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Eigenparameter dependent inverse boundary value problem for a class of Sturm-Liouville operator

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Abstract
In this work a Sturm-Liouville operator with piecewise continuous coefficient and spectral parameter in the boundary conditions is considered. The eigenvalue problem is investigated; it is shown that the eigenfunctions form a complete system and an expansion formula with respect to the eigenfunctions is obtained. Uniqueness theorems for the solution of the inverse problem with a Weyl function and spectral data are proved.
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1 Introduction

We consider the boundary value problem

$$-y'' + q(x)y = \lambda^2 \rho(x)y, \quad 0 \le x \le \pi, \tag{1}$$

$$U(y) := y'(0) + (\alpha_1 - \lambda^2\alpha_2)y(0) = 0, \tag{2}$$

$$V(y) := \lambda^2\big(\beta_4 y'(\pi) + \beta_2 y(\pi)\big) - \beta_1 y'(\pi) - \beta_3 y(\pi) = 0, \tag{3}$$

where $q(x) \in L_2(0,\pi)$ is a real valued function, λ is a complex parameter, α_i, β_j, $i = 1,2$, $j = \overline{1,4}$ are positive real numbers and

$$\rho(x) = \begin{cases} 1, & 0 \le x < a, \\ \gamma^2, & a < x \le \pi, \end{cases}$$

where $0 < \gamma \ne 1$.

Physical applications of the eigenparameter dependent Sturm-Liouville problems, *i.e.* the eigenparameter appears not only in the differential equation of the Sturm-Liouville problem but also in the boundary conditions, are given in [1–4]. Spectral analyses of these problems are examined as regards different aspects (eigenvalue problems, expansion problems with respect to eigenvalues, *etc.*) in [5–13]. Similar problems for discontinuous Sturm-Liouville problems are examined in [14–18].

Inverse problems for differential operators with boundary conditions dependent on the spectral parameter on a finite interval have been studied in [19–23]. In particular, such problems with discontinuous coefficient are studied in [24–27].

We investigate a Sturm-Liouville operator with discontinuous coefficient and a spectral parameter in boundary conditions. The theoretic formulation of the operator for the problem is given in a suitable Hilbert space in Section 2. In Section 3, an asymptotic formula for the eigenvalues is given. In Section 4, an expansion formula with respect to the eigenfunctions is obtained and Section 5 contains uniqueness theorems for the solution of the inverse problem with a Weyl function and spectral data.

2 Operator formulation

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of (1) satisfying the initial conditions

$$\varphi(0,\lambda)=1,\qquad \varphi'(0,\lambda)=\lambda^2\alpha_2-\alpha_1, \tag{4}$$

$$\psi(\pi,\lambda)=\beta_1-\lambda^2\beta_4,\qquad \psi'(\pi,\lambda)=\lambda^2\beta_2-\beta_3. \tag{5}$$

For the solution of (1), the following integral representation as $\mu^{\pm}(x)=\pm x\sqrt{\rho(x)}+a(1\mp\sqrt{\rho(x)})$ is obtained similar to [28] for all λ:

$$e(x,\lambda)=\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu^+(x)}+\frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu^-(x)}+\int_{-\mu^+(x)}^{\mu^+(x)}K(x,t)e^{i\lambda t}\,dt,$$

where $K(x,\cdot)\in L_1(-\mu^+(x),\mu^+(x))$. The following properties hold for the kernel $K(x,t)$ which has the partial derivative K_x belonging to the space $L_1(-\mu^+(x),\mu^+(x))$ for every $x\in[0,\pi]$:

$$\frac{d}{dx}K\bigl(x,\mu^+(x)\bigr)=\frac{1}{4\sqrt{\rho(x)}}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)q(x), \tag{6}$$

$$\frac{d}{dx}K\bigl(x,\mu^-(x)+0\bigr)-\frac{d}{dx}K\bigl(x,\mu^-(x)-0\bigr)=\frac{1}{4\sqrt{\rho(x)}}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)q(x). \tag{7}$$

We obtain the integral representation of the solution $\varphi(x,\lambda)$:

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+\int_0^{\mu^+(x)}A(x,t)\cos\lambda t\,dt+\bigl(\lambda^2\alpha_2-\alpha_1\bigr)\int_0^{\mu^+(x)}\tilde{A}(x,t)\frac{\sin\lambda t}{\lambda}\,dt, \tag{8}$$

where

$$A(x,t)=K(x,t)-K(x,-t),\qquad \tilde{A}(x,t)=K(x,t)+K(x,-t)$$

satisfying (6), (7).

Let us define

$$\Delta(\lambda):=\bigl\langle\varphi(x,\lambda),\psi(x,\lambda)\bigr\rangle=\varphi(x,\lambda)\psi'(x,\lambda)-\varphi'(x,\lambda)\psi(x,\lambda), \tag{9}$$

which is independent from $x \in [0,\pi]$. Substituting $x = 0$ and $x = \pi$ into (9) we get

$$\Delta(\lambda) = -U(\psi) = V(\varphi).$$

The function $\Delta(\lambda)$ is entire and has zeros at the eigenvalues of the problem (1)-(3).

In the Hilbert space $H_\rho = L_{2,\rho}(0,\pi) \oplus \mathbb{C}^2$ let an inner product be defined by

$$(f,g) := \int_0^\pi f_1(x)\overline{g_1(x)}\rho(x)\,dx + \frac{f_2\overline{g_2}}{\alpha_2} + \frac{f_3\overline{g_3}}{\delta_2},$$

where

$$f = \begin{pmatrix} f_1(x) \\ f_2 \\ f_3 \end{pmatrix} \in H_\rho, \qquad g = \begin{pmatrix} g_1(x) \\ g_2 \\ g_3 \end{pmatrix} \in H_\rho, \qquad \delta_2 := \beta_1\beta_2 - \beta_3\beta_4 > 0.$$

We define the operator

$$L(f) := \begin{pmatrix} -f_1''(x) + q(x)f_1(x) \\ f_1'(0) + \alpha_1 f_1(0) \\ \beta_1 f_1'(\pi) + \beta_3 f_1(\pi) \end{pmatrix}$$

with

$$\begin{aligned} D(L) = \bigl\{ & f \in H_\rho : f_1(x), f_1'(x) \in AC[0,\pi], l(f_1) \in L_2[0,\pi], \\ & f_2 = \alpha_2 f_1(0), f_3 = \beta_4 f_1'(\pi) + \beta_2 f_1(\pi)\bigr\}, \end{aligned}$$

where

$$l(f_1) = \frac{1}{\rho(x)}\bigl\{-f_1'' + q(x)f_1\bigr\}.$$

The boundary value problem (1)-(3) is equivalent to the equation $LY = \lambda^2 Y$. When $\lambda = \lambda_n$ are the eigenvalues, the eigenfunctions of operator L are in the form of

$$\Phi(x,\lambda_n) = \Phi_n := \begin{pmatrix} \varphi(x,\lambda_n) \\ \alpha_2\varphi(0,\lambda_n) \\ \beta_4\varphi'(\pi,\lambda_n) + \beta_2\varphi(\pi,\lambda_n) \end{pmatrix}, \quad n = 1, 2.$$

For any eigenvalue λ_n the solutions (4), (5) satisfy the relation

$$\psi(x,\lambda_n) = k_n\varphi(x,\lambda_n) \tag{10}$$

and the normalized numbers of the boundary value problem (1)-(3) are given below:

$$\begin{aligned} \alpha_n := {} & \int_0^\pi \varphi^2(x,\lambda_n)\rho(x)\,dx + \alpha_2\varphi^2(0,\lambda_n) \\ & + \frac{1}{\delta_2}\bigl(\beta_4\varphi'(\pi,\lambda_n) + \beta_2\varphi(\pi,\lambda_n)\bigr)^2. \end{aligned} \tag{11}$$

Lemma 1 *The eigenvalues of the boundary value problem* (1)-(3) *are simple, i.e.*

$$\dot{\Delta}(\lambda) = 2\lambda_n k_n \alpha_n. \tag{12}$$

Proof Since

$$-\varphi''(x,\lambda_n) + q(x)\varphi(x,\lambda_n) = \lambda_n^2 \rho(x)\varphi(x,\lambda_n),$$
$$-\psi''(x,\lambda) + q(x)\psi(x,\lambda) = \lambda^2 \rho(x)\psi(x,\lambda),$$

we get

$$\frac{d}{dx}\bigl[\varphi(x,\lambda_n)\psi'(x,\lambda) - \varphi'(x,\lambda_n)\psi(x,\lambda)\bigr] = \bigl(\lambda_n^2 - \lambda^2\bigr)\rho(x)\varphi(x,\lambda_n)\psi(x,\lambda).$$

With the help of (2), (3) we get

$$\Delta(\lambda_n) - \Delta(\lambda) = \bigl(\lambda_n^2 - \lambda^2\bigr)\int_0^\pi \varphi(x,\lambda_n)\psi(x,\lambda)\rho(x)\,dx.$$

Adding

$$\begin{aligned}&\bigl(\lambda_n^2 - \lambda^2\bigr)\alpha_2\varphi(0,\lambda_n)\psi(0,\lambda)\\ &\quad + \frac{(\lambda_n^2 - \lambda^2)}{\delta_2}\bigl(\beta_4\varphi'(\pi,\lambda_n) + \beta_2\varphi(\pi,\lambda_n)\bigr)\bigl(\beta_4\psi'(\pi,\lambda) + \beta_2\psi(\pi,\lambda)\bigr)\end{aligned}$$

to both sides of the last equation and using the relations (10), (11) we have

$$\Delta(\lambda_n) - \Delta(\lambda) = (\lambda_n + \lambda)(\lambda_n - \lambda)k_n\alpha_n.$$

Taking $\lambda \to \lambda_n$, we find (12). □

3 Asymptotic formulas of the eigenvalues

The solution of (1) satisfying the initial conditions (4) when $q(x) \equiv 0$ is in the following form:

$$\varphi_0(x,\lambda) = c_0(x,\lambda) + \bigl(\lambda^2\alpha_2 - \alpha_1\bigr)\frac{s_0(x,\lambda)}{\lambda}, \tag{13}$$

where

$$c_0(x,\lambda) = \begin{cases} \cos\lambda x, & 0 \le x < a, \\ \frac{1}{2}(1 + \frac{1}{\sqrt{\rho(x)}})\cos\lambda\mu^+(x) + \frac{1}{2}(1 - \frac{1}{\sqrt{\rho(x)}})\cos\lambda\mu^-(x), & a < x \le \pi, \end{cases}$$

and

$$s_0(x,\lambda) = \begin{cases} \frac{\sin\lambda x}{\lambda}, & 0 \le x < a, \\ \frac{1}{2}(1 + \frac{1}{\sqrt{\rho(x)}})\frac{\sin\lambda\mu^+(x)}{\lambda} + \frac{1}{2}(1 - \frac{1}{\sqrt{\rho(x)}})\frac{\sin\lambda\mu^-(x)}{\lambda}, & a < x \le \pi. \end{cases}$$

The eigenvalues λ_n^0 $(n = 0, \mp 1, \mp 2, \ldots)$ of the boundary value problem (1)-(3) when $q(x) \equiv 0$ can be found by using the equation

$$\Delta_0(\lambda) = \left(\lambda^2\beta_2 - \beta_3\right)\varphi_0(\pi,\lambda) - \left(\beta_1 - \lambda^2\beta_4\right)\varphi_0'(\pi,\lambda) = 0$$

and can be represented in the following way:

$$\lambda_n^0 = n + \psi(n), \quad n = 0, \mp 1, \mp 2, \ldots,$$

where $\sup_n |\psi(n)| < +\infty$.

Roots λ_n^0 of the function $\Delta_0(\lambda)$ are separated, *i.e.*,

$$\inf_{n \neq k} \left|\lambda_n^0 - \lambda_k^0\right| = \tau > 0.$$

Lemma 2 *The eigenvalues of the boundary value problem* (1)-(3) *are in the form of*

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{\eta_n}{n}, \quad \lambda_n > 0, \tag{14}$$

where (d_n) *is a bounded sequence,*

$$\begin{aligned} d_n = {} & \frac{1}{4\lambda_n^0 \dot{\Delta}(\lambda_n^0)} \int_0^\pi \left(1 - \frac{1}{\sqrt{\rho(t)}}\right) \frac{q(t)\sin(\lambda_n^0 \mu^-(\pi))}{\sqrt{\rho(t)}}\, dt \\ & - \frac{\alpha_1 - \alpha_2}{4\lambda_n^0 \dot{\Delta}(\lambda_n^0)} \int_0^\pi \left(1 + \frac{1}{\sqrt{\rho(t)}}\right) \frac{q(t)\cos(\lambda_n^0 \mu^-(\pi))}{\sqrt{\rho(t)}}\, dt \end{aligned}$$

and $\{\eta_n\} \in l_2$.

Proof From (8), it follows that

$$\begin{aligned} \varphi(\pi,\lambda) = {} & \varphi_0(\pi,\lambda) + \int_0^{\mu^+(\pi)} A(\pi,t)\cos\lambda t\, dt \\ & + \left(\lambda^2\alpha_2 - \alpha_1\right) \int_0^{\mu^+(\pi)} \tilde{A}(\pi,t) \frac{\sin\lambda t}{\lambda}\, dt. \end{aligned} \tag{15}$$

The expressions of $\Delta(\lambda)$ and $\Delta_0(\lambda)$ let us calculate $\Delta(\lambda) - \Delta_0(\lambda)$:

$$\begin{aligned} \Delta(\lambda) - \Delta_0(\lambda) = {} & -\lambda\tilde{A}\left(\pi,\mu^+(\pi)\right)\left(\alpha + a + \frac{\pi - 1}{2\alpha}\right)\sin\lambda\mu^+(\pi) \\ & + \left(\alpha + a + \frac{\pi - 1}{2\alpha}\right) A\left(\pi,\mu^+(\pi)\right)\cos\lambda\mu^+(\pi) + I(\lambda)\lambda^3, \end{aligned}$$

where

$$I(\lambda) = \alpha_2\beta_4 \int_0^{\mu^+(\pi)} \frac{\partial}{\partial x}\tilde{A}(\pi,t)\sin\lambda t\, dt + O\left(\frac{e^{|\operatorname{Im}\lambda|\mu^+(\pi)}}{\lambda^2}\right).$$

Therefore, for sufficiently large n, on the contours

$$\Gamma_n = \left\{\lambda : |\lambda| = \left|\lambda_n^0\right| + \frac{\tau}{2}\right\},$$

we have

$$\left|\Delta(\lambda) - \Delta_0(\lambda)\right| < \left|\Delta_0(\lambda)\right|.$$

By the Rouche theorem, we obtain the result that the number of zeros of the function

$$\left\{\Delta(\lambda) - \Delta_0(\lambda)\right\} + \Delta_0(\lambda) = \Delta(\lambda)$$

inside the contour Γ_n coincides with the number of zeros of the function $\Delta_0(\lambda)$. Moreover, applying the Rouche theorem to the circle $\gamma_n(\delta) = \{\lambda : |\lambda - \lambda_n^0| \le \delta\}$ we find, for sufficiently large n, that there exists one zero λ_n of the function $\Delta(\lambda)$ in $\gamma_n(\delta)$. Owing to the arbitrariness of $\delta > 0$ we have

$$\lambda_n = \lambda_n^0 + \epsilon_n, \qquad \epsilon_n = o(1), \quad n \to \infty. \tag{16}$$

Substituting (16) into (15), as $n \to \infty$ taking into account the equality $\Delta_0(\lambda_n^0) = 0$ and the relations $\sin \epsilon_n \mu^+(\pi) \approx \epsilon_n \mu^+(\pi)$, $\cos \epsilon_n \mu^+(\pi) \approx 1$, integrating by parts and using the properties of the kernels $A(x,t)$ and $\tilde{A}(x,t)$ we have

$$\epsilon_n \approx \frac{d_n}{\lambda_n^0 + \epsilon_n} + \frac{\eta_n}{\lambda_n^0},$$

where

$$\eta_n = \int_0^{\mu^+(\pi)} A_t(\pi,t) \sin \lambda_n^0 t \, dt + (\alpha_1 - \alpha_2) \int_0^{\mu^+(\pi)} A_t(\pi,t) \cos \lambda_n^0 t \, dt.$$

Let us show that $\eta_n \in l_2$. It is obvious that η_n can be reduced to the integral

$$\int_{-\mu^+(\pi)}^{\mu^+(\pi)} R(t) e^{i\lambda t} \, dt,$$

where $R(t) \in L_2(-\mu^+(\pi), \mu^+(\pi))$. Now, take

$$\zeta(\lambda) := \int_{-\mu^+(\pi)}^{\mu^+(\pi)} R(t) e^{i\lambda t} \, dt.$$

It is clear from [28] (p.66) that $\{\zeta_n\} = \zeta(\lambda_n) \in l_2$. By virtue of this we have $\{\eta_n\} \in l_2$. The lemma is proved. □

4 Expansion formula with respect to eigenfunctions

Denote

$$G(x,t;\lambda) := -\frac{1}{\Delta(\lambda)} \begin{cases} \varphi(t,\lambda)\psi(x,\lambda), & t \le x, \\ \psi(t,\lambda)\varphi(x,\lambda), & t \ge x \end{cases} \tag{17}$$

and consider the function

$$y(x,\lambda) := \int_0^{\pi} G(x,t;\lambda)f(t)\rho(t)\,dt - \frac{f_1}{\Delta(\lambda)}\psi(x,\lambda) + \frac{f_2}{\Delta(\lambda)}\varphi(x,\lambda). \tag{18}$$

Theorem 3 *The eigenfunctions $\Phi(x,\lambda_n)$ of the boundary value problem* (1)-(3) *form a complete system in $L_{2,\rho}(0,\pi)\oplus\mathbb{C}^2$.*

Proof With the help of (10) and (12), we can write

$$\psi(x,\lambda_n) = \frac{\dot{\Delta}(\lambda_n)}{2\lambda_n\alpha_n}\varphi(x,\lambda_n). \tag{19}$$

Using (17) and (18) we get

$$\begin{aligned}\operatorname{Res}_{\lambda=\lambda_n} y(x,\lambda) = {}& -\frac{1}{2\lambda_n\alpha_n}\varphi(x,\lambda_n)\int_0^{\pi}\varphi(t,\lambda_n)f(t)\rho(t)\,dt \\ & - \frac{1}{2\lambda_n\alpha_n}\varphi(x,\lambda_n)\left(f_1 - \frac{f_2}{k_n}\right).\end{aligned} \tag{20}$$

Now let $f(x)\in L_{2,\rho}(0,\pi)\oplus\mathbb{C}^2$ and assume

$$\begin{aligned}\bigl(\Phi(x,\lambda_n),f(x)\bigr) &= \int_0^{\pi}\varphi(x,\lambda_n)\overline{f_1(x)}\rho(x)\,dx + \varphi(0,\lambda_n)\overline{f_2} \\ &\quad + \frac{(\beta_4\varphi'(\pi,\lambda_n)+\beta_2\varphi(\pi,\lambda_n))\overline{f_3}}{\delta_2} \\ &= 0.\end{aligned} \tag{21}$$

Then from (20), we have $\operatorname{Res}_{\lambda=\lambda_n} y(x,\lambda)=0$. Consequently, for fixed $x\in[0,\pi]$ the function $y(x,\lambda)$ is entire with respect to λ. Let us denote

$$G_\delta := \bigl\{\lambda : \bigl|\lambda-\lambda_n^0\bigr| \geq \delta, n = 0,\mp 1,\mp 2,\ldots\bigr\},$$

where δ is sufficiently small positive number. It is clear that the relation below holds:

$$\bigl|\Delta(\lambda)\bigr| \geq C|\lambda|^3 e^{|\operatorname{Im}\lambda|\mu^+(\pi)}, \quad \lambda\in G_\delta, C = \text{cons}. \tag{22}$$

From (18) it follows that for fixed $\delta>0$ and sufficiently large $\lambda^*>0$ we have

$$\bigl|y(x,\lambda)\bigr| \leq \frac{C}{|\lambda|}, \quad \lambda\in G_\delta, |\lambda|\geq\lambda^*, C = \text{cons}.$$

Using maximum principle for module of analytic functions and Liouville theorem, we get $y(x,\lambda)\equiv 0$. From this we obtain $f(x)\equiv 0$ a.e. on $[0,\pi]$. Thus we conclude the completeness of the eigenfunctions $\Phi(x,\lambda_n)$ in $L_{2,\rho}(0,\pi)\oplus\mathbb{C}^2$. □

Theorem 4 *If $f(x)\in D(L)$, then the expansion formula*

$$f(x) = \sum_{n=1}^{\infty} a_n\varphi(x,\lambda_n) \tag{23}$$

is valid, where

$$a_n = \frac{1}{2\alpha_n} \int_0^\pi \varphi(t, \lambda_n) f(t) \rho(t)\, dt,$$

and the series converges uniformly with respect to $x \in [0, \pi]$. *For* $f(x) \in L_{2,\rho}(0, \pi)$, *the series converges in* $L_{2,\rho}(0, \pi)$, *moreover, the Parseval equality holds*:

$$\int_0^\pi \left|f(x)\right|^2 \rho(x)\, dx = \sum_{n=1}^\infty \alpha_n |a_n|^2.$$

Proof Since $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ are the solutions of the boundary value problem (1)-(3), we have

$$\begin{aligned} y(x, \lambda) = {} & -\frac{\psi(x, \lambda)}{\Delta(\lambda)} \left\{ \int_0^\pi \frac{[-\varphi''(t, \lambda) + q(t)\varphi(t, \lambda)] f(t)}{\lambda^2}\, dt \right\} \\ & - \frac{\varphi(x, \lambda)}{\Delta(\lambda)} \left\{ \int_\pi^x \frac{[-\psi''(t, \lambda) + q(t)\psi(t, \lambda)] f(t)}{\lambda^2}\, dt \right\} \\ & - \frac{f_1}{\Delta(\lambda)} \psi(x, \lambda) + \frac{f_2}{\Delta(\lambda)} \varphi(x, \lambda). \end{aligned} \tag{24}$$

Integrating by parts and taking into account the boundary conditions (2), (3) we obtain

$$\begin{aligned} y(x, \lambda) = {} & -\frac{1}{\lambda^2} f(x) - \frac{1}{\lambda^2} \left[Z_1(x, \lambda) + Z_2(x, \lambda) \right] \\ & - \frac{f_1}{\Delta(\lambda)} \psi(x, \lambda) + \frac{f_2}{\Delta(\lambda)} \varphi(x, \lambda), \end{aligned} \tag{25}$$

where

$$\begin{aligned} Z_1(x, \lambda) = {} & \frac{1}{\Delta(\lambda)} \psi(x, \lambda) \int_0^x \varphi'(t, \lambda) f'(t)\, dt + \frac{1}{\Delta(\lambda)} \varphi(x, \lambda) \int_x^\pi \psi'(t, \lambda) f'(t)\, dt, \\ Z_2(x, \lambda) = {} & \frac{1}{\Delta(\lambda)} \left[\left(\lambda^2 \alpha_2 - \alpha_1\right) \psi(x, \lambda) f(0) \right] - \frac{1}{\Delta(\lambda)} \left[\left(\lambda^2 \beta_2 - \beta_3\right) \varphi(x, \lambda) f(\pi) \right] \\ & + \frac{1}{\Delta(\lambda)} \psi(x, \lambda) \int_0^x \varphi(t, \lambda) q(t) f(t)\, dt + \frac{1}{\Delta(\lambda)} \varphi(x, \lambda) \int_x^\pi \psi(t, \lambda) q(t) f(t)\, dt. \end{aligned}$$

If we consider the following contour integral where Γ_n is a counter-clockwise oriented contour:

$$I_n(x) = \frac{1}{2\pi i} \oint_{\Gamma_n} \lambda y(x, \lambda)\, d\lambda,$$

and then taking into consideration (20) we get

$$\begin{aligned} I_n(x) &= \sum_{n=1}^\infty \operatorname{Res}_{\lambda = \lambda_n} \left[\lambda y(x, \lambda) \right] \\ &= \sum_{n=1}^\infty a_n \varphi(x, \lambda_n) + \sum_{n=1}^\infty \frac{\lambda_n f_1}{\dot{\Delta}(\lambda_n)} \psi(x, \lambda_n) - \sum_{n=1}^\infty \frac{\lambda_n f_2}{\dot{\Delta}(\lambda_n)} \varphi(x, \lambda_n), \end{aligned} \tag{26}$$

where

$$a_n = \frac{1}{\alpha_n}\int_0^\pi \varphi(t,\lambda_n)f(t)\rho(t)\,dt.$$

On the other hand, with the help of (25) we get

$$I_n(x) = f(x) - \frac{1}{2\pi i}\oint_{\Gamma_n}\left[Z_1(x,\lambda) + Z_2(x,\lambda)\right]d\lambda + \sum_{n=1}^{\infty}\frac{\lambda_n f_1}{\dot{\Delta}(\lambda_n)}\psi(x,\lambda_n) - \sum_{n=1}^{\infty}\frac{\lambda_n f_2}{\dot{\Delta}(\lambda_n)}\varphi(x,\lambda_n). \tag{27}$$

Comparing (26) and (27) we obtain

$$\sum_{n=1}^{\infty} a_n\varphi(x,\lambda_n) = f(x) + \epsilon_n(x),$$

where

$$\epsilon_n(x) = -\frac{1}{2\pi i}\oint_{\Gamma_n}\left[Z_1(x,\lambda) + Z_2(x,\lambda)\right]d\lambda.$$

The relations below hold for sufficiently large $\lambda^* > 0$

$$\max_{x\in[0,\pi]}\left|Z_2(x,\lambda)\right| \le \frac{C_2}{|\lambda|^2}, \quad \lambda \in G_\delta, |\lambda| \le \lambda^*, \tag{28}$$

$$\max_{x\in[0,\pi]}\left|Z_1(x,\lambda)\right| \le \frac{C_1}{|\lambda|^2}, \quad \lambda \in G_\delta, |\lambda| \le \lambda^*. \tag{29}$$

The validity of

$$\lim_{n\to\infty}\max_{x\in[0,\pi]}\left|\epsilon_n(x)\right| = 0$$

can easily be seen from (28) and (29). The last equation gives us the expansion formula

$$f(x) = \sum_{n=1}^{\infty} a_n\varphi(x,\lambda_n).$$

Since the system of $\Phi(x,\lambda_n)$ is complete and orthogonal in $L_{2,\rho}(0,\pi)\oplus\mathbb{C}^2$, the Parseval equality

$$\int_0^\pi \left|f(x)\right|^2\rho(x)\,dx = \sum_{n=1}^{\infty}\alpha_n|a_n|^2$$

holds. □

5 Uniqueness theorems

We consider the statement of the inverse problem of the reconstruction of the boundary value problem (1)-(3) from the Weyl function.

Let the functions $c(x,\lambda)$ and $s(x,\lambda)$ denote the solutions of (1) satisfying the conditions $c(0,\lambda) = 1$, $c'(0,\lambda) = 0$, $s(0,\lambda) = 0$ and $s'(0,\lambda) = 1$, respectively, and $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of (1) under the initial conditions (4), (5).

Further, let the function $\Phi(x,\lambda)$ be the solution of (1) satisfying $U(\Phi) = 1$ and $V(\Phi) = 0$. We set

$$M(\lambda) := \frac{\psi(0,\lambda)}{\Delta(\lambda)}.$$

The functions $\Phi(x,\lambda)$ and $M(\lambda)$ are called the Weyl solution and the Weyl function for the boundary value problem (1)-(3), respectively. The Weyl function is a meromorphic function having simple poles at points λ_n, eigenvalues of the boundary value problem of (1)-(3). The Wronskian

$$W(x) := \langle \varphi(x,\lambda), \Phi(x,\lambda) \rangle$$

does not depend on x. Taking $x = 0$, we get

$$W(0) = \varphi(0,\lambda)\Phi'(0,\lambda) - \varphi'(0,\lambda)\Phi(0,\lambda) = 1.$$

Hence,

$$W(x) = \langle \varphi(x,\lambda), \Phi(x,\lambda) \rangle = 1. \tag{30}$$

In view of (4) and (5), we get for $\lambda \neq \lambda_n$

$$\Phi(x,\lambda) = \frac{\psi(x,\lambda)}{\Delta(\lambda)}. \tag{31}$$

Using (31) we obtain

$$M(\lambda) = -\frac{\Delta^0(\lambda)}{\Delta(\lambda)},$$

where $\Delta^0(\lambda) = -\psi(0,\lambda)$ is the characteristic function of the boundary value problem L_0:

$$ly = \lambda^2 y, \qquad 0 \le x \le \pi,$$
$$y(0) = 0, \qquad V(y) = 0.$$

It is clear that

$$\Phi(x,\lambda) = s(x,\lambda) + M(\lambda)\varphi(x,\lambda). \tag{32}$$

Theorem 5 *The boundary value problem of* (1)-(3) *is identically denoted by the Weyl function* $M(\lambda)$.

Proof Let us denote the matrix $P(x,\lambda) = [P_{jk}(x,\lambda)]_{j,k=1,2}$ as

$$P(x,\lambda)\begin{pmatrix}\tilde{\varphi}(x,\lambda) & \tilde{\Phi}(x,\lambda)\\ \tilde{\varphi}'(x,\lambda) & \tilde{\Phi}'(x,\lambda)\end{pmatrix} = \begin{pmatrix}\varphi(x,\lambda) & \Phi(x,\lambda)\\ \varphi'(x,\lambda) & \Phi'(x,\lambda)\end{pmatrix}. \tag{33}$$

Then we have

$$\begin{aligned}&\varphi(x,\lambda) = P_{11}(x,\lambda)\tilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}'(x,\lambda),\\ &\Phi(x,\lambda) = P_{11}(x,\lambda)\tilde{\Phi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\Phi}'(x,\lambda)\end{aligned} \tag{34}$$

or

$$\begin{aligned}&P_{11}(x,\lambda) = \varphi(x,\lambda)\tilde{\Phi}'(x,\lambda) - \tilde{\varphi}'(x,\lambda)\Phi(x,\lambda),\\ &P_{12}(x,\lambda) = \tilde{\varphi}(x,\lambda)\Phi(x,\lambda) - \varphi(x,\lambda)\tilde{\Phi}(x,\lambda).\end{aligned} \tag{35}$$

Taking (31) into consideration in (35) we get

$$\begin{aligned}&P_{11}(x,\lambda) = 1 + \frac{1}{\Delta(\lambda)}\psi(x,\lambda)\bigl[\varphi'(x,\lambda) - \tilde{\varphi}'(x,\lambda)\bigr] + \frac{1}{\Delta(\lambda)}\varphi(x,\lambda)\bigl[\tilde{\psi}'(x,\lambda) - \psi'(x,\lambda)\bigr],\\ &P_{12}(x,\lambda) = \frac{1}{\Delta(\lambda)}\bigl[\tilde{\varphi}(x,\lambda)\psi(x,\lambda) - \varphi(x,\lambda)\tilde{\psi}(x,\lambda)\bigr].\end{aligned} \tag{36}$$

From the estimates as $|\lambda| \to \infty$

$$\begin{aligned}&\left|\frac{\varphi'(x,\lambda) - \tilde{\varphi}'(x,\lambda)}{\Delta(\lambda)}\right| = O\left(\frac{1}{|\lambda|^2}e^{|\operatorname{Im}\lambda|\mu^+(x)}\right),\\ &\left|\frac{\tilde{\psi}'(x,\lambda) - \psi'(x,\lambda)}{\Delta(\lambda)}\right| = O\left(\frac{1}{|\lambda|^2}e^{|\operatorname{Im}\lambda|(\mu^+(\pi)-\mu^+(x))}\right),\end{aligned}$$

we have from (36)

$$\lim_{|\lambda|\to\infty}\max_{x\in[0,\pi]}\bigl|P_{11}(x,\lambda) - 1\bigr| = \lim_{|\lambda|\to\infty}\max_{x\in[0,\pi]}\bigl|P_{12}(x,\lambda)\bigr| = 0 \tag{37}$$

for $\lambda \in G_\delta$.

Now, if we take into consideration (32) and (35), we have

$$\begin{aligned}&P_{11}(x,\lambda) = \varphi(x,\lambda)\tilde{s}'(x,\lambda) - \tilde{\varphi}'(x,\lambda)s(x,\lambda) + \tilde{\varphi}'(x,\lambda)\varphi(x,\lambda)\bigl[\tilde{M}(\lambda) - M(\lambda)\bigr],\\ &P_{12}(x,\lambda) = \tilde{\varphi}(x,\lambda)s(x,\lambda) - \varphi(x,\lambda)\tilde{s}(x,\lambda) + \varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\bigl[M(\lambda) - \tilde{M}(\lambda)\bigr].\end{aligned}$$

Therefore if $M(\lambda) = \tilde{M}(\lambda)$, one has

$$\begin{aligned}&P_{11}(x,\lambda) = \varphi(x,\lambda)\tilde{s}'(x,\lambda) - s(x,\lambda)\tilde{\varphi}'(x,\lambda),\\ &P_{12}(x,\lambda) = \varphi(x,\lambda)\tilde{s}(x,\lambda) - s(x,\lambda)\tilde{\varphi}(x,\lambda).\end{aligned}$$

Thus, for every fixed x functions $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire functions for λ. It can easily be seen from (37) that $P_{11}(x,\lambda) = 1$ and $P_{12}(x,\lambda) = 0$. Consequently, we get $\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda)$ and $\Phi(x,\lambda) \equiv \tilde{\Phi}(x,\lambda)$ for every x and λ. Hence, we arrive at $q(x) \equiv \tilde{q}(x)$. □

The validity of the equation below can be seen analogously to [29]:

$$M(\lambda) = M(0) + \sum_{n=1}^{\infty} \frac{\lambda^2}{\alpha_n \lambda_n^2 (\lambda^2 - \lambda_n^2)}. \tag{38}$$

Theorem 6 *The spectral data identically define the boundary value problem* (1)-(3).

Proof From (38), it is clear that the function $M(\lambda)$ can be constructed by λ_n. Since $\tilde{\lambda}_n = \lambda_n$ for every $n \in \mathbb{N}$, we can say that $M(\lambda) = \tilde{M}(\lambda)$. Then from Theorem 5, it is obvious that $L = \tilde{L}$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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