0)$ to itself. It follows from the contractive mapping principle that the mapping Γ has a unique fixed point u in $B_{M_0}(0)$. This completes the proof. □

Lemma 3.4 *Let function $u(t,x)$ be a solution of problem* (3), $s \ge \frac{n}{2}$ *and the initial value* $u_0(x) \in H^s(R^n)$. *For* $r \in (0, s-1]$, *there is a constant c depending only on the coefficients of the first equation of system* (3) *such that*

$$\begin{aligned}
\int_R \left(\Lambda^{r+1} u \right)^2 dx &\le \int_R \left(\Lambda^{r+1} u_0 \right)^2 dx \\
&\quad + c \int_0^t \left(1 + \|u\|_{L^\infty(R^n)}^{q-1} + \|u\|_{L^\infty(R^n)}^{m-1} \right) \|u\|_{H^{r+1}(R^n)}^2 \, d\tau. \qquad (11)
\end{aligned}$$

Proof Using $\Delta = -\Lambda^2 + 1$ and the Parseval equality gives rise to

$$\int_R \Lambda^r u \Lambda^r \Delta u \, dx = -\int_R \left(\Lambda^{r+1} u\right) \Lambda^{r+1} u \, dx + \int_R \left(\Lambda^r u\right)^2 dx.$$

For $r \in (0, s-1]$, applying $(\Lambda^r u)\Lambda^r$ on both sides of the first equation of system (3), noting the above equality and integrating the resultant equation with respect to x by parts, we obtain the equation

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left[\int_R \left(\left(\Lambda^r u\right)^2 + k\left(\Lambda^r u_x\right)^2\right) dx\right] &= -\int_{R^n} \left(\Lambda^{r+1} u\right) \Lambda^{r+1} u \, dx + \int_{R^n} \left(\Lambda^r u\right)^2 dx \\ &\quad + \alpha \int_{R^n} \left(\Lambda^r u\right) \Lambda^r \left(u^q\right) dx + \beta \int_{R^n} \left(\Lambda^r u\right) \Lambda^r f(u) \, dx \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{12}$$

For the terms I_1 and I_2, we have

$$|I_1| \le \|u\|^2_{H^{r+1}(R^n)} \tag{13}$$

and

$$|I_2| \le \|u\|^2_{H^{r+1}(R^n)}. \tag{14}$$

For the terms I_3 and I_4, using Lemma 3.2 gives rise to

$$\begin{aligned} |I_3| &\le \left\| \Lambda^r u \right\|_{L^2(R^n)} \left\| \Lambda^r \left(u^q\right) \right\|_{L^2(R^n)} \\ &\le c\|u\|_{H^r(R^n)} \|u\|^{q-1}_{L^\infty(R^n)} \|u\|_{H^r(R^n)} \\ &\le c\|u\|^{q-1}_{L^\infty(R^n)} \|u\|^2_{H^{r+1}(R^n)} \end{aligned} \tag{15}$$

and

$$\begin{aligned} |I_4| &\le c\left\| \Lambda^r u \right\|_{L^2(R^n)} \left\| \Lambda^r \left[Df(u)\right] \right\|_{L^2(R^n)} \\ &\le c\left\| \Lambda^r u \right\|_{L^2(R^n)} \left\| \Lambda^{r+1} f(u) \right\|_{L^2(R^n)} \\ &\le c\|u\|_{H^r(R^n)} \left(1 + \|u\|^{m-1}_{L^\infty(R^n)}\right) \|u\|_{H^{r+1}(R^n)} \\ &\le c\left(1 + \|u\|^{m-1}_{L^\infty(R^n)}\right) \|u\|^2_{H^{r+1}(R^n)}. \end{aligned} \tag{16}$$

It follows from (12)-(16) that

$$\begin{aligned} &\frac{1}{2}\int_R \left[\left(\Lambda^r u\right)^2 + k\left(\Lambda^r u_x\right)^2\right] dx - \frac{1}{2}\int_R \left[\left(\Lambda^r u_0\right)^2 + k\left(\Lambda^r u_{0x}\right)^2\right] dx \\ &\quad \le c\int_0^t \left(1 + \|u\|^{q-1}_{L^\infty(R^n)} + \|u\|^{m-1}_{L^\infty(R^n)}\right) \|u\|^2_{H^{r+1}} \, d\tau, \end{aligned}$$

which results in (11). □

Proof of Theorem 2.1 Using Lemma 3.4, for any $s > \frac{n}{2}$, we have

$$\|u\|_{H^s(R^n)} \leq c\|u_0\|_{H^s(R^n)} e^{\int_0^t [1+\|u\|_{L^\infty(R^n)}^{q-1} + \|u\|_{L^\infty(R^n)}^{m-1}]} dt. \tag{17}$$

For $s > \frac{n}{2}$, the Sobolev imbedding theorem yields

$$\|u\|_{L^\infty(R^n)} \leq c\|u\|_{H^s(R^n)}. \tag{18}$$

Applying the inequalities (17), (18), and Lemma 3.3 completes the proof. □

Proof of Theorem 2.2 For the space dimension $n = 1$, we write problem (3) in the form

$$\begin{cases} u_t - ku_{txx} = u_{xx} + \alpha u^q + \beta[f(u)]_x, & x \in R, t > 0, \\ u(0,x) = u_0(x), & x \in R. \end{cases} \tag{19}$$

Using $\int_R u^j u_x \, dx = 0$ for any integer j and integration by parts, we have

$$\begin{aligned} &\frac{1}{2}\int_R u^2 \, dx \\ &\quad = \int_R uu_t \, dx \\ &\quad = \int_R u\bigl[ku_{txx} + u_{xx} + \alpha u^q + \beta\bigl[f(u)\bigr]_x\bigr] \, dx \\ &\quad = \int_R \bigl[-ku_x u_{tx} - u_x^2 + \alpha u^{q+1}\bigr] \, dx, \end{aligned} \tag{20}$$

which results in

$$\frac{1}{2}\frac{d}{dt}\int_R \bigl(u^2 + ku_x^2\bigr) \, dx + \int_R u_x^2 \, dx - \alpha \int_R u^{q+1} \, dx = 0, \tag{21}$$

from which we obtain

$$\frac{1}{2}\int_R \bigl(u^2 + ku_x^2\bigr) \, dx + \int_0^t \int_R \bigl[u_x^2 - \alpha u^{q+1}\bigr] \, dx \, dt = \frac{1}{2}\int_R \bigl(u_0^2 + ku_{0x}^2\bigr) \, dx. \tag{22}$$

If q is an odd integer, $\alpha \leq 0$, and $u_0 \in H^1(R)$, we get

$$\bigl\|u(t,\cdot)\bigr\|_{L^\infty(R)} \leq c\|u_0\|_{H^1(R)}. \tag{23}$$

Using the conclusion of Theorem 2.1, we finish the proof of Theorem 2.2. □

Proof of Theorem 2.3 For $n = 1$ and $q = 1$, using (22) yields

$$\frac{1}{2}\int_R \bigl(u^2 + ku_x^2\bigr) \, dx + \int_0^t \int_R \bigl[u_x^2 - \alpha u^2\bigr] \, dx \, dt = \frac{1}{2}\int_R \bigl(u_0^2 + ku_{0x}^2\bigr) \, dx. \tag{24}$$

Since

$$\left|\int_R \bigl[u_x^2 - \alpha u^2\bigr] \, dx\right| \leq \bigl(1 + |\alpha|\bigr)\|u\|_{H^1(R)}^2, \tag{25}$$

it follows from (24) and (25) that

$$\|u\|_{H^1(R)}^2 \le \|u_0\|_{H^1(R)}^2 e^{(1+|\alpha|)t}, \tag{26}$$

from which we obtain

$$\|u\|_{L^\infty(R)} \le \|u_0\|_{H^1(R)} e^{(1+|\alpha|)t}, \tag{27}$$

which together with Theorem 2.1 completes the proof of Theorem 2.3. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The article is a joint work of three authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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