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Positive solutions for a class of superlinear semipositone systems on exterior domains

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Abstract

We study the existence of a positive radial solution to the nonlinear eigenvalue problem $-\Delta u = \lambda K_1(|x|)f(v)$ in Ω_e, $-\Delta v = \lambda K_2(|x|)g(u)$ in Ω_e, $u(x) = v(x) = 0$ if $|x| = r_0$ (> 0), $u(x) \to 0$, $v(x) \to 0$ as $|x| \to \infty$, where $\lambda > 0$ is a parameter, $\Delta u = \mathrm{div}(\nabla u)$ is the Laplace operator, $\Omega_e = \{x \in \mathbb{R}^n \mid |x| > r_0, n > 2\}$, and $K_i \in C^1([r_0,\infty),(0,\infty))$; $i = 1,2$ are such that $K_i(|x|) \to 0$ as $|x| \to \infty$. Here $f, g : [0,\infty) \to \mathbb{R}$ are C^1 functions such that they are negative at the origin (semipositone) and superlinear at infinity. We establish the existence of a positive solution for λ small via degree theory and rescaling arguments. We also discuss a non-existence result for $\lambda \gg 1$ for the single equations case.

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1 Introduction

We consider the nonlinear elliptic boundary value problem

$$\left.\begin{aligned} -\Delta u &= \lambda K_1(|x|)f(v) && \text{in } \Omega_e,\\ -\Delta v &= \lambda K_2(|x|)g(u) && \text{in } \Omega_e,\\ u(x) &= v(x) = 0 && \text{if } |x| = r_0\ (> 0),\\ u(x) &\to 0,\ v(x) \to 0 && \text{as } |x| \to \infty, \end{aligned}\right\} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $\Delta u = \mathrm{div}(\nabla u)$ is the Laplace operator, and $\Omega_e = \{x \in \mathbb{R}^n \mid |x| > r_0, n > 2\}$ is an exterior domain. Here the nonlinearities $f, g : [0,\infty) \to \mathbb{R}$ are C^1 functions which satisfy:

(H_1) $f(0) < 0$ and $g(0) < 0$ (semipositone).
(H_2) For $i = 1,2$ there exist $b_i > 0$ and $q_i > 1$ such that $\lim_{s\to\infty} \frac{f(s)}{s^{q_1}} = b_1$, and $\lim_{s\to\infty} \frac{g(s)}{s^{q_2}} = b_2$.

Further, for $i = 1,2$, the weight functions $K_i \in C^1([r_0,\infty),(0,\infty))$ are such that $K_i(|x|) \to 0$ as $|x| \to \infty$. In particular, we are interested in the challenging case, where K_i do not decay too fast. Namely, we assume

(H_3) There exist $\tilde{d}_1 > 0$, $\tilde{d}_2 > 0$, $\rho \in (0, n-2)$ such that for $i = 1,2$

$$\frac{\tilde{d}_1}{|x|^{n+\rho}} \le K_i\bigl(|x|\bigr) \le \frac{\tilde{d}_2}{|x|^{n+\rho}} \quad \text{for } |x| \gg 1.$$

We then establish the following.

Theorem 1.1 *Let* (H_1)-(H_3) *hold. Then* (1.1) *has a positive radial solution* (u, v) $(u > 0, v > 0$ *in* $\Omega_e)$ *when* λ *is small, and* $\|u\|_\infty \to \infty$, $\|v\|_\infty \to \infty$ *as* $\lambda \to 0$.

We prove this result via the Leray-Schauder degree theory, by arguments similar to those used in [1] and [2]. The study of such eigenvalue problems with semipositone structure has been documented to be mathematically challenging (see [3, 4]), yet a rich history is developing starting from the 1980s (see [5–7]) until recently (see [8–12]). In [1, 2] the authors studied such superlinear semipositone problems on bounded domains. In particular, in [12] the authors studied the system

$$\left.\begin{array}{ll} -\Delta u = \lambda f(v) & \text{in } \Omega, \\ -\Delta v = \lambda g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array}\right\}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, and establish an existence result when λ is small. The main motivation of this paper is to extend this study in the case of exterior domains (see Theorem 1.1).

We also discuss a non-existence result for the single equation model:

$$\left.\begin{array}{ll} -\Delta u = \lambda K_1(|x|)\tilde{f}(u) & \text{in } \Omega_e, \\ u(x) = 0 & \text{if } |x| = r_0 \ (> 0), \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{array}\right\} \tag{1.2}$$

for large values of λ, when $\tilde{f}$, K_1 satisfy the following hypotheses:

(H_4) $\tilde{f} \in C^1([0,\infty), \mathbb{R})$, $\tilde{f}'(z) > 0$ for all $z > 0$, $\tilde{f}(0) < 0$, and there exists $m_0 > 0$ such that $\lim_{z\to\infty} \frac{\tilde{f}(z)}{z} \geq m_0$.

(H_5) The weight function $K_1 \in C^1([r_0,\infty), (0,\infty))$ is such that $s^{\frac{-2(n-1)}{n-2}} K_1(r_0 s^{\frac{1}{2-n}})$ is decreasing for $s \in (0, 1]$.

We establish the following.

Theorem 1.2 *Let* (H_3)-(H_5) *hold. Then* (1.2) *has no nonnegative radial solution for* $\lambda \gg 1$.

We establish Theorem 1.2 by recalling various useful properties of solutions established in [13], where the authors prove a uniqueness result for $\lambda \gg 1$ for such an equation in the case when $\tilde{f}$ is sublinear at ∞. However, the properties we recall from [13] are independent of the growth behavior of $\tilde{f}$ at ∞. Non-existence results for such superlinear semipositone problems on bounded domain also have a considerable history starting from the work in the 1980s in [14] leading to the recent work in [15]. Here we discuss such a result for the first time on exterior domains.

Finally, we note that the study of radial solutions $(u(r), v(r))$ (with $r = |x|$) of (1.1) corresponds to studying

$$\left.\begin{array}{ll} -(r^{n-1}u'(r))' = \lambda r^{n-1}K_1(r)f(v(r)) & \text{for } r > r_0, \\ -(r^{n-1}v'(r))' = \lambda r^{n-1}K_2(r)g(u(r)) & \text{for } r > r_0, \\ u(r) = v(r) = 0 & \text{if } r = r_0\ (> 0), \\ u(r) \to 0,\ v(r) \to 0 & \text{as } r \to \infty, \end{array}\right\}$$

which can be reduced to the study of solutions $(u(s), v(s))$; $s \in [0,1]$ to the singular system:

$$\left.\begin{array}{ll} -u''(s) = \lambda h_1(s)f(v(s)), & 0 < s < 1, \\ -v''(s) = \lambda h_2(s)g(u(s)), & 0 < s < 1, \\ u(0) = u(1) = 0,\ v(0) = v(1) = 0, & \end{array}\right\} \tag{1.3}$$

via the Kelvin transformation $s = (\frac{r}{r_0})^{2-n}$, where $h_i(s) = \frac{r_0^2}{(n-2)^2} s^{\frac{-2(n-1)}{(n-2)}} K_i(r_0 s^{\frac{1}{2-n}})$, $i = 1,2$ (see [16]).

Remark 1.3 The assumption (H_3) implies that $\lim_{s\to 0^+} h_i(s) = \infty$, for $i = 1,2$, $\hat{h} = \inf_{t\in(0,1)}\{h_1(t), h_2(t)\} > 0$, and there exist $d > 0$, $\eta \in (0,1)$ such that $h_i(s) \le \frac{d}{s^\eta}$ for $s \in (0,1]$, and for $i = 1,2$. When in addition (H_5) is satisfied, h_1 is decreasing in $(0,1]$.

We will prove Theorem 1.1 in Section 2 by studying the singular system (1.3), and Theorem 1.2 in Section 3 by studying the corresponding single equation

$$\left.\begin{array}{ll} -u''(s) = \lambda h_1(s)\tilde{f}(u(s)), & 0 < s < 1, \\ u(0) = u(1) = 0. & \end{array}\right\} \tag{1.4}$$

2 Existence result

We first establish some useful results for solutions to the system

$$\left.\begin{array}{ll} -u''(s) = b_1 h_1(s)|v(s) + l|^{q_1}, & 0 < s < 1, \\ -v''(s) = b_2 h_2(s)|u(s) + l|^{q_2}, & 0 < s < 1, \\ u(0) = u(1) = 0,\ v(0) = v(1) = 0, & \end{array}\right\} \tag{2.1}$$

where $l \ge 0$ is a parameter. (Clearly, any solution (u_l, v_l) of (2.1) for $l > 0$ must satisfy $u_l(s) > 0$, $v_l(s) > 0$ for $s \in (0,1)$. This is also true for any nontrivial solution when $l = 0$.) We prove the following.

Lemma 2.1

(i) *There exists $l_0 > 0$ such that* 2.1 *has no solution if $l \ge l_0$.*

(ii) *For each $l \in [0, l_0)$, there exists $M > 0$ (independent of l) such that if (u_l, v_l) is a solution of* (2.1), *then* $\max\{\|u_l\|_\infty, \|v_l\|_\infty\} \le M$.

Proof of (i) Let $\lambda_1 := \pi^2$, $\phi_1 := \sin(\pi s)$. Here λ_1 is the principal eigenvalue and ϕ_1 a corresponding eigenfunction of $-\phi''(s) = \lambda\phi(s)$ in $(0,1)$ with $\phi(0) = 0 = \phi(1)$. Let $a > \frac{\lambda_1}{\sqrt{b_1 b_2 \hat{h}}}$, $c > 0$ be such that $(s + l)^{q_i} \ge as - c$ for all $s \ge 0$ and for $i = 1,2$. Now let (u_l, v_l) be a solution of

(2.1). Multiplying (2.1) by ϕ_1 and integrating, we obtain

$$\lambda_1 \int_0^1 u_l \phi_1 \, ds = b_1 \int_0^1 h_1(s)(v_l + l)^{q_1} \phi_1 \, ds \geq b_1 \int_0^1 h_1(s)(a v_l - c)\phi_1 \, ds$$

and

$$\lambda_1 \int_0^1 v_l \phi_1 \, ds = b_2 \int_0^1 h_2(s)(u_l + l)^{q_2} \phi_1 \, ds \geq b_2 \int_0^1 h_2(s)(a u_l - c)\phi_1 \, ds.$$

By Remark 1.3, $\hat{h} = \inf_{t\in(0,1)}\{h_1(t), h_2(t)\} > 0$, and $\|h_i\|_1 := \int_0^1 h_i(s)\, ds < \infty$ for $i = 1, 2$. Then from the above inequalities we obtain

$$\int_0^1 v_l \phi_1 \, ds \leq \frac{1}{a b_1 \hat{h}} \left(\lambda_1 \int_0^1 u_l \phi_1 \, ds + b_1 c \|h_1\|_1 \right)$$

and

$$\int_0^1 u_l \phi_1 \, ds \leq \frac{1}{a b_2 \hat{h}} \left(\lambda_1 \int_0^1 v_l \phi_1 \, ds + b_2 c \|h_2\|_1 \right).$$

Hence we deduce that

$$\int_0^1 u_l \phi_1 \, ds \leq \frac{m_1}{m} := m_2,$$

where $m := (a b_2 \hat{h} - \frac{\lambda_1^2}{a b_1 \hat{h}})$, and $m_1 := \frac{\lambda_1 c \|h_1\|_1}{a \hat{h}} + b_2 c \|h_2\|_1$. This implies

$$\int_0^1 (v_l + l)^{q_1} \phi_1 \, ds \leq \frac{\lambda_1 m_2}{b_1 \hat{h}} := m_3.$$

In particular, this implies $\int_{\frac{1}{4}}^{\frac{3}{4}} l^{q_1} \mathrm{d}s \leq \frac{m_3}{\inf_{[\frac{1}{4},\frac{3}{4}]} \phi_1}$. Since m_3 is independent of l, clearly this is a contradiction for $l \gg 1$, and hence there must exists an $l_0 > 0$ such that for $l \geq l_0$, (2.1) has no solution.

Proof of (ii) Assume the contrary. Then without loss of generality we can assume there exists $\{l_n\} \subset (0, l_0)$ such that $\|u_{l_n}\|_\infty \to \infty$ as $n \to \infty$. Clearly $u''_{l_n}(s) < 0$, and $v''_{l_n}(s) < 0$ for all $s \in (0,1)$. Let $s_{1(l_n)} \in (0,1)$, $s_{2(l_n)} \in (0,1)$ be the points at which u_{l_n} and v_{l_n} attain their maximums. Now since $u''_{l_n}(s) < 0$ for all $s \in (0,1)$, we have

$$u_{l_n}(s) \geq \begin{cases} \frac{s u_{l_n}(s_{1(l_n)})}{s_{1(l_n)}} & \text{for } s \in (0, s_{1(l_n)}), \\ \frac{(1-s) u_{l_n}(s_{1(l_n)})}{1 - s_{1(l_n)}} & \text{for } s \in (s_{1(l_n)}, 1). \end{cases}$$

Hence $u_{l_n}(s) \geq \min\{\frac{s\|u_{l_n}\|_\infty}{s_{1(l_n)}}, \frac{(1-s)\|u_{l_n}\|_\infty}{1-s_{1(l_n)}}\}$, and in particular, for $s \in [\frac{1}{4}, \frac{3}{4}]$,

$$u_{l_n}(s) \geq \min\left\{ \frac{1}{4}\|u_{l_n}\|_\infty, \frac{1}{4}\|u_{l_n}\|_\infty \right\} = \frac{1}{4}\|u_{l_n}\|_\infty.$$

Let $\tilde{s_{l_n}}, \bar{s_{l_n}} \in [\frac{1}{4}, \frac{3}{4}]$ be such that $\min_{[\frac{1}{4},\frac{3}{4}]} u_{l_n}(s) = u_{l_n}(\tilde{s_{l_n}})$, and $\min_{[\frac{1}{4},\frac{3}{4}]} v_{l_n}(s) = v_{l_n}(\bar{s_{l_n}})$. Now for $s \in [\frac{1}{4}, \frac{3}{4}]$,

$$v_{l_n}(s) \geq b_2 \hat{h} \tilde{m} \int_{\frac{1}{4}}^{\frac{3}{4}} \left|u_{l_n}(t) + l\right|^{q_2} \, dt,$$

where $\tilde{m} := \min_{[\frac{1}{4},\frac{3}{4}]\times[\frac{1}{4},\frac{3}{4}]} G(s,t)$ (> 0), and G is the Green's function of $-Z''$ with $Z(0) = 0 = Z(1)$. In particular, $v_{l_n}(\bar{s_{l_n}}) \geq b_2 \hat{h} \frac{\tilde{m}}{2} (u_{l_n}(\tilde{s_{l_n}}))^{q_2}$. Similarly $u_{l_n}(\tilde{s_{l_n}}) \geq b_1 \hat{h} \frac{\tilde{m}}{2} (v_{l_n}(\bar{s_{l_n}}))^{q_1}$. Hence, there exists a constant $A > 0$ such that

$$u_{l_n}(\tilde{s_{l_n}}) \geq A\left(u_{l_n}(\tilde{s_{l_n}})\right)^{q_1 q_2}.$$

This is a contradiction since $q_1 q_2 > 1$ and $u_{l_n}(\tilde{s_{l_n}}) \geq \frac{1}{4}\|u_{l_n}\|_\infty \to \infty$ as $n \to \infty$. Thus (ii) holds. $\square$

Proof of Theorem 1.1 We first extend f and g as even functions on $\mathbb{R}$ by setting $f(-s) = f(s)$ and $g(-s) = g(s)$. Then we use the rescaling, $\lambda = \gamma^\delta$, $w_1 = \gamma u$, and $w_2 = \gamma^\theta v$ with $\gamma > 0$, $\theta = \frac{q_2+1}{q_1+1}$, and $\delta = \frac{q_1 q_2 - 1}{q_1 + 1}$. With this rescaling, (1.3) reduces to

$$\left.\begin{array}{ll} -w_1''(s) = F(s,\gamma,w_2), & 0 < s < 1, \\ -w_2''(s) = G(s,\gamma,w_1), & 0 < s < 1, \\ w_1(0) = w_1(1) = 0,\ w_2(0) = w_2(1) = 0, & \end{array}\right\} \tag{2.2}$$

where

$$F(s,\gamma,w_2) := \gamma^{1+\delta} h_1(s)\left(f\left(\frac{w_2}{\gamma^\theta}\right) - b_1\left|\frac{w_2}{\gamma^\theta}\right|^{q_1}\right) + b_1|w_2|^{q_1} h_1(s), \quad \text{and}$$

$$G(s,\gamma,w_1) := \gamma^{\theta+\delta} h_2(s)\left(g\left(\frac{w_1}{\gamma}\right) - b_2\left|\frac{w_1}{\gamma}\right|^{q_2}\right) + b_2|w_1|^{q_2} h_2(s).$$

Note that by our hypothesis (H_2), $F(s,\gamma,w_2) \to b_1|w_2|^{q_1} h_1(s)$ and $G(s,\gamma,w_1) \to b_2|w_1|^{q_2} \times h_2(s)$ as $\gamma \to 0$. Hence we can continuously extend $F(s,\gamma,w_2)$ and $G(s,\gamma,w_1)$ to $F(s,0,w_2) = b_1|w_2|^{q_1} h_1(s)$ and $G(s,0,w_1) = b_2|w_1|^{q_2} h_2(s)$, respectively. Note that proving (1.3) has a positive solution for λ small is equivalent to proving (2.2) has a solution (w_1, w_2) with $w_1 > 0$, $w_2 > 0$ in $(0,1)$ for small $\gamma > 0$. We will achieve this by establishing that the limiting equation (when $\gamma = 0$)

$$\left.\begin{array}{ll} -w_1''(s) = F(s,0,w_2) = b_1 h_1(s)|w_2|^{q_1}, & 0 < s < 1, \\ -w_2''(s) = G(s,0,w_1) = b_2 h_2(s)|w_1|^{q_2}, & 0 < s < 1, \\ w_1(0) = w_1(1) = 0,\ w_2(0) = w_2(1) = 0 & \end{array}\right\} \tag{2.3}$$

(which is the same as (2.1) with $l = 0$) has a positive solution $w_1 > 0$, $w_2 > 0$ in $(0,1)$ that persists for small $\gamma > 0$.

Let $X = C_0[0,1] \times C_0[0,1]$ be the Banach space equipped with $\|\underline{w}\|_X = \|(w_1,w_2)\|_X = \max\{\|w_1\|_\infty, \|w_2\|_\infty\}$, where $\|\cdot\|_\infty$ denotes the usual supremum norm in $C_0([0,1])$. Then for fixed $\gamma \geq 0$, we define the map $S(\gamma,\cdot) : X \to X$ by

$$S(\gamma,\underline{w}) := \underline{w} - \left(K\left(F(s,\gamma,w_2)\right), K\left(G(s,\gamma,w_1)\right)\right),$$

where $K(H(s,\gamma,Z(s))) = \int_0^1 G(t,s)H(t,\gamma,Z(t))\,dt$. Note that $F(s,\gamma,\cdot), G(s,\gamma,\cdot) : C_0([0,1]) \to L^1(0,1)$ are continuous and $K : L^1(0,1) \to C_0^1([0,1])$ is compact. Hence $S(\gamma,\cdot)$ is a compact perturbation of the identity. Clearly for $\gamma > 0$, if $S(\gamma,\underline{w}) = \underline{0}$, then $\underline{w} = (w_1,w_2)$ is a solution of (2.2), and if $S(0,\underline{w}) = \underline{0}$, then $\underline{w} = (w_1,w_2)$ is a solution of (2.3).

We first establish the following.

Lemma 2.2 *There exists $R > 0$ such that $S(0,\underline{w}) \neq \underline{0}$ for all $\underline{w} = (w_1,w_2) \in X$ with $\|\underline{w}\|_X = R$ and* $\deg(S(0,\cdot), B_R(\underline{0}), \underline{0}) = 0$.

Proof Define $S^l(0,\underline{w}) : X \to X$ by

$$S^l(0,\underline{w}) := \underline{w} - \bigl(K\bigl(b_1h_1(s)|w_2 + l|^{q_1}\bigr), K\bigl(b_2h_2(s)|w_1 + l|^{q_2}\bigr)\bigr)$$

for $l \geq 0$. (Note $S^0(0,\underline{w}) = S(0,\underline{w})$.) By Lemma 2.1, if $l \geq l_0$ then $S^l(0,\underline{w}) \neq \underline{0}$ and if $S^l(0,\underline{w}) = \underline{0}$ for $l \in [0,l_0)$, then $\|w\|_X \leq M$. This implies that there exists $R \gg 1$ such that $S^l(0,\underline{w}) \neq \underline{0}$ for $\underline{w} \in \partial B_R(\underline{0})$ for any $l \geq 0$. Also, since (2.1) has no solution for $l \geq l_0$, $\deg(S^{l_0}(0,\cdot), B_R(\underline{0}), \underline{0}) = 0$. Hence, using the homotopy invariance of degree with the parameter $l \in [0,l_0]$ we get

$$\deg\bigl(S(0,\cdot), B_R(\underline{0}), \underline{0}\bigr) = \deg\bigl(S^{l_0}(0,\cdot), B_R(\underline{0}), \underline{0}\bigr) = 0. \qquad \square$$

Next we establish the following.

Lemma 2.3 *There exists $r \in (0,R)$ small enough such that $S(0,\underline{w}) \neq \underline{0}$ for all $\underline{w} = (w_1,w_2) \in X$ with $\|\underline{w}\|_X = r$ and* $\deg(S(0,\cdot), B_r(\underline{0}), \underline{0}) = 1$.

Proof Define $T^\tau(0,\underline{w}) : X \to X$ by

$$T^\tau(0,\underline{w}) := \underline{w} - \bigl(K\bigl(\tau b_1h_1(s)|w_2|^{q_1}\bigr), K\bigl(\tau b_2h_2(s)|w_1|^{q_2}\bigr)\bigr)$$

for $\tau \in [0,1]$. Clearly $T^1(0,\underline{w}) = S(0,\underline{w})$, and $T^0(0,\underline{w}) = I$ is the identity operator. Note that $T^\tau(0,\underline{w}) = 0$ if $\underline{w} = (w_1,w_2)$ is a solution of

$$\left.\begin{array}{ll} -w_1''(s) = \tau b_1h_1(s)|w_2|^{q_1}, & 0 < s < 1, \\ -w_2''(s) = \tau b_2h_2(s)|w_1|^{q_2}, & 0 < s < 1, \\ w_1(0) = w_1(1) = 0,\ w_2(0) = w_2(1) = 0, & \end{array}\right\} \tag{2.4}$$

and for $\tau = 1$, (2.4) coincides with (2.3). Assume to the contrary that (2.4) has a solution $\underline{w} = (w_1,w_2)$ with $\|\underline{w}\|_X = \tilde{r} > 0$. Without loss of generality assume $\|w_1\|_\infty = \tilde{r}$. Now,

$$w_1(s) = \tau \int_0^1 G(s,t)b_1h_1(s)|w_2|^{q_1}\,ds.$$

Then $\|w_1\|_\infty \leq \tilde{C}\|w_2\|_\infty^{q_1}$ for some constant $\tilde{C} > 0$ independent of $\tau \in [0,1]$. Similarly $\|w_2\|_\infty \leq \hat{C}\|w_1\|_\infty^{q_2}$ for some constant $\hat{C} > 0$. This implies that

$$\tilde{r} = \|w_1\|_\infty \leq C\|w_1\|_\infty^{q_1q_2} = C\tilde{r}^{q_1q_2}$$

for some constant $C > 0$. But $q_1 q_2 > 1$, and hence this is a contradiction if $\tilde{r} > 0$ is small. Thus there exists small $r > 0$ such that (2.4) has no solution $\underline{w}$ with $\|\underline{w}\|_X = r$ for all $\tau \in [0,1]$. Now using the homotopy invariance of degree with the parameter $\tau \in [0,1]$, in particular using the values $\tau = 1$ and $\tau = 0$, we obtain

$$\deg\bigl(S(0,\cdot), B_r(\underline{0}), \underline{0}\bigr) = \deg\bigl(T^1(0,\cdot), B_r(\underline{0}), \underline{0}\bigr) = \deg\bigl(T^0(0,\cdot), B_r(\underline{0}), \underline{0}\bigr) = 1. \qquad \square$$

By Lemma 2.2 and Lemma 2.3, with $0 < r < R$, we conclude that

$$\deg\bigl(S(0,\cdot), B_R(\underline{0}) \setminus \overline{B_r}(\underline{0}), \underline{0}\bigr) = -1,$$

and hence (2.3) has a solution $\underline{w} = (w_1, w_2)$ with $w_1 > 0$, $w_2 > 0$ in $(0,1)$, and $r < \|w\|_X < R$. Now we show that the solution obtained above (when $\gamma = 0$) persists for small $\gamma > 0$ and remains positive componentwise.

Lemma 2.4 *Let R, r be as in Lemmas* 2.2, 2.3, *respectively. Then there exists $\gamma_0 > 0$ such that*:

(i) $\deg(S(\gamma,\cdot), B_R(\underline{0}) \setminus \overline{B_r}(\underline{0}), \underline{0}) = -1$ *for all* $\gamma \in [0,\gamma_0]$.

(ii) *If* $S(\gamma,\underline{w}) = \underline{0}$ *for* $\gamma \in [0,\gamma_0]$ *with* $r < \|\underline{w}\|_X < R$, *then* $w_1 > 0$, $w_2 > 0$ *in* $(0,1)$.

Proof of (i) We first show that there exists $\gamma_0 > 0$ such that $S(\gamma,\underline{w}) \neq \underline{0}$ for all $\underline{w} = (w_1, w_2) \in X$ with $\|\underline{w}\|_X \in \{R, r\}$, for all $\gamma \in [0,\gamma_0]$. Suppose to the contrary that there exists $\{\gamma_n\}$ with $\gamma_n \to 0$, $S(\gamma_n, \underline{w_n}) = \underline{0}$ and $\|\underline{w_n}\|_X \in \{r, R\}$. Since $\underline{K} = (K,K) : L^1(0,1) \times L^1(0,1) \to C_0^1([0,1]) \times C_0^1([0,1])$ is compact, and $\{F(s,\gamma_n, w_{2n}), G(s,\gamma_n, w_{1n})\}$ are bounded in $L^1(0,1) \times L^1(0,1)$, $\underline{w_n} \to \underline{Z} = (Z_1, Z_2) \in C_0^1([0,1]) \times C_0^1([0,1])$ (up to a subsequence) with $\|\underline{Z}\|_X = R$ or r and $S(0,\underline{Z}) = \underline{0}$. This is a contradiction to Lemma 2.2 or 2.3 and hence there exists a small $\gamma_0 > 0$ satisfying the assertions. Now, by the homotopy invariance of degree with respect to $\gamma \in [0,\gamma_0]$,

$$\deg\bigl(S(\gamma,\cdot), B_R(\underline{0}) \setminus \overline{B_r}(\underline{0}), \underline{0}\bigr) = \deg\bigl(S(0,\cdot), B_R(\underline{0}) \setminus \overline{B_r}(\underline{0}), \underline{0}\bigr) = -1$$

for all $\gamma \in [0,\gamma_0]$.

Proof of (ii) Assume to the contrary that there exists $\gamma_n \to 0$ and a corresponding solution $\underline{w_n} = (w_{1n}, w_{2n})$ such that $r < \|\underline{w_n}\|_X < R$ and

$$\Omega_n := \bigl\{x \in (0,1) \mid w_{1n}(x) \le 0 \text{ or } w_{2n}(x) \le 0\bigr\} \neq \emptyset.$$

Arguing as before, $\underline{w_n} \to \underline{Z} \in C_0^1([0,1]) \times C_0^1([0,1])$ with $S(0,\underline{Z}) = \underline{0}$ (up to a subsequence). Note that $\underline{Z} \not\equiv \underline{0}$ since $\|\underline{Z}\|_X \ge r > 0$. By the strong maximum principle $Z_1 > 0$, $Z_2 > 0$, $Z_1'(0) > 0$, $Z_2'(0) > 0$, $Z_1'(1) < 0$ and $Z_2'(1) < 0$. Now suppose there exists $\{x_n\} \in (0,1)$ with $\{x_n\} \in \Omega_n$ and $w_{1n}(x_n) \le 0$. Then $\{x_n\}$ must have a subsequence (renamed as $\{x_n\}$ itself) such that $x_n \to \tilde{x} \in [0,1]$. But $Z_1 > 0$ in $(0,1)$ implies that $\tilde{x} \in \{0,1\}$. Suppose $\tilde{x} = 0$. Since $w_{1n}(x_n) \le 0$ and $w_{1n}(0) = 0$, there exists $y_n \in (0,x_n)$ such that $w_{1n}'(y_n) \le 0$, and hence taking the limit as $n \to \infty$ we will have $Z_1'(0) \le 0$, which is a contradiction since $Z_1'(0) > 0$. A similar contradiction follows if $\tilde{x} = 1$, using the fact that $Z_1'(1) < 0$. Further, contradictions can be achieved if there exists $\{x_n\} \in \Omega$ with $\{x_n\} \in \Omega_n$ and $w_{2n}(x_n) \leq 0$ using the facts that $Z_2'(0) > 0$ and $Z_2'(1) < 0$. This completes the proof of the lemma. $\square$

We now easily conclude the proof of Theorem 1.1. From Lemma 2.4, since $\underline{w} = (w_1, w_2)$ is a positive solution of (2.2) for γ small, $(u, v) = (\gamma^{-1}w_1, \gamma^{-\theta}w_2)$ with $\theta = \frac{q_2+1}{q_1+1}$ is a positive solution of (1.3) for $\lambda = \gamma^\delta$ where $\delta = \frac{q_1q_2-1}{q_1+1}$. Further, since $w_1 > 0$ and $w_2 > 0$ in $(0,1)$ for $\gamma \in [0, \gamma_0]$, $\|u\|_\infty \to \infty$ and $\|v\|_\infty \to \infty$ as $\lambda(= \gamma^\delta) \to 0$. This completes the proof of Theorem 1.1. $\square$

3 Non-existence result

We first recall from [13] that, when (H_5) is satisfied, one can prove via an energy analysis that a nonnegative solution u of (1.4) must be positive in $(0,1)$ and have a unique interior maximum with maximum value greater than θ, where θ is the unique positive zero of $\tilde{F}(s) = \int_0^s \tilde{f}(y)\,dy$. Further, for $\lambda \gg 1$ and $s_1, \hat{s}_1 \in (0,1)$ such that $\hat{s}_1 > s_1$, $u(s_1) = u(\hat{s}_1) = \beta$ (see Figure 1), where $\beta > 0$ is the unique zero of $\tilde{f}$, there exists a constant C such that $s_1 \leq C\lambda^{-\frac{1}{2}}$ and $(1 - \hat{s}_1) \leq C\lambda^{-\frac{1}{2}}$. Hence we can assume $(\hat{s}_1 - s_1) > \frac{1}{2}$ for $\lambda \gg 1$. Now we provide the proof of Theorem 1.2.

Proof of Theorem 1.2 Let $v := u - \beta$. Then $v > 0$ in $(s_1, \hat{s}_1)$ and satisfies

$$\left.\begin{array}{l} -v'' = \lambda h_1(s)\frac{\tilde{f}(u)}{u-\beta}v, \quad s_1 < s < \hat{s}_1, \\ v(s_1) = v(\hat{s}_1) = 0. \end{array}\right\}$$

Note that $\phi(s) = -(\sin(\frac{\pi(s-s_1)}{(\hat{s}_1-s_1)})) > 0$ in $(s_1, \hat{s}_1)$, $\phi(s_1) = \phi(\hat{s}_1) = 0$, and it satisfies $-\phi'' = \frac{\pi^2}{(\hat{s}_1-s_1)^2}\phi$ in $(s_1, \hat{s}_1)$. Hence using the fact that $\int_{s_1}^{\hat{s}_1}(-\phi v'' + v\phi'')\,ds = 0$, we obtain

$$\int_{s_1}^{\hat{s}_1}\left(\lambda\frac{\tilde{f}(u)}{u-\beta}h_1(s) - \frac{\pi^2}{(\hat{s}_1 - s_1)^2}\right)v\phi\,ds = 0.$$

In particular,

$$\lambda\frac{\tilde{f}(u(s_\lambda))}{u(s_\lambda) - \beta}h_1(s_\lambda) = \frac{\pi^2}{(\hat{s}_1 - s_1)^2}, \quad \text{for some } s_\lambda \in (s_1, \hat{s}_1). \tag{3.1}$$

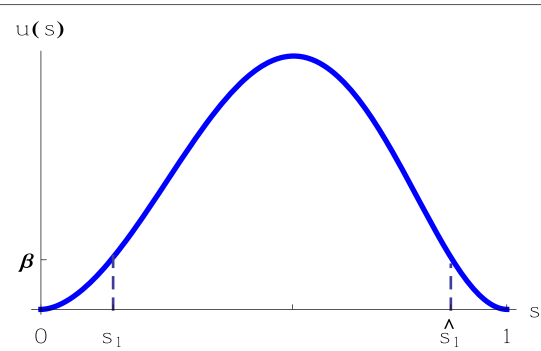

Figure 1 Graph of *u*.

But $\hat{h} = \inf_{(0,1)} h_1(s) > 0$, and $(\hat{s_1} - s_1) > \frac{1}{2}$ for $\lambda \gg 1$. Thus clearly (3.1) can hold when $\lambda \to \infty$, only if $Z = u(s_\lambda) \to \infty$ with $\frac{\tilde{f}(u(s_\lambda))}{u(s_\lambda)-\beta} \to 0$. But by ($H_4$), this is not possible since $\lim_{Z\to\infty} \frac{\tilde{f}(Z)}{Z} \geq m_0 > 0$. Hence the nonnegative solution cannot exist for $\lambda \gg 1$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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