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Multiplicity of positive solutions of superlinear semi-positone singular Neumann problems

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Abstract

Introduction: Neumann boundary value problems have been studied by many authors. We are mainly interested in the semi-positone case. This paper deals with the existence and multiplicity of positive solutions of a superlinear semi-positone singular Neumann boundary value problem.

Preliminaries: The proof of our main results relies on a nonlinear alternative of Leray-Schauder type, the method of upper and lower solutions and on a well-known fixed point theorem in cones.

Main results: We obtained the existence of at least two different positive solutions.

Keywords: positive solutions; superlinear; semi-positone; singular; Neumann problem

1 Introduction

We will be concerned with the existence and multiplicity of positive solutions of the superlinear singular Neumann boundary value problem in the semi-positone case

$$\begin{cases} -(p(x)u')' + q(x)u = g(x, u), & x \in I = [0, 1], \\ u'(0) = 0, & u'(1) = 0. \end{cases} \quad (1.1)$$

Here the type of perturbations $g(x, u)$ may be singular near $u = 0$ and $g(x, u)$ is superlinear near $u = +\infty$. From the physical point of view, $g(x, u)$ has an attractive singularity near $u = 0$ if

$$\lim_{u \rightarrow 0^+} g(x, u) = +\infty \quad \text{uniformly in } x$$

and the superlinearity of $g(x, u)$ means that

$$\lim_{u \rightarrow +\infty} g(x, u)/u = +\infty \quad \text{uniformly in } x.$$

By the semi-positone case of (1.1), we mean that $g(x, u)$ may change sign and satisfies $F(x, u) = g(x, u) + M \geq 0$ where $M > 0$ is a constant.

It is well known that the existence of positive solutions of boundary value problems has been studied by many authors in [1–6] and references therein. They mainly considered the case of $p(x) \equiv 1$ and $q(x) \equiv 0$. In [7], the authors studied positive solutions of Neumann boundary problems of second order impulsive differential equations in the positone case, based on a nonlinear alternative principle of Leray-Schauder type and a well-known fixed point theorem in cones. This paper attempts to study the existence and multiplicity of positive solutions of second order superlinear singular Neumann boundary value problems in the semi-positone case. The techniques we employ here involve a nonlinear result of Leray-Schauder, the well-known fixed point theorem in cones and the method of upper and lower solutions. We prove that problem (1.1) has at least two different positive solutions. Moreover, we do not take the restrictions $p(x) \equiv 1$ or $q(x) \equiv 0$.

Throughout this paper, we assume that the perturbed part $g(x, u)$ satisfies the following hypotheses:

- (H₁) $g(x, u) \in C(I \times \mathbb{R}^+, \mathbb{R}^+)$, $p(x) \in C^1(I)$, $q(x) \in C(I)$, $p(x) > 0$, $q(x) > 0$.
- (H₂) There exists a constant $M > 0$ such that $F(x, u) = g(x, u) + M \geq 0$ for all $x \in I$ and $u \in (0, \infty)$.

In Section 2, we perform a study of the sign of the Green’s function of the corresponding linear problems

$$\begin{cases} -(p(x)u')' + q(x)u = h(x), & x \in I, \\ u'(0) = 0, & u'(1) = 0. \end{cases} \tag{1.2}$$

In detail, we construct the Green’s function $G(x, y)$ and give a sufficient condition to ensure $G(x, y)$ is positive. This fact is crucial for our arguments. We denote

$$A = \min_{(x,y) \in I \times I} G(x, y), \quad B = \max_{(x,y) \in I \times I} G(x, y), \quad \sigma = A/B. \tag{1.3}$$

We also use $\omega(x)$ to denote the unique solution of (1.2) with $h(x) = 1$, $\omega(x) = \int_0^1 G(x, y) dy$. In Section 3, we state and prove the main results of this paper.

2 Preliminaries

For the reader’s convenience we introduce some results of Green’s functions. Let $Q = I \times I$, $Q_1 = \{(x, y) \in Q | 0 \leq x \leq y \leq 1\}$, $Q_2 = \{(x, y) \in Q | 0 \leq y \leq x \leq 1\}$.

Considering the homogeneous boundary value problem

$$\begin{cases} -(p(x)u')' + q(x)u = 0, & x \in I, \\ u'(0) = 0, & u'(1) = 0, \end{cases} \tag{2.1}$$

and let $G(x, y)$ be the Green’s function of problem (2.1). Then $G(x, y)$ can be written as

$$G(x, y) = \begin{cases} \frac{m(x)n(y)}{\omega}, & (x, y) \in Q_1, \\ \frac{m(y)n(x)}{\omega}, & (x, y) \in Q_2, \end{cases} \tag{2.2}$$

where m and n are linearly independent, and m , n and ω satisfy the following lemma.

Lemma 2.1 [7] *Suppose that (H_1) holds and problem (2.1) has only zero solution, then there exist two functions $m(x)$ and $n(x)$ satisfying:*

- (i) $m(x) \in C^2(I, R)$ is increasing and $m(x) > 0, x \in I$;
- (ii) $n(x) \in C^2(I, R)$ is decreasing and $n(x) > 0, x \in I$;
- (iii) $Lm \equiv -(p(x)m')' + q(x)m = 0, m(0) = 1, m'(0) = 0$;
- (iv) $Ln \equiv -(p(x)n')' + q(x)n = 0, n(1) = 1, n'(1) = 0$;
- (v) $\omega \equiv p(x)(m'(x)n(x) - m(x)n'(x))$ is a positive constant.

Lemma 2.2 [7] *The Green's function $G(x, y)$ defined by (2.2) has the following properties:*

- (i) $G(x, y)$ is continuous in Q ;
- (ii) $G(x, y)$ is symmetrical on Q ;
- (iii) $G(x, y)$ has continuous partial derivatives on Q_1, Q_2 ;
- (iv) For each fixed $y \in I, G(x, y)$ satisfies $LG(x, y) = 0$ for $x \neq y, x \in I$. Moreover, $G'_x(0, y) = G'_x(1, y) = 0$ for $y \in (0, 1)$.
- (v) For $x = y, G'_x$ has discontinuity point of the first kind, and

$$G'_x(y + 0, y) - G'_x(y - 0, y) = -\frac{1}{p(y)}, \quad y \in (0, 1).$$

Lemma 2.3 [8] *Suppose that conditions in Lemma 2.1 hold and $h : I \rightarrow R$ is continuous. Then the problem*

$$\begin{cases} -(p(x)u')' + q(x)u = h(x), & x \in I, \\ u'(0) = 0, & u'(1) = 0, \end{cases} \quad (2.3)$$

has a unique solution, which can be written as

$$u(x) = \int_0^1 G(x, y)h(y) dy. \quad (2.4)$$

Next we state the theorem of fixed points in cones, which will be used in Section 3.

Theorem 2.1 [9] *Let X be a Banach space and $K (\subset X)$ be a cone. Assume that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let*

$$T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$$

be a continuous and compact operator such that either

- (i) $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_2$; or
- (ii) $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

In applications below, we take $X = C(I)$ with the supremum norm $\|\cdot\|$ and define

$$K = \left\{ u \in X : u(x) \geq 0 \text{ and } \min_{x \in I} u(x) \geq \sigma \|u\| \right\}. \quad (2.5)$$

One may readily verify that K is a cone in X . Now suppose that $F : I \times R \rightarrow [0, \infty)$ is continuous and define an operator $T : X \rightarrow X$ by

$$(Tu)(x) = \int_0^1 G(x, y)F(y, u(y)) dy \tag{2.6}$$

for $u \in X$ and $x \in [0, 1]$.

Lemma 2.4 *T is well defined and maps X into K. Moreover, T is continuous and completely continuous.*

3 Main results

In this section we establish the existence and multiplicity of positive solutions to (1.1). Since we are mainly interested in the attractive-superlinear nonlinearities $g(x, u)$ in the semi-positone case, we assume that the hypotheses of the following theorem are satisfied.

Theorem 3.1 *Suppose that (H₁) and (H₂) hold. Furthermore, assume the following:*

(H₃) *There exist continuous, non-negative functions f(u) and g(u) such that*

$$F(x, u) = g(x, u) + M \leq f(u) + h(u) \quad \text{for all } (x, u) \in I \times (0, \infty),$$

and f(u) > 0 is non-increasing and h(u)/f(u) is non-decreasing in u ∈ (0, ∞).

(H₄) *There exists r > $\frac{M\|\omega\|}{\sigma}$ such that $\frac{r}{f(\sigma r - M\|\omega\|)\{1 + \frac{h(r)}{f(r)}\}} > \|\omega\|$.*

(H₅) *There exists a constant A > M, ε > 0 such that*

$$F(x, u) \geq A, \quad f(u) > A \quad \text{for all } (x, u) \in I \times (0, \varepsilon].$$

Then problem (1.1) has at least one positive solution v ∈ C(I) with 0 < \|v + M\omega\| < r.

Before we present the proof of Theorem 3.1, we state and prove some facts.

First, it is easy to see that we can take $c > 0$ and $n_0 > 1$ such that

$$c\|\omega\| < \min \left\{ \varepsilon, \frac{A - M}{\|q\|} \right\}, \tag{3.1}$$

$$\frac{1}{n_0} < \min \left\{ \varepsilon, \frac{\varepsilon}{M}, c\sigma\|\omega\|, \sigma r - M\|\omega\| \right\}. \tag{3.2}$$

Lemma 3.1 *Suppose that (H₁)-(H₅) hold, then $\alpha(x) = (M + c)\omega(x)$ is a strict lower solution to the problem*

$$\begin{cases} -(p(x)u')' + q(x)u = F_n(x, u - M\omega(x)), & x \in I, n > n_0, \\ u'(0) = 0, & u'(1) = 0, \end{cases} \tag{3.3}$$

where $F_n(x, u) = F(x, \max\{u, \frac{1}{n}\})$, $(x, u) \in I \times R$.

Proof It is easy to see that $\alpha'(0) = (M + c)\omega'(0) = 0$ and $\alpha'(1) = (M + c)\omega'(1) = 0$.

Since $\alpha(x) - M\omega(x) = c\omega(x) \geq c\sigma\|\omega\| > \frac{1}{n_0} \geq \frac{1}{n}$, and using (3.1), we have $\varepsilon > \alpha(x) - M\omega(x) = c\omega(x) \geq \frac{1}{n} > 0$.

By assumption (H₅), we have

$$F_n(x, \alpha(x) - M\omega(x)) > A, \quad \forall n > n_0.$$

This implies that $\alpha(x)$ is a strict lower solution to (3.3). □

Lemma 3.2 *Suppose that (H₁)-(H₅) hold. Then the problem*

$$\begin{cases} -(p(x)u')' + q(x)u = f_n(u - M\omega(x))(1 + \frac{h(r)}{f(r)}), & x \in I, \\ u'(0) = 0, \quad u'(1) = 0, \end{cases} \tag{3.4}$$

has at least one positive solution $\beta_n(x)$ with $\|\beta_n\| < r$.

Proof The existence is proved using the Leray-Schauder alternative principle together with a truncation technique.

Since (H₄) holds, we have

$$\|\omega\|f(\sigma r - M\|\omega\|)(1 + h(r)/f(r)) < r.$$

Consider the family of problems

$$\begin{cases} -(p(x)u')' + q(x)u = \lambda f_n(u - M\omega(x))(1 + \frac{h(r)}{f(r)}), & x \in I, \\ u'(0) = 0, \quad u'(1) = 0, \end{cases} \tag{3.5}$$

where $\lambda \in I$ and $f_n(u) = f(\max\{u, 1/n\})$, $(x, u) \in I \times R$. $f_n(u)$ is non-increasing.

Problem (3.5) is equivalent to the following fixed point problem in $C[0, 1]$

$$\beta = \lambda T_n \beta, \tag{3.6}$$

where T_n is defined by

$$T_n(\beta(x)) = \int_0^1 G(x, y) f_n(\beta(y) - M\omega(y)) (1 + h(r)/f(r)) dy. \tag{3.7}$$

We claim that any fixed point β of (3.6) for any $\lambda \in [0, 1]$ must satisfy $\|\beta\| \neq r$. Otherwise, assume that β is a solution of (3.6) for some $\lambda \in [0, 1]$ such that $\|\beta\| = r$. Note that $f_n(x, u) \geq 0$. By Lemma 3.1, for all x , $\beta(x) - M\omega(x) \geq \sigma r - M\|\omega\| \geq 1/n$. Hence, for all x ,

$$\beta(x) - M\omega(x) \geq 1/n \quad \text{and} \quad \beta(x) - M\omega(x) \geq \sigma r - M\|\omega\|. \tag{3.8}$$

Then we have, for all x ,

$$\begin{aligned} \beta(x) &= \lambda \int_0^1 G(x, y) f_n(\beta(y) - M\omega(y)) \left(1 + \frac{h(r)}{f(r)}\right) dy \\ &\leq \int_0^1 G(x, y) f(\beta(y) - M\omega(y)) \left(1 + \frac{h(r)}{f(r)}\right) dy \end{aligned}$$

$$\begin{aligned} &\leq \int_0^1 G(x, y) f(\sigma r - M\|\omega\|)(1 + h(r)/f(r)) dy \\ &\leq \|\omega\| f(\sigma r - M\|\omega\|)(1 + h(r)/f(r)). \end{aligned} \tag{3.9}$$

Therefore,

$$r = \|\beta\| \leq \|\omega\| f(\sigma r - M\|\omega\|)(1 + h(r)/f(r)) < r.$$

This is a contradiction and the claim is proved. □

From this claim, the nonlinear alternative of Leray-Schauder guarantees that problem (3.6) (with $\lambda = 1$) has a fixed point, denoted by β_n , in B_r , i.e., problem (3.4) has a positive solution β_n with $\|\beta_n\| < r$. (In fact, it is easy to see that $\beta_n(x) \geq 1/n$ with $\|\beta_n\| \neq r$.)

Lemma 3.3 *Suppose that (H₁)-(H₅) hold, then $\beta_n(x)$ is an upper solution of problem (3.3).*

Proof By Lemma 3.2 we know that $\beta_n(x)$ is a solution to equation (3.4).

If $\beta_n(x) - M\omega(x) \geq \frac{1}{n}$, then

$$\begin{aligned} F_n(x, \beta_n(x) - M\omega(x)) &= F(x, \beta_n(x) - M\omega(x)) \\ &\leq f(\beta_n(x) - M\omega(x)) \left(1 + \frac{h(\beta_n(x) - M\omega(x))}{f(\beta_n(x) - M\omega(x))} \right) \\ &\leq f_n(\beta_n(x) - M\omega(x)) \left(1 + \frac{h(r)}{f(r)} \right). \end{aligned} \tag{3.10}$$

If $\beta_n(x) - M\omega(x) \leq \frac{1}{n}$, then

$$\begin{aligned} F_n(x, \beta_n(x) - M\omega(x)) &= F\left(x, \frac{1}{n}\right) \leq f\left(\frac{1}{n}\right) \left(1 + \frac{h(\frac{1}{n})}{f(\frac{1}{n})} \right) \\ &\leq f_n(\beta_n(x) - M\omega(x)) \left(1 + \frac{h(r)}{f(r)} \right). \end{aligned} \tag{3.11}$$

Since $\beta'_n(0) = \beta'_n(1) = 0$, we have

$$\begin{cases} -(p(x)\beta'_n(x))' + q(x)\beta_n(x) \geq F_n(x, \beta_n(x) - M\omega(x)), & x \in I, \\ \beta'_n(0) = 0, & \beta'_n(1) = 0. \end{cases}$$

This implies that $\beta_n(x)$ is an upper solution of problem (3.3). □

Lemma 3.4 *Suppose that (H₁)-(H₅) hold, then $\beta_n(x) \geq \alpha(x)$ ($n > n_0$).*

Proof Let $z(x) = \alpha(x) - \beta_n(x)$, we will prove $z(x) \leq 0$. If this is not true for $n > n_0$, there exists $x_0 \in [0, 1]$ such that $z(x_0) = \max z(x) > 0$, $z'(x_0) = 0$, $z''(x_0) \leq 0$. Then $(p(x_0)z'(x_0))' \leq 0$.

Since $\alpha(x_0) - M\omega(x_0) = c\omega(x_0) \geq c\sigma \|\omega\| > \frac{1}{n_0} \geq \frac{1}{n}$, $\alpha(x_0) - M\omega(x_0) \leq c\|\omega\| < \varepsilon$, and $f_n(u)$ is non-increasing, we have

$$\begin{aligned} f_n(\beta(x_0) - M\omega(x_0)) &\geq f_n(\alpha(x_0) - M\omega(x_0)) \\ &= f(\alpha(x_0) - M\omega(x_0)) \\ &> A \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} -(p(x_0)z'(x_0))' + q(x_0)z(x_0) &= M + c - f_n(\beta_n(x_0) - M\omega(x_0)) \left(1 + \frac{h(r)}{f(r)}\right) \\ &\leq M + c - f_n(\alpha(x_0) - M\omega(x_0)) \left(1 + \frac{h(r)}{f(r)}\right) \\ &\leq M + c - A \left(1 + \frac{h(r)}{f(r)}\right) < 0. \end{aligned} \tag{3.13}$$

This is a contradiction and completes the proof of Lemma 3.4. □

Proof of Theorem 3.1 To show (1.1) has a positive solution, we will show

$$\begin{cases} -(p(x)u')' + q(x)u = F(x, u(x) - M\omega(x)), & x \in I, \\ u'(0) = u'(1) = 0 \end{cases} \tag{3.14}$$

has a solution $u \in C(I)$, $u(x) > M\omega(x)$, $x \in I$.

If this is true, then $v(x) = u(x) - M\omega(x)$ is a positive solution of (1.1) since

$$\begin{aligned} -(p(x)v')' + q(x)v &= -(p(x)u'(x) - p(x)M\omega'(x))' + q(x)u(x) - Mq(x)\omega(x) \\ &= -(p(x)u'(x))' + q(x)u(x) - M \\ &= F(x, u(x) - M\omega(x)) - M \\ &= g(x, u(x) - M\omega(x)) \\ &= g(x, v(x)). \end{aligned}$$

As a result, we will only concentrate our study on (3.14).

By Lemmas 3.1-3.4 and the upper and lower solutions method, we know that (3.3) has a solution u_n with $(M + c)\omega(x) = \alpha(x) \leq u_n(x) \leq \beta_n(x) < r$. Thus we have $u_n(x) - M\omega(x) \geq c\sigma \|\omega\|$, $u_n(x) \leq \beta_n(x) < r$.

By the fact that u_n is a bounded and equi-continuous family on $[0, 1]$, the Arzela-Ascoli theorem guarantees that $\{u_n\}_{n \in \mathbb{N}_0}$ has a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$, which converges uniformly on $[0, 1]$ to a function $u \in C[0, 1]$. Then u satisfies $u(x) - M\omega(x) \geq c\sigma \|\omega\|$, $u(x) < r$ for all x . Moreover, u_{n_k} satisfies the integral equation

$$u_{n_k}(x) = \int_0^1 G(x, y)F(y, u_{n_k}(y) - M\omega(y)) dy.$$

Letting $k \rightarrow \infty$, we arrive at

$$u(x) = \int_0^1 G(x, y)F(y, u(y) - M\omega(y)) dy,$$

where the uniform continuity of $F(x, u(x) - M\omega(x))$ on $[0, 1] \times [c\sigma\|\omega\|, r]$ is used. Therefore, u is a positive solution of (3.14).

Finally, it is not difficult to show that $\|u\| < r$. Assume otherwise: note that $F(x, u) \geq 0$. By Lemma 2.4, for all x , $u(x) \geq 1/n$ and $r \geq u(x) - M\omega(x) \geq \sigma r - M\|\omega\| \geq 1/n$. Hence, for all x ,

$$u(x) - M\omega(x) \geq 1/n \quad \text{and} \quad r \geq u(x) - M\omega(x) \geq \sigma r - M\|\omega\|. \tag{3.15}$$

Then we have for all x ,

$$\begin{aligned} u(x) &= \int_0^1 G(x, y)F(y, u(y) - M\omega(y)) dy \\ &\leq \int_0^1 G(x, y)f(u(y) - M\omega(y)) \left(1 + \frac{h(u(y) - M\omega(y))}{f(u(y) - M\omega(y))}\right) dy \\ &\leq \int_0^1 G(x, y)f(\sigma r - M\|\omega\|)(1 + h(r)/f(r)) dy \\ &\leq \|\omega\|f(\sigma r - M\|\omega\|)(1 + h(r)/f(r)). \end{aligned} \tag{3.16}$$

Therefore,

$$r = \|u\| \leq \|\omega\|f(\sigma r - M\|\omega\|)(1 + h(r)/f(r)).$$

This is a contradiction and completes the proof of Theorem 3.1. □

Corollary 3.1 *Let us consider the following boundary value problem*

$$\begin{cases} -(p(x)u')' + q(x)u = \mu(u^{-\alpha} + u^\beta + k(x)), & x \in I, \\ u'(0) = u'(1) = 0, \end{cases} \tag{3.17}$$

where $\alpha > 0$, $\beta > 0$ and $k : [0, 1] \rightarrow R$ is continuous, $\mu > 0$ is chosen such that

$$\mu < \sup_{u \in (\frac{M\|\omega\|}{\sigma}, \infty)} \frac{u(\sigma u - M\|\omega\|)^\alpha}{\|\omega\|\{1 + 2Hu^\alpha + u^{\alpha+\beta}\}}, \tag{3.18}$$

here $H = \|k\|$. Then problem (3.17) has a positive solution $u \in C[0, 1]$.

Proof We will apply Theorem 3.1 with $M = \mu H$ and

$$f(u) = f_1(u) = \mu u^{-\alpha}, \quad h(u) = \mu(u^\beta + 2H), \quad h_1(u) = \mu u^\beta.$$

Clearly, (H_1) - (H_3) and (H_5) are satisfied.

Set

$$T(u) = \frac{u(\sigma u - M\|\omega\|)^\alpha}{\|\omega\|\{1 + 2Hu^\alpha + u^{\alpha+\beta}\}}, \quad u \in \left(\frac{M\|\omega\|}{\sigma}, +\infty\right).$$

Since $T(\frac{M\|\omega\|}{\sigma}) = 0$, $T(\infty) = 0$, then there exists $r \in (\frac{M\|\omega\|}{\sigma}, \infty)$ such that

$$T(r) = \sup_{u \in (\frac{M\|\omega\|}{\sigma}, \infty)} \frac{u(\sigma u - M\|\omega\|)^\alpha}{\|\omega\|\{1 + 2Hu^\alpha + u^{\alpha+\beta}\}}.$$

This implies that there exists $r \in (\frac{M\|\omega\|}{\sigma}, \infty)$ such that $\mu < \frac{r(\sigma r - M\|\omega\|)^\alpha}{\|\omega\|\{1+r^{\alpha+\beta}+2Hr^\alpha\}}$, so (H_4) is satisfied.

Since $\beta > 1$. Thus all the conditions of Theorem 3.1 are satisfied, so the existence is guaranteed. \square

Next we will find another positive solution to problem (1.1) by using Theorem 2.1.

Theorem 3.2 *Suppose that conditions (H_1) - (H_5) hold. In addition, it is assumed that the following two conditions are satisfied:*

(H_6) $F(x, u) = g(x, u) + M \geq f_1(u) + h_1(u)$ for some continuous non-negative functions $f_1(u)$ and $h_1(u)$ with the properties that $f_1(u) > 0$ is non-increasing and $h_1(u)/f_1(u)$ is non-decreasing.

(H_7) There exists $R > r$ such that $\frac{R}{\sigma f_1(R)\{1 + \frac{h_1(\sigma R - M\|\omega\|)}{f_1(\sigma R - M\|\omega\|)}\}} < \|\omega\|$.

Then, besides the solution u constructed in Theorem 3.1, problem (1.1) has another positive solution $\tilde{v} \in C[0, 1]$ with $r < \|\tilde{v} + M\omega\| \leq R$.

Proof To show (1.1) has a positive solution, we will show (3.14) has a solution $\tilde{u} \in C[0, 1]$ with $\tilde{u}(x) > M\omega(x)$ for $x \in [0, 1]$ and $r \leq \|\tilde{u}\| \leq R$.

Let $X = C[0, 1]$ and K be a cone in X defined by (2.5). Let

$$\Omega_r = \{\tilde{u} \in U : \|\tilde{u}\| < r\}, \quad \Omega_R = \{\tilde{u} \in X : \|\tilde{u}\| < R\}$$

and define the operator $T : K \cap (\bar{\Omega}_R \setminus \Omega_r) \rightarrow K$ by

$$(T\tilde{u})(x) = \int_0^1 G(x, y)F(y, \tilde{u}(y) - M\omega(y)) dy, \quad 0 \leq x \leq 1, \tag{3.19}$$

where $G(x, y)$ is as in (2.2).

For each $\tilde{u} \in K \cap (\bar{\Omega}_R \setminus \Omega_r)$, $r \leq \|\tilde{u}\| \leq R$, we have $0 < \sigma r - M\|\omega\| \leq \tilde{u}(x) - M\omega(x) \leq R$. Since $F : [0, 1] \times [\sigma r - M\|\omega\|, R] \rightarrow [0, \infty)$ is continuous, it follows from Lemma 2.4 that the operator $T : K \cap (\bar{\Omega}_R \setminus \Omega_r) \rightarrow K$ is well defined, is continuous and completely continuous.

First we show

$$\|T\tilde{u}\| < \|\tilde{u}\| \quad \text{for } \tilde{u} \in K \cap \partial\Omega_r. \tag{3.20}$$

In fact, if $\tilde{u} \in K \cap \partial\Omega_r$, then $\|\tilde{u}\| = r$ and $\tilde{u}(x) \geq \sigma r > M\|\omega\|$ for $x \in I$. So we have

$$\begin{aligned} (T\tilde{u})(x) &= \int_0^1 G(x, y)F(y, \tilde{u}(y) - M\omega(y)) \, dy \\ &\leq \int_0^1 G(x, y)f(\tilde{u}(y) - M\omega(y)) \left\{ 1 + \frac{h(\tilde{u}(y) - M\omega(y))}{f(\tilde{u}(y) - M\omega(y))} \right\} \, dy \\ &\leq \int_0^1 G(x, y)f(\sigma r - M\|\omega\|) \left\{ 1 + \frac{h(r)}{f(r)} \right\} \, dy \\ &= \omega(x)f(\sigma r - M\|\omega\|) \left\{ 1 + \frac{h(r)}{f(r)} \right\} \\ &\leq \|\omega\|f(\sigma r - M\|\omega\|) \left\{ 1 + \frac{h(r)}{f(r)} \right\} \\ &< r = \|\tilde{u}\|. \end{aligned}$$

This implies $\|T\tilde{u}\| < \|\tilde{u}\|$, i.e., (3.20) holds.

Next we show

$$\|T\tilde{u}\| > \|\tilde{u}\| \quad \text{for } \tilde{u} \in K \cap \partial\Omega_R. \tag{3.21}$$

To see this, let $\tilde{u} \in K \cap \partial\Omega_R$, then $\|\tilde{u}\| = R$ and $\tilde{u}(x) \geq \sigma R > M\|\omega\|$ for $x \in I$. As a result, it follows from (H₆) and (H₇) that, for $x \in I$,

$$\begin{aligned} (T\tilde{u})(x) &= \int_0^1 G(x, y)F(y, \tilde{u}(y) - M\omega(y)) \, dy \\ &\geq \int_0^1 G(x, y)f_1(\tilde{u}(y) - M\omega(y)) \left\{ 1 + \frac{h_1(\tilde{u}(y) - M\omega(y))}{f_1(\tilde{u}(y) - M\omega(y))} \right\} \, dy \\ &\geq \int_0^1 G(x, y)f_1(R) \left\{ 1 + \frac{h_1(\sigma R - M\|\omega\|)}{f_1(\sigma R - M\|\omega\|)} \right\} \, dy \\ &= \omega(x)f_1(R) \left\{ 1 + \frac{h_1(\sigma R - M\|\omega\|)}{f_1(\sigma R - M\|\omega\|)} \right\} \\ &\geq \sigma \|\omega\|f_1(R) \left\{ 1 + \frac{h_1(\sigma R - M\|\omega\|)}{f_1(\sigma R - M\|\omega\|)} \right\} \\ &> R = \|\tilde{u}\|. \end{aligned}$$

Now (3.20), (3.21) and Theorem 2.1 guarantee that T has a fixed point $\tilde{u} \in K \cap (\bar{\Omega}_R \setminus \Omega_r)$ with $r \leq \|\tilde{u}\| \leq R$. Clearly, this \tilde{u} is a positive solution of (3.14). This completes the proof of Theorem 3.2. \square

Let us consider again example (3.17) in Corollary 3.1 for the superlinear case, i.e., $\alpha > 0$, $\beta > 1$ and $k : [0, 1] \rightarrow R$ is continuous, $\mu > 0$ is chosen such that (3.18) holds, here $H = \|k\|$. Then problem (3.17) has a positive solution $\tilde{u} \in C[0, 1]$. Clearly, (H₁)-(H₆) are satisfied.

Since $\beta > 1$, then (H₇) is satisfied for R large enough because when $R \rightarrow \infty$,

$$\frac{R}{\sigma f_1(R) \left\{ 1 + \frac{h_1(\sigma R - M\|\omega\|)}{f_1(\sigma R - M\|\omega\|)} \right\}} = \frac{R^{\alpha+1}}{\sigma \mu (1 + (\sigma R - M\|\omega\|)^{\alpha+\beta})} \rightarrow 0.$$

Thus all the conditions of Theorem 3.2 are satisfied, so the existence is guaranteed.

Corollary 3.2 *Assume that $\alpha > 0$, $\beta > 1$ and $k : I \rightarrow R$ is continuous, $\mu > 0$ is chosen such that (3.18) holds. Take $H = \|k\|$. Then problem (3.17) has at least two different positive solutions.*

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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