Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On the regularity criterion for the 3D generalized MHD equations in Besov spaces

Shuanghu Zhang[1] and Hua Qiu[2*]

*Correspondence: tsiuhua@scau.edu.cn
[2]Department of Applied Mathematics, South China Agricultural University, Wushan, Guangzhou, 510642, China
Full list of author information is available at the end of the article

Abstract
In this paper, we consider the three-dimensional generalized MHD equations, a system of equations resulting from replacing the Laplacian $-\Delta$ in the usual MHD equations by a fractional Laplacian $(-\Delta)^{\alpha}$. We obtain a regularity criterion of the solution for the generalized MHD equations in terms of the summation of the velocity field u and the magnetic field b by means of the Littlewood-Paley theory and the Bony paradifferential calculus, which extends the previous result.

MSC: 76B03; 76D03

Keywords: generalized MHD equations; regularity criterion; Littlewood-Paley decomposition

1 Introduction

In this paper, we are concerned with the following three-dimensional generalized MHD equations:

$$\begin{cases} \frac{\partial u}{\partial t} + u\cdot\nabla u - b\cdot\nabla b + (-\Delta)^{\alpha}u + \nabla P = 0, \\ \frac{\partial b}{\partial t} + u\cdot\nabla b - b\cdot\nabla u + (-\Delta)^{\beta}b = 0, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \qquad b(x,0) = b_0(x), \end{cases} \qquad x\in\mathbb{R}^3, t>0, \tag{1}$$

where $u(x,t)$ denotes the fluid velocity vector field, b is the magnetic field, $P = P(x,t)$ is the scalar pressure; while $u_0(x)$ and $b_0(x)$ are the given initial velocity and initial magnetic fields, respectively, in the sense of distributions, with $\nabla\cdot u_0 = \nabla b_0 = 0$; $\alpha,\beta > 0$ are the parameters. The generalized MHD equations generalize the usual MHD equations by replacing the Laplacian $-\Delta$ by a general fractional Laplacian $(-\Delta)^{\alpha}$, $(-\Delta)^{\beta}$. As $\alpha = \beta = 1$, the generalized MHD equations reduce to the usual MHD equations; when $\alpha = \beta = 1$ and $b = 0$, the generalized MHD equations reduce to the Navier-Stokes equations. Moreover, it has similar scaling properties and energy estimate to the Navier-Stokes equations and the MHD equations. The study of system (1) will improve our understanding of the Navier-Stokes equations and the MHD equations.

For the 3D generalized MHD equations (1), Wu [1] showed that the system (1) possesses a global weak solutions corresponding to any L^2 initial data. Yet, just like the 3D Navier-Stokes equations and the 3D MHD equations, whether there exists a global smooth solution for the 3D generalized MHD equations (1) or not is an open problem. Recently, many authors studied the regularity problem for the 3D generalized MHD equations (1) intensively. Wu [1, 2] obtained some regularity criteria only relying on the velocity u. Zhou [3] considered the following two cases: $1 \le \alpha = \beta \le \frac{3}{2}$ and $1 \le \beta \le \frac{5}{4} \le \alpha < \frac{5}{2}$ and established the Serrin-type criteria involving velocity u. Wu [4] obtained the classical Beal-Kato-Majda criterion for the system (1). By means of the Fourier localization technique and the Bony paraproduct decomposition, Yuan [5] extended the Serrin-type criterion to

$$u \in L^q\big(0,T;B^s_{p,\infty}\big(\mathbb{R}^3\big)\big), \tag{2}$$

with $\frac{2\alpha}{q} + \frac{3}{p} \le 2\alpha - 1 + s$, $\frac{3}{2\alpha-1+s} < p \le \infty$, $-1 < s \le 1$, $(p,s) \ne (\infty,1)$ provided that $1 \le \alpha = \beta \le \frac{5}{4}$.

On the other hand, suggested by the results in [6, 7], one may presume upon that there should have some cancelation properties between the velocity field u and the magnetic field b. When $\alpha = \beta$, plus and minus the first equation of (1) and the second one, respectively, the system (1) can be rewritten as

$$\begin{cases} \frac{\partial W^+}{\partial t} + W^- \cdot \nabla W^+ + (-\Delta)^\alpha W^+ + \nabla P = 0, \\ \frac{\partial W^-}{\partial t} + W^+ \cdot \nabla W^- + (-\Delta)^\alpha W^- + \nabla P = 0, \\ \nabla \cdot W^+ = \nabla \cdot W^- = 0, \\ W^+(x,0) = W_0^+(x), \qquad W^-(x,0) = W_0^-(x), \end{cases} \tag{3}$$

where

$$W^\pm = u \pm b, \qquad W_0^\pm(x) = u_0(x) \pm b_0(x).$$

In this paper, we are interested in what kind of confluence the integrability of W^+ or W^- brings to the weak solution (u,b) of the system (1). Furthermore, we shall make efforts to establish some new regularity criterion of weak solutions of the system (3) in terms of W^+ or W^-. When $\alpha = \beta = 1$, He and Wang [8], Gala [9], Dong *et al.* [10] establish some regularity criteria in Lorentz spaces, the multiplier space, and the nonhomogeneous Besov space, respectively.

The purpose of this paper is to deal with the case $\alpha = \beta$ of the system (3), to establish certain kind of regularity criteria. The tools we use here are the Littlewood-Paley theory and the Bony paraproduct decomposition. Before stating our main result, we firstly recall the definition of weak solutions to the 3D generalized MHD equations (1) as follows.

Definition 1.1 Suppose that $u_0, b_0 \in L^2(\mathbb{R}^3)$. The vector-valued function (u,b) is called a weak solution to the system (1) on $\mathbb{R}^3 \times (0,T)$, if it satisfies the following properties:

(1) $u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^\alpha(\mathbb{R}^3))$, $b \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^\beta(\mathbb{R}^3))$;

(2) $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$ in the sense of a distribution;

(3) for any $\phi, \varphi \in C_0^\infty(\mathbb{R}^3 \times [0,T))$ with $\nabla \cdot \phi = 0$ and $\nabla \cdot \varphi = 0$, one has

$$\int_0^T \int_{\mathbb{R}^3} \left(\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi \right) u \, dx \, dt + \int_{\mathbb{R}^3} u_0 \cdot \phi(x,0) \, dx = \int_0^T \int_{\mathbb{R}^3} \big(u \Lambda^{2\alpha} \phi + b \cdot \nabla \phi b \big) \, dx \, dt,$$

and

$$\int_0^T \int_{\mathbb{R}^3} \left(\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi \right) b \, dx \, dt + \int_{\mathbb{R}^3} b_0 \cdot \varphi(x,0) \, dx = \int_0^T \int_{\mathbb{R}^3} \left(b \Lambda^{2\beta} \varphi + b \cdot \nabla \varphi u \right) dx \, dt,$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$.

The weak solution (W^+, W^-) to the system (3) can be defined in a similar way as follows.

Definition 1.2 Suppose that $W_0^+, W_0^- \in L^2(\mathbb{R}^3)$. The vector-valued function (W^+, W^-) is called a weak solution to the system (3) on $\mathbb{R}^3 \times (0,T)$, if it satisfies the following properties:

(1) $W^+, W^- \in L^\infty(0,T; L^2(\mathbb{R}^3)) \cap L^2(0,T; H^\alpha(\mathbb{R}^3))$;
(2) $\nabla \cdot W^+ = 0$ and $\nabla \cdot W^- = 0$ in the sense of a distribution;
(3) for any $\phi, \varphi \in C_0^\infty(\mathbb{R}^3 \times [0,T))$ with $\nabla \cdot \phi = 0$ and $\nabla \cdot \varphi = 0$, one has

$$\int_0^T \int_{\mathbb{R}^3} \left(W^+ \cdot \frac{\partial \phi}{\partial t} + \nabla \phi : \left(W^- \otimes W^+ \right) + W^+ \cdot \Lambda^{2\alpha} \phi \right) dx \, dt + \int_{\mathbb{R}^3} W_0^+ \cdot \phi(x,0) \, dx = 0$$

and

$$\int_0^T \int_{\mathbb{R}^3} \left(W^- \cdot \frac{\partial \varphi}{\partial t} + \nabla \varphi : \left(W^+ \otimes W^- \right) + W^- \cdot \Lambda^{2\alpha} \varphi \right) dx \, dt + \int_{\mathbb{R}^3} W_0^- \cdot \varphi(x,0) \, dx = 0,$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$.

From the above two definitions as regards weak solutions, the systems (1) and (3) are equivalent. It is easy to see that (W^+, W^-) also verifies the system (3) in the sense of distribution, provided that (u,b) is a weak solution to the system (1) as $\alpha = \beta$.

Now our main result can be stated.

Theorem 1.3 *Suppose that* $1 \le \alpha = \beta \le \frac{5}{4}$, *the initial velocity and magnetic field* $(u_0, b_0) \in H^1(\mathbb{R}^3)$ *and* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ *in the sense of distribution. Assume that* (u,b) *is a weak solution to the system* (1) *on some interval* $[0,T]$ *with* $0 < T \le \infty$. *If* W^+ *satisfies the following condition*:

$$W^+ \in L^q\left(0,T; B^s_{p,\infty}\left(\mathbb{R}^3\right)\right), \tag{4}$$

with $\frac{2\alpha}{q} + \frac{3}{p} \le 2\alpha - 1 + s$, $\frac{3}{2\alpha - 1 + s} < p \le \infty$, $-1 < s \le 1$, $(p,s) \ne (\infty, 1)$. *Then the weak solution* (u,b) *remains smooth on* $\mathbb{R}^3 \times (0,T]$.

Remark 1.4 It should be mentioned that our condition (4) on W^+ does not seem comparable with (2) on u at least there is no inclusion between them.

Remark 1.5 Our result here improves the recent result obtained by Dong *et al.* [10] as $\alpha = \beta = 1$ except that $s = 1$. However, the method of [10] cannot also be applied to the case $s = 1$ in this paper. We shall consider this problem in the future.

Notation Throughout the paper, C stands for generic constant. We use the notation $A \lesssim B$ to denote the relation $A \leq CB$, and the notation $A \approx B$ to denote the relations $A \lesssim B$ and $B \lesssim A$. For convenience, given a Banach space X, we denote its norm by $\|\cdot\|_X$. If there is no ambiguity, we omit the domain of function spaces.

This paper is structured as follows. In Section 2, we introduce the Littlewood-Paley decomposition and the Bony paradifferential calculus. In Section 3, we give the proof of Theorem 1.3 by means of the Littlewood-Paley theory and the Bony paradifferential calculus.

2 The Littlewood-Paley theory

In this section, we will provides the Littlewood-Paley theory and the related facts.

Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwarz functions of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transformation $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\,dx.$$

Take two nonnegative radial functions $\mathcal{X}, \psi \in \mathcal{S}(\mathbb{R}^3)$ supported respectively in $\mathcal{B} = \{\xi \in \mathbb{R}^3, |\xi| \leq \frac{4}{3}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^3, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that

$$\mathcal{X}(\xi) + \sum_{j\geq 0} \psi\left(2^{-j}\xi\right) = 1, \quad \xi \in \mathbb{R}^3. \tag{5}$$

Let $h = \mathcal{F}^{-1}\psi$ and $\tilde{h} = \mathcal{F}^{-1}\mathcal{X}$. The frequency localization operator is defined by

$$\begin{aligned}
&\Delta_j f = \psi\left(2^{-j}D\right)f = 2^{3j} \int_{\mathbb{R}^3} h\left(2^j y\right) f(x-y)\,dy, \quad j \geq 0, \\
&S_j f = \mathcal{X}\left(2^{-j}D\right)f = \sum_{-1\leq k\leq j-1} \Delta_k f = 2^{3j} \int_{\mathbb{R}^3} \tilde{h}\left(2^j y\right) f(x-y)\,dy, \\
&\Delta_{-1} f = S_0 f, \qquad \Delta_j f = 0 \quad \text{for } j \leq -2.
\end{aligned}$$

Formally, Δ_j is a frequency projection to the annulus $\{|\xi| \approx 2^j\}$, and S_j is a frequency projection to the ball $\{|\xi| \lesssim 2^j\}$. The above dyadic decomposition has a nice quasi-orthogonality, with the choice of $\mathcal{X}$ and ψ; namely, for any $f, g \in \mathcal{S}(\mathbb{R}^3)$, we have the following properties:

$$\begin{aligned}
&\Delta_i \Delta_j f \equiv 0, \quad |i-j| \geq 2, \\
&\Delta_i (S_{j-1} f \Delta_j g) \equiv 0, \quad |i-j| \geq 5.
\end{aligned} \tag{6}$$

Details of the Littlewood-Paley decomposition theory can be found in [11, 12].

Let $s \in \mathbb{R}$, the homogeneous Sobolev space is defined by

$$H^s\left(\mathbb{R}^3\right) = \left\{f \in \mathcal{S}'\left(\mathbb{R}^3\right); \|f\|_{H^s} < +\infty\right\},$$

where

$$\|f\|_{H^s}^2 = \sum_{j=-1}^{\infty} 2^{2js}\|\Delta_j f\|_{L^2}^2$$

and the set $\mathcal{S}'(\mathbb{R}^2)$ of temperate distributions is the dual set of $\mathcal{S}$ for the usual pairing.

When dealing with our problems, we will use some paradifferential calculus [12, 13]. It is a nice way to define a generalized product between temperate distributions, which is continuous in fractional Sobolev spaces, and which yet does not make any sense for the usual product. Let f, g be two temperate distributions. We denote

$$T_f g \triangleq \sum_{i\le j} \Delta_i f\,\Delta_j g = \sum_j S_{j-1}f\,\Delta_j g, \qquad R(f,g) \triangleq \sum_{|i-j|\le 2} \Delta_i f\,\Delta_j g. \tag{7}$$

At least, we have the following the Bony decomposition:

$$fg = T_f g + T_g f + R(f,g), \tag{8}$$

where the paraproduct T is a bilinear continuous operator. For simplicity, we denote

$$T'_f g = T_f g + R(f,g).$$

We now introduce the inhomogeneous Besov spaces.

Definition 2.1 Let $s \in \mathbb{R}$, $1 \le p,q \le \infty$. The inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^3)$ is defined by

$$B^s_{p,q}(\mathbb{R}^3) = \{f \in \mathcal{S}'(\mathbb{R}^3); \|f\|_{B^s_{p,q}} < \infty\},$$

where

$$\|f\|_{B^s_{p,q}} = \begin{cases} (\sum_{j=-1}^{\infty} 2^{jsq}\|\Delta_j f\|_{L^p}^q)^{\frac{1}{q}}, & \text{for } q < \infty, \\ \sup_{j\ge -1} 2^{js}\|\Delta_j f\|_{L^p}, & \text{for } q = \infty. \end{cases}$$

The following lemmas will be useful in the proof of our main result.

Lemma 2.2 (Bernstein Inequality, [13]) *Let $1 \le p \le q$. Assume that $f \in L^p(\mathbb{R}^3)$, then there exists a constant C independent of f, j such that*

$$\operatorname{supp}\hat{f} \subset \{\xi : |\xi| \approx 2^j\} \quad \Longrightarrow \quad \|f\|_{L^p} \le C2^{-j|\alpha|} \sup_{|\beta|=|\alpha|} \|\partial^\beta f\|_{L^p},$$

$$\operatorname{supp}\hat{f} \subset \{\xi : |\xi| \lesssim 2^j\} \quad \Longrightarrow \quad \|\partial^\alpha f\|_{L^q} \le C2^{j|\alpha|+3j(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}.$$

Lemma 2.3 (Embedding Results, [14])

(1) *Let $1 \le p \le \infty$, $1 \le q, q_1 \le \infty$, and $s \ge s_1 > 0$. Assume that either $s > s_1$ or $s = s_1$ and $q \le q_1$. Then $B^s_{p,q}(\mathbb{R}^3) \hookrightarrow B^{s_1}_{q,q_1}(\mathbb{R}^3)$.*

(2) *If $1 \le p \le p_1 \le \infty$ and $s = s_1 + 3(\frac{1}{p} - \frac{1}{p_1})$, then $B^s_{p,q}(\mathbb{R}^3) \hookrightarrow B^{s_1}_{p_1,q}(\mathbb{R}^3)$.*

3 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. Since $B_{p,\infty}^{\frac{2\alpha}{q}+\frac{3}{p}-(2\alpha-1)}(\mathbb{R}^3) \hookrightarrow B_{\infty,\infty}^{\frac{2\alpha}{q}-(2\alpha-1)}(\mathbb{R}^3)$ by Lemma 2.3, we only need to prove that Theorem 1.3 holds in the case $p=\infty$, that is, to prove that if $W^+ \in B_{\infty,\infty}^{s}(\mathbb{R}^3)$ satisfies $\frac{2\alpha}{q} \le 2\alpha - 1 + s$, then the weak solution (u,b) to the system (1) is regular on $\mathbb{R}^3 \times (0,T]$.

Now, we denote $u_k = \Delta_k u$, $\theta_k = \Delta_k\theta$, $P_k = \Delta_k P$, and take $\gamma = \frac{2\alpha}{2\alpha-1+s}$. Then we have $s = \frac{2\alpha}{\gamma} - (2\alpha-1)$. Applying the operator Δ_k to both sides of (3), we get

$$\begin{aligned} &\frac{\partial W_k^+}{\partial t} + \Delta_k\bigl(W^- \cdot \nabla W^+\bigr) + (-\Delta)^\alpha W_k^+ + \nabla P_k = 0, \\ &\frac{\partial W_k^-}{\partial t} + \Delta_k\bigl(W^+ \cdot \nabla W^-\bigr) + (-\Delta)^\alpha W_k^- + \nabla P_k = 0. \end{aligned} \tag{9}$$

Then multiplying the first and the second equation of (9) by W_k^+, W_k^-, respectively, and by Lemma 2.2, we obtain for $k \ge 0$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|W_k^+\|_{L^2}^2 + c2^{2k\alpha}\|W_k^+\|_{L^2}^2 = -\bigl\langle \Delta_k\bigl(W^- \cdot \nabla W^+\bigr), W_k^+\bigr\rangle, \\ &\frac{1}{2}\frac{d}{dt}\|W_k^-\|_{L^2}^2 + c2^{2k\alpha}\|W_k^-\|_{L^2}^2 = -\bigl\langle \Delta_k\bigl(W^+ \cdot \nabla W^-\bigr), W_k^-\bigr\rangle. \end{aligned} \tag{10}$$

Denote $\Theta_k(t) \triangleq (\|W_k^+(t)\|_{L^2}^2 + \|W_k^-(t)\|_{L^2}^2)^{\frac{1}{2}}$. Adding the two equations in (10), we have

$$\frac{1}{2}\frac{d}{dt}\Theta_k^2(t) + c2^{2k\alpha}\Theta_k^2(t) = -\bigl\langle \Delta_k\bigl(W^- \cdot \nabla W^+\bigr), W_k^+\bigr\rangle - \bigl\langle \Delta_k\bigl(W^+ \cdot \nabla W^-\bigr), W_k^-\bigr\rangle. \tag{11}$$

Note that

$$\bigl\langle W^- \cdot \nabla W^+, W^+\bigr\rangle = 0, \qquad \bigl\langle W^+ \cdot \nabla W^-, W^-\bigr\rangle = 0.$$

Then (11) can be rewritten as

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Theta_k^2(t) + c2^{2k\alpha}\Theta_k^2(t) &= \bigl\langle \bigl[W^-, \Delta_k\bigr]\nabla W^+, W_k^+\bigr\rangle + \bigl\langle \bigl[W^+, \Delta_k\bigr]\nabla W^-, W_k^-\bigr\rangle \\ &\triangleq I + II, \end{aligned} \tag{12}$$

where the commutator operator $[A,B] = AB - BA$. By the Bony decomposition (8), the term I can be rewritten as

$$\begin{aligned} I &= \bigl\langle [T_{(W^-)^i}, \Delta_k]\partial_i W^+, W_k^+\bigr\rangle + \bigl\langle T'_{\Delta_k\partial_i W^+}\bigl(W^-\bigr)^i, W_k^+\bigr\rangle \\ &\quad - \bigl\langle \Delta_k\bigl(T_{\partial_i W^+}\bigl(W^-\bigr)^i\bigr), W_k^+\bigr\rangle - \bigl\langle \Delta_k\bigl(R\bigl(\bigl(W^-\bigr)^i, \partial_i W^+\bigr)\bigr), W_k^+\bigr\rangle \\ &\triangleq I_1 + I_2 + I_3 + I_4. \end{aligned}$$

From the definition of Δ_k, we have

$$\begin{aligned} &[T_{(W^-)^i}, \Delta_k]\partial_i W^+ \\ &\quad = \sum_{|k-k'|\le 4} \bigl[S_{k'-1}\bigl(W^-\bigr)^i, \Delta_k\bigr]\partial_i W_{k'}^+ \end{aligned}$$

$$
\begin{aligned}
&= \sum_{|k-k'|\le 4} \bigl[S_{k'-1}\bigl(\bigl(W^-\bigr)^i\bigr)\Delta_k\partial_i W^+_{k'} - \Delta_k\bigl(S_{k'-1}\bigl(\bigl(W^-\bigr)^i\bigr)\partial_i W^+_{k'}\bigr)\bigr] \\
&= \sum_{|k-k'|\le 4} 2^{3k}\int_{\mathbb{R}^3} h\bigl(2^k(x-y)\bigr)\bigl[S_{k'-1}\bigl(W^-\bigr)^i(x) - S_{k'-1}\bigl(W^-\bigr)^i(y)\bigr]\partial_i W^+_{k'}(y)\,dy \\
&= \sum_{|k-k'|\le 4} 2^{4k}\int_{\mathbb{R}^3}\int_0^1 y\cdot\nabla S_{k'-1}\bigl(W^-\bigr)^i(x-\tau y)\,d\tau\,\partial_i h\bigl(2^k y\bigr)W^+_{k'}(x-y)\,dy.
\end{aligned}
$$

Then by the Minkowski inequality, we get

$$
\begin{aligned}
|I_1| &\lesssim \|W^+_k\|_{L^2}\sum_{|k-k'|\le 4}\|\nabla S_{k'-1}W^-\|_{L^2}\|W^+_{k'}\|_{L^\infty} \\
&\lesssim \sum_{k''\le k'-2}\sum_{|k-k'|\le 4} 2^{k''}\|W^-_{k''}\|_{L^2}\|W^+_k\|_{L^2}\|W^+_{k'}\|_{L^\infty} \\
&\lesssim \sum_{k''\le k'-2}\sum_{|k-k'|\le 4} 2^{k''}2^{k'(2\alpha-1-\frac{2\alpha}{\gamma})}\Theta_k\Theta_{k''}\|W^+\|_{B^s_{\infty,\infty}} \\
&\lesssim \sum_{k'\le k-2} 2^{k'}2^{k(2\alpha-1-\frac{2\alpha}{\gamma})}\Theta_k\Theta_{k'}\|W^+\|_{B^s_{\infty,\infty}}.
\end{aligned}
$$

For the term I_2,

$$
\begin{aligned}
T'_{\Delta_k\partial_i W^+}\bigl(W^-\bigr)^i &= \sum_j S_{j-1}\bigl(\Delta_k\partial_i W^+\bigr)\Delta_j\bigl(W^-\bigr)^i + \sum_{|j-j'|\le 1}\Delta_{j'}\bigl(\Delta_k\partial_i W^+\bigr)\Delta_j\bigl(W^-\bigr)^i \\
&= \sum_j\biggl(\sum_{-1\le j'\le j-2}\Delta_{j'}\bigl(\Delta_k\partial_i W^+\bigr) \\
&\quad + \bigl(\Delta_{j-1}\bigl(\Delta_k\partial_i W^+\bigr) + \Delta_j\bigl(\Delta_k\partial_i W^+\bigr) + \Delta_{j+1}\bigl(\Delta_k\partial_i W^+\bigr)\bigr)\biggr)\cdot\Delta_j\bigl(W^-\bigr)^i \\
&= \sum_j\biggl(\sum_{-1\le j'\le j+1}\Delta_{j'}\bigl(\Delta_k\partial_i W^+\bigr)\biggr)\Delta_j\bigl(W^-\bigr)^i \\
&= \sum_{k'\ge k-2} S_{k'+2}\bigl(\Delta_k\partial_i W^+\bigr)\Delta_{k'}\bigl(W^-\bigr)^i,
\end{aligned}
$$

where j is replaced by k' in the last equality. Noticing that $S_{k'+2}\Delta_k W^+ = \Delta_k W^+$ for $k' > k$, then by Lemma 2.2 and the Minkowski inequality, we have

$$
\begin{aligned}
|I_2| &= \biggl|\sum_{k'\ge k-2}\bigl\langle S_{k'+2}\Delta_k\partial_i W^+\Delta_{k'}\bigl(W^-\bigr)^i, W^+_k\bigr\rangle\biggr| \\
&\lesssim \sum_{|k-k'|\le 2}\bigl|\bigl\langle S_{k'+2}\Delta_k\partial_i W^+\Delta_{k'}\bigl(W^-\bigr)^i, W^+_k\bigr\rangle\bigr| \\
&\quad + \sum_{k'\ge k-2}\bigl|\bigl\langle S_{k'+2}\Delta_k\partial_i W^+\Delta_{k'}\bigl(W^-\bigr)^i, W^+_k\bigr\rangle\bigr| \\
&\lesssim \|W^+_k\|_{L^2}\sum_{|k'-k|\le 2}\|\nabla S_{k'-1}W^+\|_{L^\infty}\|W^-_{k'}\|_{L^2} + 2^k\|W^+_k\|_{L^\infty}\sum_{k'\ge k-2}\|W^+_{k'}\|_{L^2}\|W^-_{k'}\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sum_{k''\le k'-2}\sum_{|k-k'|\le 2} 2^{2k''\alpha(1-\frac{1}{\gamma})}\Theta_k\Theta_{k'}\|W^+\|_{B^s_{\infty,\infty}} + \sum_{k'\ge k-2} 2^{2k\alpha(1-\frac{1}{\gamma})}\Theta_{k'}^2\|W^+\|_{B^s_{\infty,\infty}} \\
&\lesssim \sum_{k'\le k-2} 2^{2k'\alpha(1-\frac{1}{\gamma})}\Theta_k^2\|W^+\|_{B^s_{\infty,\infty}} + \sum_{k'\ge k-2} 2^{2k\alpha(1-\frac{1}{\gamma})}\Theta_{k'}^2\|W^+\|_{B^s_{\infty,\infty}}.
\end{aligned}
$$

Using the support of the Fourier transformation of the term $T_{\partial_i W^+}(W^-)^i$, we get

$$
\begin{aligned}
\Delta_k\big(T_{\partial_i W^+}\big(W^-\big)^i\big) &= \Delta_k\bigg(\sum_{k'} S_{k'-1}\big(\partial_i W^+\big)\Delta_{k'}\big(W^-\big)^i\bigg) \\
&= \sum_{|k-k'|\le 4} \Delta_k\big(S_{k'-1}\big(\partial_i W^+\big)\big(W^-_{k'}\big)^i\big).
\end{aligned}
$$

By the Minkowski inequality and Lemma 2.2, we have

$$
\begin{aligned}
|I_3| &\lesssim \sum_{|k-k'|\le 4} \big\|\Delta_k\big(S_{k'-1}\big(\partial_i W^+\big)\big(W^-_{k'}\big)^i\big)\big\|_{L^2}\big\|W_k^+\big\|_{L^2} \\
&\lesssim \big\|W_k^+\big\|_{L^2}\sum_{|k-k'|\le 4}\big\|\nabla S_{k'-1}W^+\big\|_{L^\infty}\big\|W^-_{k'}\big\|_{L^2} \\
&\lesssim \sum_{k''\le k'-2}\sum_{|k-k'|\le 4} 2^{2k''\alpha(1-\frac{1}{\gamma})}\Theta_k\Theta_{k'}\|W^+\|_{B^s_{\infty,\infty}} \\
&\lesssim \sum_{k'\le k-2} 2^{2k'\alpha(1-\frac{1}{\gamma})}\Theta_k^2\|W^+\|_{B^s_{\infty,\infty}}.
\end{aligned}
$$

With the incompressibility condition $\nabla\cdot u = 0$, we have

$$
\begin{aligned}
\Delta_k R\big(\big(W^-\big)^i, \partial_i W^+\big) &= \Delta_k\bigg(\sum_{|k'-k''|\le 1}\Delta_{k'}\big(W^-\big)^i\Delta_{k''}\big(\partial_i W^+\big)\bigg) \\
&= \sum_{k',k''\ge k-2;|k'-k''|\le 1}\Delta_k\big(\Delta_{k'}\big(W^-\big)^i\Delta_{k''}\big(\partial_i W^+\big)\big) \\
&= \sum_{k',k''\ge k-2;|k'-k''|\le 1}\partial_i\Delta_k\big(\Delta_{k'}\big(W^-\big)^i\Delta_{k''}W^+\big),
\end{aligned}
$$

then by Lemma 2.2, we get

$$
\begin{aligned}
|I_4| &\lesssim 2^k\big\|W_k^+\big\|_{L^\infty}\sum_{k'\ge k-2}\big\|W^+_{k'}\big\|_{L^2}\big\|W^-_{k'}\big\|_{L^2} \\
&\lesssim \sum_{k'\ge k-2} 2^{2k\alpha(1-\frac{1}{\gamma})}\Theta_{k'}^2\|W^+\|_{B^s_{\infty,\infty}}.
\end{aligned}
$$

Combining the above estimates for I_1-I_4, we have

$$
\begin{aligned}
|I| \lesssim{}& \sum_{k'\le k-2} 2^{k'}2^{k(2\alpha-1-\frac{2\alpha}{\gamma})}\Theta_k\Theta_{k'}\|W^+\|_{B^s_{\infty,\infty}} + \sum_{k'\le k-2} 2^{2k'\alpha(1-\frac{1}{\gamma})}\Theta_k^2\|W^+\|_{B^s_{\infty,\infty}} \\
&+ \sum_{k'\ge k-2} 2^{2k\alpha(1-\frac{1}{\gamma})}\Theta_{k'}^2\|W^+\|_{B^s_{\infty,\infty}}. \qquad (13)
\end{aligned}
$$

For the second term II of (12), by similar arguments to the ones used to derive (13) one can get

$$
\begin{aligned}
|II| \lesssim{}& \sum_{k'\le k-2} 2^{2k'\alpha(1-\frac{1}{\gamma})}\Theta_k^2 \|W^+\|_{B^s_{\infty,\infty}} \\
&+ \sum_{k'\le k-2} 2^{k'}2^{k(2\alpha-1-\frac{2\alpha}{\gamma})}\Theta_k\Theta_{k'}\|W^+\|_{B^s_{\infty,\infty}} \\
&+ \sum_{k'\ge k-2} 2^{k}2^{k'(2\alpha-1-\frac{2\alpha}{\gamma})}\Theta_k\Theta_{k'}\|W^+\|_{B^s_{\infty,\infty}}.
\end{aligned}
\tag{14}
$$

Inserting (13) and (14) into (12) one infers that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Theta_k^2(t) + 2^{2k\alpha}\Theta_k^2(t) \\
&\quad\lesssim \sum_{k'\le k-2} 2^{2k'\alpha(1-\frac{1}{\gamma})}\Theta_k^2\|W^+\|_{B^s_{\infty,\infty}} + \sum_{k'\ge k-2} 2^{2k\alpha(1-\frac{1}{\gamma})}\Theta_{k'}^2\|W^+\|_{B^s_{\infty,\infty}} \\
&\qquad+ \sum_{k'\le k-2} 2^{k'}2^{k(2\alpha-1-\frac{2\alpha}{\gamma})}\Theta_k\Theta_{k'}\|W^+\|_{B^s_{\infty,\infty}} \\
&\qquad+ \sum_{k'\ge k-2} 2^{k}2^{k'(2\alpha-1-\frac{2\alpha}{\gamma})}\Theta_k\Theta_{k'}\|W^+\|_{B^s_{\infty,\infty}}.
\end{aligned}
\tag{15}
$$

The proof of Theorem 1.3 in the rest of this section is divided into two cases.

Case I: $s\in[0,1)$. Take $\sigma\in(0,\frac{1}{\alpha})$, then $1-\sigma\alpha>0$. Let

$$
\varphi(t) \triangleq \sup_{k\ge -1} 2^{k\sigma\alpha}\Theta_k(t), \qquad \omega(t) \triangleq \sup_{k\ge -1} 2^{2k\alpha(\sigma+1)}\int_0^t \Theta_k^2(\tau)\,d\tau.
$$

Multiplying (15) with $2^{2k\sigma\alpha}$ and integrating with respect to t, we have

$$
\begin{aligned}
&2^{2k\sigma\alpha}\Theta_k^2(t) - 2^{2k\sigma\alpha}\Theta_k^2(0) + 2^{2k\alpha(\sigma+1)}\int_0^t \Theta_k^2(\tau)\,d\tau \\
&\quad\lesssim \int_0^t \sum_{k'\le k-2} 2^{2k\sigma\alpha}2^{2k'\alpha(1-\frac{1}{\gamma})}\Theta_k^2\|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\qquad+ \int_0^t \sum_{k'\ge k-2} 2^{2k\sigma\alpha}2^{2k\alpha(1-\frac{1}{\gamma})}\Theta_{k'}^2\|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\qquad+ \int_0^t \sum_{k'\le k-2} 2^{2k\sigma\alpha}2^{k'}2^{k(2\alpha-1-\frac{2\alpha}{\gamma})}\Theta_k\Theta_{k'}\|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\qquad+ \int_0^t \sum_{k'\ge k-2} 2^{2k\sigma\alpha}2^{k}2^{k'(2\alpha-1-\frac{2\alpha}{\gamma})}\Theta_k\Theta_{k'}\|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\quad\triangleq \mathbb{A}_1 + \mathbb{A}_2 + \mathbb{A}_3 + \mathbb{A}_4.
\end{aligned}
\tag{16}
$$

By Lemma 2.2 and the Hölder inequality, we have

$$\begin{aligned}
\mathbb{A}_1 &= \int_0^t \sum_{k'\le k-2} \left(2^{k\sigma\alpha}\Theta_k\right)^{\frac{2}{\gamma}} \left(2^{k(\sigma+1)\alpha}\Theta_k\right)^{2-\frac{2}{\gamma}} 2^{(k'-k)(2-\frac{2}{\gamma})} \|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\lesssim \int_0^t \Big(\sup_{k\ge -1} 2^{k\sigma\alpha}\Theta_k\Big)^{\frac{2}{\gamma}} \Big(\sup_{k\ge -1} 2^{k\alpha(\sigma+1)}\Theta_k\Big)^{2-\frac{2}{\gamma}} \|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\lesssim \left(\int_0^t \|W^+(\tau)\|^{\gamma}_{B^s_{\infty,\infty}} \varphi^2(\tau)\,d\tau\right)^{\frac{1}{\gamma}} \omega^{1-\frac{1}{\gamma}}(t), \\
\mathbb{A}_2 &= \int_0^t \sum_{k'\ge k-2} \left(2^{k'\sigma\alpha}\Theta_{k'}\right)^{\frac{2}{\gamma}} \left(2^{k'(\sigma+1)\alpha}\Theta_{k'}\right)^{2-\frac{2}{\gamma}} 2^{2\alpha(k-k')(\sigma+1-\frac{2}{\gamma})} \|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\lesssim \int_0^t \Big(\sup_{k'\ge -1} 2^{k'\sigma\alpha}\Theta_{k'}\Big)^{\frac{2}{\gamma}} \Big(\sup_{k'\ge -1} 2^{k'\alpha(\sigma+1)}\Theta_{k'}\Big)^{2-\frac{2}{\gamma}} \|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\lesssim \left(\int_0^t \|W^+(\tau)\|^{\gamma}_{B^s_{\infty,\infty}} \varphi^2(\tau)\,d\tau\right)^{\frac{1}{\gamma}} \omega^{1-\frac{1}{\gamma}}(t), \\
\mathbb{A}_3 &= \int_0^t \sum_{k'\le k-2} \left(2^{k\sigma\alpha}\Theta_k\right)^{\frac{2}{\gamma}-1} \left(2^{k'\sigma\alpha}\Theta_{k'}\right) \left(2^{k(\sigma+1)\alpha}\Theta_k\right)^{2-\frac{2}{\gamma}} 2^{(k'-k)(1-\sigma\alpha)} \|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\lesssim \int_0^t \Big(\sup_{k\ge -1} 2^{k\sigma\alpha}\Theta_k\Big)^{\frac{2}{\gamma}} \Big(\sup_{k\ge -1} 2^{k\alpha(\sigma+1)}\Theta_k\Big)^{2-\frac{2}{\gamma}} \|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\lesssim \left(\int_0^t \|W^+(\tau)\|^{\gamma}_{B^s_{\infty,\infty}} \varphi^2(\tau)\,d\tau\right)^{\frac{1}{\gamma}} \omega^{1-\frac{1}{\gamma}}(t),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{A}_4 &= \int_0^t \sum_{k'\ge k-2} \left(2^{k'\sigma\alpha}\Theta_{k'}\right) \left(2^{k\sigma\alpha}\Theta_k\right)^{\frac{2}{\gamma}-1} \left(2^{k(\sigma+1)\alpha}\Theta_k\right)^{2-\frac{2}{\gamma}} \\
&\quad \cdot 2^{(k-k')(1+\sigma\alpha+\frac{2\alpha}{\gamma}-2\alpha)} \|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\lesssim \int_0^t \Big(\sup_{k'\ge -1} 2^{k'\sigma\alpha}\Theta_{k'}\Big)^{\frac{2}{\gamma}} \Big(\sup_{k'\ge -1} 2^{k'\alpha(\sigma+1)}\Theta_{k'}\Big)^{2-\frac{2}{\gamma}} \|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\lesssim \left(\int_0^t \|W^+(\tau)\|^{\gamma}_{B^s_{\infty,\infty}} \varphi^2(\tau)\,d\tau\right)^{\frac{1}{\gamma}} \omega^{1-\frac{1}{\gamma}}(t).
\end{aligned}$$

Combining the above estimates for $\mathbb{A}_1$-$\mathbb{A}_4$ with (16) and the Young inequality, we have

$$\begin{aligned}
\varphi^2(t) - \varphi^2(0) + \omega(t) &\le C\left(\int_0^t \|W^+(\tau)\|^{\gamma}_{B^s_{\infty,\infty}} \varphi^2(\tau)\,d\tau\right)^{\frac{1}{\gamma}} \omega^{1-\frac{1}{\gamma}}(t) \\
&\le C\int_0^t \|W^+(\tau)\|^{\gamma}_{B^s_{\infty,\infty}} \varphi^2(\tau)\,d\tau + \omega(t),
\end{aligned}$$

which implies that

$$\varphi^2(t) \le \varphi^2(0) + C\int_0^t \|W^+(\tau)\|^{\gamma}_{B^s_{\infty,\infty}} \varphi^2(\tau)\,d\tau.$$

Then the Gronwall inequality gives

$$\varphi^2(t) \le C\varphi^2(0)\exp\left(C\int_0^t \|W^+(\tau)\|^{\gamma}_{B^s_{\infty,\infty}}\,d\tau\right). \tag{17}$$

Case II: $s \in (-1, 0)$. Let

$$\varphi(t) \triangleq \sup_{k\ge -1} 2^{k(1-\frac{1}{\gamma})}\Theta_k(t), \qquad \omega(t) \triangleq \sup_{k\ge -1} 2^{2k\alpha\gamma}\int_0^t \Theta_k^{2\gamma}(\tau)\,d\tau.$$

Multiplying (15) by $2^{2k\alpha(\gamma-1)}\Theta_k^{2(\gamma-1)}$, and integrating with respect to t, it follows that

$$\begin{aligned}
&2^{2k\alpha(\gamma-1)}\Theta_k^{2\gamma}(t) - 2^{2k\alpha(\gamma-1)}\Theta_k^{2\gamma}(0) + 2^{2k\alpha\gamma}\int_0^t \Theta_k^{2\gamma}(\tau)\,d\tau \\
&\quad\lesssim \int_0^t \sum_{k'\le k-2} 2^{2k\alpha(\gamma-1)}\Theta_k^{2(\gamma-1)} 2^{2k'\alpha(1-\frac{1}{\gamma})}\Theta_k^2 \|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\qquad + \int_0^t \sum_{k'\ge k-2} 2^{2k\alpha(\gamma-1)}\Theta_k^{2(\gamma-1)} 2^{2k\alpha(1-\frac{1}{\gamma})}\Theta_{k'}^2 \|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\qquad + \int_0^t \sum_{k'\le k-2} 2^{2k\alpha(\gamma-1)}\Theta_k^{2(\gamma-1)} 2^{k'} 2^{k(2\alpha-1-\frac{2\alpha}{\gamma})}\Theta_k\Theta_{k'} \|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\qquad + \int_0^t \sum_{k'\ge k-2} 2^{2k\alpha(\gamma-1)}\Theta_k^{2(\gamma-1)} 2^{k} 2^{k'(2\alpha-1-\frac{2\alpha}{\gamma})}\Theta_k\Theta_{k'} \|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\quad\triangleq \mathbb{B}_1 + \mathbb{B}_2 + \mathbb{B}_3 + \mathbb{B}_4.
\end{aligned} \tag{18}$$

By Lemma 2.2 and the Hölder inequality, we have

$$\begin{aligned}
\mathbb{B}_1 &= \int_0^t \sum_{k'\le k-2} \left(2^{2k\alpha\gamma}\Theta_k^{2\gamma}\right)^{1-\frac{1}{\gamma}} \left(2^{k\alpha(1-\frac{1}{\gamma})}\Theta_k\right)^2 2^{2\alpha(k'-k)(1-\frac{1}{\gamma})}\|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\lesssim \int_0^t \left(\sup_{k\ge -1} 2^{2k\alpha\gamma}\Theta_k^{2\gamma}\right)^{1-\frac{1}{\gamma}} \left(\sup_{k\ge -1} 2^{k\alpha(1-\frac{1}{\gamma})}\Theta_k\right)^2 \|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\lesssim \left(\int_0^t \|W^+(\tau)\|^{\gamma}_{B^s_{\infty,\infty}}\varphi^{2\gamma}(\tau)\,d\tau\right)^{\frac{1}{\gamma}} \omega^{1-\frac{1}{\gamma}}(t), \\
\mathbb{B}_2 &= \int_0^t \sum_{k'\ge k-2} \left(2^{2k\alpha\gamma}\Theta_k^{2\gamma}\right)^{1-\frac{1}{\gamma}} \left(2^{k'\alpha(1-\frac{1}{\gamma})}\Theta_{k'}\right)^2 2^{2\alpha(k-k')(1-\frac{1}{\gamma})}\|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\lesssim \int_0^t \left(\sup_{k'\ge -1} 2^{2k'\alpha\gamma}\Theta_{k'}^{2\gamma}\right)^{1-\frac{1}{\gamma}} \left(\sup_{k'\ge -1} 2^{k'\alpha(1-\frac{1}{\gamma})}\Theta_{k'}\right)^2 \|W^+\|_{B^s_{\infty,\infty}}\,d\tau \\
&\lesssim \left(\int_0^t \|W^+(\tau)\|^{\gamma}_{B^s_{\infty,\infty}}\varphi^{2\gamma}(\tau)\,d\tau\right)^{\frac{1}{\gamma}} \omega^{1-\frac{1}{\gamma}}(t), \\
\mathbb{B}_3 &= \int_0^t \sum_{k'\le k-2} \left(2^{2k\alpha\gamma}\Theta_k^{2\gamma}\right)^{1-\frac{1}{\gamma}} \cdot 2^{k\alpha(1-\frac{1}{\gamma})}\Theta_k \cdot 2^{k'\alpha(1-\frac{1}{\gamma})}\Theta_{k'} \\
&\qquad \cdot 2^{(k'-k)(1-\alpha+\frac{\alpha}{\gamma})}\|W^+\|_{B^s_{\infty,\infty}}\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^t \Big(\sup_{k\ge -1} 2^{2k\alpha\gamma}\Theta_k^{2\gamma}\Big)^{1-\frac{1}{\gamma}}\Big(\sup_{k\ge -1} 2^{k\alpha(1-\frac{1}{\gamma})}\Theta_k\Big)^2 \|W^+\|_{B^s_{\infty,\infty}}\,d\tau\\
&\lesssim \bigg(\int_0^t \|W^+(\tau)\|^\gamma_{B^s_{\infty,\infty}}\varphi^{2\gamma}(\tau)\,d\tau\bigg)^{\frac{1}{\gamma}}\omega^{1-\frac{1}{\gamma}}(t),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{B}_4 &= \int_0^t \sum_{k'\ge k-2}\big(2^{2k\alpha\gamma}\Theta_k^{2\gamma}\big)^{1-\frac{1}{\gamma}}\cdot 2^{k\alpha(1-\frac{1}{\gamma})}\Theta_k\cdot 2^{k'\alpha(1-\frac{1}{\gamma})}\Theta_{k'}\\
&\quad\cdot 2^{(k-k')(1-\alpha+\frac{\alpha}{\gamma})}\|W^+\|_{B^s_{\infty,\infty}}\,d\tau\\
&\lesssim \int_0^t \Big(\sup_{k'\ge -1} 2^{2k'\alpha\gamma}\Theta_{k'}^{2\gamma}\Big)^{1-\frac{1}{\gamma}}\Big(\sup_{k'\ge -1} 2^{k'\alpha(1-\frac{1}{\gamma})}\Theta_{k'}\Big)^2 \|W^+\|_{B^s_{\infty,\infty}}\,d\tau\\
&\lesssim \bigg(\int_0^t \|W^+(\tau)\|^\gamma_{B^s_{\infty,\infty}}\varphi^{2\gamma}(\tau)\,d\tau\bigg)^{\frac{1}{\gamma}}\omega^{1-\frac{1}{\gamma}}(t).
\end{aligned}
$$

Inserting the above estimates for $\mathbb{B}_1$-$\mathbb{B}_4$ into (18) and by the Young inequality, we have

$$
\begin{aligned}
\varphi^{2\gamma}(t)-\varphi^{2\gamma}(0)+\omega(t) &\le C\bigg(\int_0^t \|W^+(\tau)\|^\gamma_{B^s_{\infty,\infty}}\varphi^{2\gamma}(\tau)\,d\tau\bigg)^{\frac{1}{\gamma}}\omega^{1-\frac{1}{\gamma}}(t)\\
&\le C\int_0^t \|W^+(\tau)\|^\gamma_{B^s_{\infty,\infty}}\varphi^{2\gamma}(\tau)\,d\tau+\omega(t),
\end{aligned}
$$

which gives

$$
\varphi^{2\gamma}(t)\le\varphi^{2\gamma}(0)+C\int_0^t \|W^+(\tau)\|^\gamma_{B^s_{\infty,\infty}}\varphi^{2\gamma}(\tau)\,d\tau.
$$

Again, by the Gronwall inequality, we have

$$
\sup_{t\in(0,T]}\varphi^{2\gamma}(t)\le\varphi^{2\gamma}(0)\exp\bigg(C\int_0^t \|W^+(\tau)\|^\gamma_{B^s_{\infty,\infty}}\,d\tau\bigg). \tag{19}
$$

By Lemma 2.3, as $s\in[0,1)$, for some $\eta<\sigma\alpha$, the following embedding holds:

$$
B^{\sigma\alpha}_{2,\infty}(\mathbb{R}^3)\hookrightarrow B^{\eta}_{2,2}(\mathbb{R}^3)=H^\eta(\mathbb{R}^3).
$$

According to the local existence with the condition [5] $\eta>\frac{5}{2}-2\alpha$, thus σ can be taken as $\frac{\frac{5}{2}-2\alpha}{\alpha}<\sigma<\frac{1}{\alpha}$. When $s\in(-1,0)$, for $\alpha\in(1,\frac{5}{4})$, one has $\frac{1-s}{2}\ge\frac{5}{2}-2\alpha$, which infers that

$$
B^{\frac{1-s}{2}}_{2,\infty}(\mathbb{R}^3)\hookrightarrow B^{\eta}_{2,2}(\mathbb{R}^3)=H^\eta(\mathbb{R}^3).
$$

Then for any $q\ge\gamma$, if $W^+\in L^q(0,T;B^s_{\infty,\infty}(\mathbb{R}^3))$ with $\frac{2\alpha}{q}\le\frac{2\alpha}{\gamma}=2\alpha-1+s$, by the standard Picard method [15, 16], we can easily show that the solution (u,b) remains smooth at time $t=T$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Southwest University, Tiansheng Road, Chongqing, 400715, China. [2]Department of Applied Mathematics, South China Agricultural University, Wushan, Guangzhou, 510642, China.

Acknowledgements
The authors are highly grateful for the referees' careful reading and comments on this paper. The work of Zhang is partially supported by the NNSF of China under the grant 11101337, Doctoral Foundation of Ministry of Education of China grant-20110182120013, Fundamental Research Funds for the Central Universities grant-XDJK2011C046, and Doctoral Fund of Southwest University (SWU110035). The work of Qiu is partially supported by the NNSF of China grant-11126266, the NSF of Guangdong grant-S2013010013608, Foundation for Distinguished Young Talents in Higher Education of Guangdong, China grant 2012LYM_0030 and Pearl River New Star Program grant-2012J2200016.

Received: 12 May 2014 Accepted: 2 July 2014 Published online: 24 September 2014

References

1. Wu, J: Generalized MHD equations. J. Differ. Equ. **195**, 284-312 (2003)
2. Wu, J: Regularity criteria for the generalized MHD equations. Commun. Partial Differ. Equ. **33**, 285-360 (2008)
3. Zhou, Y: Regularity criteria for the generalized viscous MHD equations. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **24**, 491-505 (2007)
4. Wu, G: Regularity criteria for the 3D generalized MHD equations in terms of vorticity. Nonlinear Anal. **71**, 4251-4258 (2009)
5. Yuan, J: Existence theorem and regularity criteria for the generalized MHD equations. Nonlinear Anal., Real World Appl. **11**, 1640-1649 (2010)
6. Hasegawa, A: Self-organization processes in continuous media. Adv. Phys. **34**, 1-42 (1985)
7. Politano, H, Pouquet, A, Sulem, PL: Current and vorticity dynamics in three-dimensional magnetohydrodynamics turbulence. Phys. Plasmas **2**, 2931-2939 (1995)
8. He, C, Wang, Y: Remark on the regularity for weak solutions to the magnetohydrodynamic equations. Math. Methods Appl. Sci. **31**, 1667-1684 (2008)
9. Gala, S: Extension criterion on regularity for weak solutions to the 3D MHD equations. Math. Methods Appl. Sci. **33**, 1496-1503 (2010)
10. Dong, B, Jia, Y, Zhang, W: An improved regularity criterion of three-dimensional magnetohydrodynamic equations. Nonlinear Anal., Real World Appl. **13**, 1159-1169 (2012)
11. Cannone, M: Harmonic analysis tools for solving the incompressible Navier-Stokes equations. In: Handbook of Mathematical Fluid Dynamics, vol. 13, pp. 161-244 (2005)
12. Bahouri, H, Chemin, JY, Danchin, R: Fourier Analysis and Nonlinear Partial Differential Equations. Springer, Berlin (2011)
13. Chemin, JY: Perfect Incompressible Fluids. Oxford University Press, New York (1998)
14. Brenner, P: Besov Spaces and Applications to Difference Methods for Initial Value Problems. Lecture Notes in Math., vol. 75. Springer, Berlin (1975)
15. Fujita, H, Kato, T: On the Navier-Stokes initial value problem I. Arch. Ration. Mech. Anal. **16**, 269-315 (1964)
16. Kato, T: Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$. J. Funct. Anal. **9**, 296-305 (1972)

doi:10.1186/s13661-014-0178-3
Cite this article as: Zhang and Qiu: **On the regularity criterion for the 3D generalized MHD equations in Besov spaces.** *Boundary Value Problems* 2014 **2014**:178.