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Global existence and blow-up phenomena for p-Laplacian heat equation with inhomogeneous Neumann boundary conditions

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Abstract

In this paper, we consider a p-Laplacian heat equation with inhomogeneous Neumann boundary condition. We establish respectively the conditions on the nonlinearities to guarantee that the solution $u(\boldsymbol{x},t)$ exists globally or blows up at some finite time. If blow-up occurs, we obtain upper and lower bounds of the blow-up time by differential inequalities.

MSC: Primary 35K55; secondary 35K60

Keywords: p-Laplacian heat equation; inhomogeneous; global existence; blow-up

1 Introduction

In this paper, we deal with the initial-boundary value problem

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - f(u), & \boldsymbol{x} \in \Omega, t \in (0,t^*), \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \boldsymbol{n}} = g(u), & (\boldsymbol{x},t) \in \partial\Omega \times (0,t^*), \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \geq 0, & \boldsymbol{x} \in \Omega, \end{cases} \tag{1.1}$$

where $p \geq 2$ is a real number, div denotes the scalar divergence operator, Ω is a bounded star-sharped region of R^N $(N \geq 2)$ with smooth boundary $\partial\Omega$, $\boldsymbol{n}$ is the unit outward normal on $\partial\Omega$, $\frac{\partial u}{\partial \boldsymbol{n}}$ is the outward normal derivative of u on $\partial\Omega$ and t^* is the blow-up time if blow-up occurs, or else $t^* = +\infty$.

The blow-up phenomena of solutions to various nonlinear problems, particularly for hyperbolic and parabolic systems, have received considerable attention in the recent literature. For work in this area, the reader can refer to [1–4]. Other contributions in the field can be found in [5–14] and the references cited therein. A variety of methods have been used to determine the blow-up of solutions and to indicate an upper bound for the blow-up time. To our knowledge, the first work on lower bound for t^* was shown by Weissler [5, 6], but during the past several years a number of papers deriving lower bound for t^* in various problems have appeared (see [7]).

The *homogeneous Dirichlet* problems of nonlinear parabolic equations were considered in [8–13].

The blow-up and global existence phenomena for nonlinear parabolic equations with *Neumann boundary* conditions have received considerable attention in [14–19]. Payne and Schaefer [14] considered

$$u_t = \Delta u \quad \text{in } \Omega \times \bigl(0, t^*\bigr). \tag{1.2}$$

Under suitable conditions on the nonlinearities, they determined a lower bound on the blow-up time when blow-up occurs. In addition, a sufficient condition which implies that blow-up does occur was determined. Ding and Guo [15] studied the global solution and blow-up solution of the equation

$$\bigl(h(u)\bigr)_t = \nabla \cdot \bigl(a(u,t)b(x)\nabla u\bigr) + g(t)f(u) \quad \text{in } D \times (0, T), \tag{1.3}$$

where $D \subset R^N$ is a bounded domain with smooth boundary ∂D. Under appropriate assumptions on the functions a, b, f, g and h, by constructing auxiliary functions and using maximum principles, the sufficient conditions for the existence of global solution or blow-up solution, an upper estimate of the global solution, an upper bound of the blow-up time and an upper estimate of the blow-up rate were specified. Mizoguchi [16] studied the semilinear heat equation

$$u_t = \Delta u + u^p \quad \text{in } \Omega \times (0, T) \tag{1.4}$$

and showed that if u blows up at $t = T$, then $|u(t)|_\infty \le C(T - t)^{-\frac{1}{p-1}}$ for some $C > 0$. Ishige and Yagisita [17] considered the blow-up problem for the semilinear heat equation

$$u_t = D\Delta u + u^p \quad \text{in } \Omega \times (0, T_D), \tag{1.5}$$

where Ω is a bounded smooth domain in R^N, $T_D > 0$, $D > 0$, $p > 1$, and studied the blow-up time, the location of the blow-up set, and the blow-up profile of the blow-up solution for sufficiently large D. In particular, they proved that for almost all initial data, if D is sufficiently large, then the solution blows up only near the maximum points of the orthogonal projection of the initial data from $L^2(\Omega)$ onto the second Neumann eigenspace.

In recent paper, Payne *et al.* [18] considered

$$u_t = \Delta u - f(u), \quad \boldsymbol{x} \in \Omega, t \in \bigl(0, t^*\bigr) \tag{1.6}$$

and established conditions on nonlinearities sufficient to guarantee that $u(\boldsymbol{x}, t)$ exists for all time $t > 0$ or blows up at some finite time t^*. Moreover, an upper bound for t^* was derived. Under somewhat more restrictive conditions, a lower bound for t^* was derived. Moreover, in [19], Payne *et al.* investigated

$$u_t = \nabla \cdot \bigl(|\nabla u|^{2p} \nabla u\bigr), \quad \boldsymbol{x} \in \Omega, t \in \bigl(0, t^*\bigr) \tag{1.7}$$

and showed that blow-up occurs at some finite time under certain conditions on the nonlinearities and the data, upper and lower bounds for the blow-up time were obtained when blow-up occurs. Li and Li [20] investigated

$$\begin{cases} u_t = \sum_{i,j=1}^{N}(a^{ij}(\boldsymbol{x})u_{x_i})_{x_j} - f(u), & \boldsymbol{x} \in \Omega, t \in (0,t^*), \\ \sum_{i,j=1}^{N} a^{ij}(\boldsymbol{x})u_{x_i}\boldsymbol{n}_j = g(u), & (\boldsymbol{x},t) \in \partial\Omega \times (0,t^*), \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \geq 0, & \boldsymbol{x} \in \Omega \end{cases} \tag{1.8}$$

and established respectively the conditions on nonlinearities to guarantee that $u(x,t)$ exists globally or blows up at some finite time. If blow-up occurs, we obtain upper and lower bounds of the blow-up time.

Li [21] considered the p-Laplacian heat-conduction model

$$u_t = \operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr) \tag{1.9}$$

and showed the backward uniqueness in time for solutions to Neumann and Dirichlet problems by energy methods and gave reasonable physical interpretation for the obtained conclusions.

Motivated by the above work, we intend to study the global existence and the blow-up phenomena for problem (1.1). It is well known that the data f and g may greatly affect the behavior of $u(\boldsymbol{x},t)$ with the development of time. From the physical standpoint, $-f$ is the heat source function, $|\nabla u|^{p-2}$ is the variational heat-conduction coefficient, $g(u)$ is the heat-conduction function transmitting into interior of Ω from the boundary of Ω. We can deduce that if $f < 0$ and $g > 0$, the blow-up phenomena of solution of (1.1) occur early under some conditions. Under the conditions that f and g are nonnegative functions, we can deduce that the solution of (1.1) is nonnegative and smooth. In this paper, by using differential inequalities, we establish the conditions on the nonlinearities to guarantee that $u(\boldsymbol{x},t)$ exists globally or blows up at some finite time, respectively. If blow-up occurs, we obtain the upper and lower bounds of the blow-up time. The main innovational and novel points of this paper are: (a) the model is representative, for example, the model is the equation in [8, 14, 16–18] if $p = 2$ with suitable f; (b) the problem considered in this paper is a nonlinear equation with *inhomogeneous* Neumann boundary dissipation, this problem is significant; (c) we give the reason and process of the definition of auxiliary functional; (d)since the model is general, the estimates are concise and precise.

The present work is organized as follows. In Section 2, we establish the conditions on the nonlinearities to guarantee that $u(\boldsymbol{x},t)$ exists globally. In Section 3, we show the conditions on the nonlinearities which ensure that the solution blows up at some finite time and obtain the upper bound for the blow-up time. Section 4 is devoted to showing the lower bound of the blow-up time under some assumptions.

2 The global solution

In this section, we establish the conditions on the nonlinearities to guarantee that $u(\boldsymbol{x},t)$ exists globally. We state our result as follows.

Theorem 2.1 *Let u be a classical solution of* (1.1), *assume that the nonnegative functions f and g satisfy the following conditions*:

$$f(s) \geq k_1 s^{\alpha}, \tag{2.1}$$

$$g(s) \le k_2 s^{\beta}, \tag{2.2}$$

where $k_1 > 0$, $k_2 \ge 0$, $s \ge 0$, $\alpha > \beta > 1$ *and*

$$2\beta < \alpha + 1. \tag{2.3}$$

Then $u(\boldsymbol{x}, t)$ *does not blow up, that is,* $u(\boldsymbol{x}, t)$ *exists for all time* $t > 0$.

In order to prove this theorem, we give the following lemma.

Lemma 2.1 *Let* Ω *be a bounded star-sharped region in* R^N, $N \ge 2$. *Then, for any nonnegative* C^1 *function* w *and* $r > 0$, *we have*

$$\int_{\partial\Omega} w^r\, dS \le \frac{N}{\rho_0} \int_{\Omega} w^r\, d\boldsymbol{x} + \frac{rd}{\rho_0} \int_{\Omega} w^{r-1} |\nabla w|\, d\boldsymbol{x},$$

where

$$\rho_0 := \min_{\boldsymbol{x}\in\partial\Omega} (\boldsymbol{x} \cdot \boldsymbol{n}), \qquad d := \max_{\boldsymbol{x}\in\partial\Omega} |\boldsymbol{x}|.$$

Proof The proof see [19, 20]. □

Proof of Theorem 2.1 Set

$$\Phi(t) := \int_{\Omega} u^2\, d\boldsymbol{x}. \tag{2.4}$$

Differentiating (2.4) and using (2.1), we obtain

$$\begin{aligned} \Phi'(t) &= 2 \int_{\Omega} u u_t\, d\boldsymbol{x} \\ &= 2 \int_{\Omega} u\big[\operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big) - f(u)\big]\, d\boldsymbol{x} \\ &\le 2 \int_{\Omega} u\big[\operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big)\big]\, d\boldsymbol{x} - 2k_1 \int_{\Omega} u^{\alpha+1}\, d\boldsymbol{x}. \end{aligned} \tag{2.5}$$

By the divergence theorem and (2.2), we obtain

$$\begin{aligned} \int_{\Omega} u\big[\operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big)\big]\, d\boldsymbol{x} &= \int_{\partial\Omega} u |\nabla u|^{p-2} \frac{\partial u}{\partial \boldsymbol{n}}\, dS - \int_{\Omega} |\nabla u|^p\, d\boldsymbol{x} \\ &\le k_2 \int_{\partial\Omega} u^{\beta+1}\, dS - \int_{\Omega} |\nabla u|^p\, d\boldsymbol{x}. \end{aligned} \tag{2.6}$$

Applying Lemma 2.1, we have

$$\int_{\partial\Omega} u^{\beta+1}\, dS \le \frac{N}{\rho_0} \int_{\Omega} u^{\beta+1}\, d\boldsymbol{x} + \frac{(\beta+1)d}{\rho_0} \int_{\Omega} u^{\beta} |\nabla u|\, d\boldsymbol{x}. \tag{2.7}$$

From (2.5)-(2.7), we get

$$\begin{aligned}\Phi'(t) \le{}& \frac{2Nk_2}{\rho_0}\int_\Omega u^{\beta+1}\,d\boldsymbol{x} + \frac{2(\beta+1)k_2 d}{\rho_0}\int_\Omega u^\beta|\nabla u|\,d\boldsymbol{x} \\ &- 2\int_\Omega |\nabla u|^p\,d\boldsymbol{x} - 2k_1\int_\Omega u^{\alpha+1}\,d\boldsymbol{x}. \end{aligned} \tag{2.8}$$

Clearly, for all $\rho > 0$,

$$\int_\Omega u^\beta|\nabla u|\,d\boldsymbol{x} \le \frac{\rho}{2}\int_\Omega u^{2\beta}\,d\boldsymbol{x} + \frac{1}{2\rho}\int_\Omega |\nabla u|^2\,d\boldsymbol{x}. \tag{2.9}$$

Applying Hölder's inequality and Young's inequality, we get

$$\int_\Omega |\nabla u|^2\,d\boldsymbol{x} \le \left(\int_\Omega |\nabla u|^p\,d\boldsymbol{x}\right)^{\frac{2}{p}}|\Omega|^{\frac{p-2}{p}} \le \frac{2}{p}\int_\Omega |\nabla u|^p\,d\boldsymbol{x} + \frac{p-2}{p}|\Omega|, \tag{2.10}$$

where $|\Omega|$ denotes the measure of Ω. Combining (2.9), (2.10) with (2.8), we have

$$\begin{aligned}\Phi'(t) \le{}& \frac{2Nk_2}{\rho_0}\int_\Omega u^{\beta+1}\,d\boldsymbol{x} + \frac{\rho(\beta+1)k_2 d}{\rho_0}\int_\Omega u^{2\beta}\,d\boldsymbol{x} + \left[\frac{2k_2(\beta+1)d}{\rho\rho_0 p} - 2\right]\int_\Omega |\nabla u|^p\,d\boldsymbol{x} \\ &+ \frac{k_2(\beta+1)(p-2)}{\rho\rho_0 p}|\Omega| - 2k_1\int_\Omega u^{\alpha+1}\,d\boldsymbol{x}. \end{aligned} \tag{2.11}$$

Choose $\rho = \frac{k_2(\beta+1)d}{\rho_0 p}$. Equation (2.11) implies

$$\Phi'(t) \le \frac{2Nk_2}{\rho_0}\int_\Omega u^{\beta+1}\,d\boldsymbol{x} + p\rho^2\int_\Omega u^{2\beta}\,d\boldsymbol{x} + (p-2)|\Omega| - 2k_1\int_\Omega u^{\alpha+1}\,d\boldsymbol{x}. \tag{2.12}$$

Using Hölder's inequality and Young's inequality, we get

$$\begin{aligned}\int_\Omega u^{2\beta}\,d\boldsymbol{x} &= \int_\Omega u^{\frac{(\beta-1)(\alpha+1)}{\alpha-\beta} + \frac{(\alpha+1-2\beta)(\beta+1)}{\alpha-\beta}}\,d\boldsymbol{x} \\ &\le \left(\int_\Omega u^{\beta+1}\,d\boldsymbol{x}\right)^\gamma \left(\int_\Omega u^{\alpha+1}\,d\boldsymbol{x}\right)^{1-\gamma} \\ &= \left(\varepsilon\int_\Omega u^{\alpha+1}\,d\boldsymbol{x}\right)^{1-\gamma}\left(\varepsilon^{\frac{\gamma-1}{\gamma}}\int_\Omega u^{\beta+1}\,d\boldsymbol{x}\right)^\gamma \\ &\le (1-\gamma)\varepsilon\int_\Omega u^{\alpha+1}\,d\boldsymbol{x} + \gamma\varepsilon^{\frac{\gamma-1}{\gamma}}\int_\Omega u^{\beta+1}\,d\boldsymbol{x}, \end{aligned} \tag{2.13}$$

where $\gamma = \frac{\alpha+1-2\beta}{\alpha-\beta}$. We note that $\gamma \in (0,1)$ in view of (2.3) and $\alpha > \beta > 1$.

Inserting (2.13) into (2.12), we obtain

$$\Phi'(t) \le M_1\int_\Omega u^{\beta+1}\,d\boldsymbol{x} - M_2\int_\Omega u^{\alpha+1}\,d\boldsymbol{x} + (p-2)|\Omega|, \tag{2.14}$$

with $M_1 = \frac{2Nk_2}{\rho_0} + p\rho^2\gamma\varepsilon^{\frac{\gamma-1}{\gamma}} > 0$, $M_2 = 2k_1 - p\rho^2(1-\gamma)\varepsilon > 0$ for $\varepsilon > 0$ small enough.

Application of Hölder's inequality leads to

$$\int_\Omega u^{\beta+1}\,d\boldsymbol{x} \le \left(\int_\Omega u^{\alpha+1}\,d\boldsymbol{x}\right)^{\frac{\beta+1}{\alpha+1}} |\Omega|^{\frac{\alpha-\beta}{\alpha+1}}. \tag{2.15}$$

Combining (2.14) with (2.15), we obtain

$$\Phi'(t) \le M_1\left(\int_\Omega u^{\alpha+1}\,d\boldsymbol{x}\right)^{\frac{\beta+1}{\alpha+1}} \left\{|\Omega|^{\frac{\alpha-\beta}{\alpha+1}} - M_2\left(\int_\Omega u^{\alpha+1}\,d\boldsymbol{x}\right)^{\frac{\alpha-\beta}{\alpha+1}}\right\} + (p-2)|\Omega|. \tag{2.16}$$

Using Hölder's inequality, we have

$$\Phi(t) = \int_\Omega u^2\,d\boldsymbol{x} \le \left(\int_\Omega u^{\alpha+1}\,d\boldsymbol{x}\right)^{\frac{2}{\alpha+1}} |\Omega|^{\frac{\alpha-1}{\alpha+1}}. \tag{2.17}$$

Combining (2.17) with (2.16), we obtain

$$\Phi'(t) \le M_1\left(\int_\Omega u^{\alpha+1}\,d\boldsymbol{x}\right)^{\frac{\beta+1}{\alpha+1}} \left\{|\Omega|^{\frac{\alpha-\beta}{\alpha+1}} - M_2|\Omega|^{\frac{(\alpha-\beta)(1-\alpha)}{2(\alpha+1)}}\Phi^{\frac{\alpha-\beta}{2}}\right\} + (p-2)|\Omega|. \tag{2.18}$$

From (2.18), we can conclude that $\Phi(t)$ remains bounded for all time under the condition in Theorem 2.1. In fact, if $u(\boldsymbol{x},t)$ blows up at finite time t^*, then $\Phi(t)$ is unbounded near t^*, which forces $\Phi'(t) \le 0$ in some interval $[t_0, t^*)$. So we have $\Phi(t) \le \Phi(t_0)$ in $[t_0, t^*)$, which implies that $\Phi(t)$ is bounded in $[t_0, t^*)$, this is a contradiction.

The proof of Theorem 2.1 is completed. □

3 Blow-up and upper bound estimation of t^*

In this section, we do not need Ω to be star-sharped. We establish the conditions to ensure that the solution of (1.1) blows up at finite time t^* and derive an upper bound for the blow-up time t^*. Now we state the result as follows.

Theorem 3.1 *Let $u(\boldsymbol{x},t)$ be the nonnegative solution of problem* (1.1), *and assume that the nonnegative integrable functions f and g satisfy the following conditions*:

$$\xi g(\xi) \ge pG(\xi), \quad \xi \ge 0,$$
$$\xi f(\xi) \le 2(1+\theta)F(\xi), \quad \xi \ge 0,$$

where

$$F(\xi) = \int_0^\xi f(s)\,ds, \qquad G(\xi) = \int_0^\xi g(s)\,ds,$$

and θ satisfies

$$0 \le \theta \le \frac{p-2}{2}.$$

Moreover, we assume $H(0) > 0$ *with*

$$H(t) = p\int_{\partial\Omega} G(u)\,dS - \int_\Omega |\nabla u|^p\,d\boldsymbol{x} - p\int_\Omega F(u)\,d\boldsymbol{x}.$$

Then $u(\boldsymbol{x},t)$ *blows up at time* $t^* < T$, *with*

$$T = \frac{\Phi(0)}{(p-2)H(0)} \quad (p > 2),$$

where $\Phi(t) = \int_\Omega u^2\,d\boldsymbol{x}$ *is defined as* (2.4). *When* $p = 2$ $(\theta = 0)$, *we have* $t^* = \infty$.

Proof Using the divergence theorem and the assumptions on f, g, we obtain

$$\begin{aligned}
\Phi'(t) &= 2\int_\Omega uu_t\,d\boldsymbol{x} \\
&= 2\int_\Omega u\bigl[\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr) - f(u)\bigr]\,d\boldsymbol{x} \\
&= 2\int_{\partial\Omega} ug(u)\,dS - 2\int_\Omega |\nabla u|^p\,d\boldsymbol{x} - 2\int_\Omega uf(u)\,d\boldsymbol{x} \\
&\ge 2\int_{\partial\Omega} pG(u)\,dS - 2\int_\Omega |\nabla u|^p\,d\boldsymbol{x} - 2\int_\Omega 2(1+\theta)F(u)\,d\boldsymbol{x} \\
&\ge 2\biggl\{p\int_{\partial\Omega} G(u)\,dS - \int_\Omega |\nabla u|^p\,d\boldsymbol{x} - p\int_\Omega F(u)\,d\boldsymbol{x}\biggr\} \\
&= 2H(t),
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
H'(t) &= p\int_{\partial\Omega} g(u)u_t\,dS - p\int_\Omega |\nabla u|^{p-2}\nabla u\nabla u_t\,d\boldsymbol{x} - p\int_\Omega f(u)u_t\,d\boldsymbol{x} \\
&= p\int_{\partial\Omega} g(u)u_t\,dS - p\int_{\partial\Omega} u_t|\nabla u|^{p-2}\frac{\partial u}{\partial \boldsymbol{n}}\,dS \\
&\quad + p\int_\Omega \operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr)u_t\,d\boldsymbol{x} - p\int_\Omega f(u)u_t\,d\boldsymbol{x} \\
&= p\int_\Omega |u_t|^2\,d\boldsymbol{x},
\end{aligned} \tag{3.2}$$

which with $H(0) > 0$ imply $H(t) > 0$ for all $t \in (0, t^*)$.

Using (3.1), (3.2) and Hölder's inequality, we have

$$H(t)\Phi'(t) \le \frac{(\Phi'(t))^2}{2} = 2\biggl(\int_\Omega uu_t\,d\boldsymbol{x}\biggr)^2 \le 2\int_\Omega u^2\,d\boldsymbol{x}\int_\Omega u_t^2\,d\boldsymbol{x} = \frac{2H'(t)\Phi(t)}{p}. \tag{3.3}$$

Multiplying (3.3) by $\Phi^{-\frac{p}{2}-1}$, we deduce

$$\bigl(H(t)\Phi^{\frac{-p}{2}}(t)\bigr)' \ge 0. \tag{3.4}$$

Integrating (3.4) over $[0,t]$ implies

$$H(t)\Phi^{\frac{-p}{2}}(t) \geq H(0)\Phi^{\frac{-p}{2}}(0) =: M > 0. \tag{3.5}$$

Combining (3.1) with (3.5), we obtain

$$\Phi'(t)\Phi^{\frac{-p}{2}}(t) \geq 2H(0)\Phi^{\frac{-p}{2}}(0) \geq 2M. \tag{3.6}$$

If $p > 2$, (3.6) can be written as

$$\left(\Phi^{1-\frac{p}{2}}(t)\right)' \leq 2M\left(1 - \frac{p}{2}\right). \tag{3.7}$$

Integrating (3.7) over $[0,t]$, we obtain

$$\Phi^{1-\frac{p}{2}}(t) \leq 2M\left(1 - \frac{p}{2}\right)t + \Phi^{1-\frac{p}{2}}(0),$$

that is,

$$\Phi^{\frac{p-2}{2}}(t) \geq \frac{1}{M(2-p)t + \Phi^{\frac{2-p}{2}}(0)},$$

which with (3.5) implies $\Phi(t) \to +\infty$ as $t \to T = \frac{\Phi(0)}{(p-2)H(0)}$. Therefore, for $p > 2$,

$$t^* \leq T = \frac{\Phi(0)}{(p-2)H(0)}.$$

If $p = 2$, we have $\theta = 0$ (by $0 \leq \theta \leq \frac{p-2}{2}$). Furthermore, by (3.6) we conclude that $\Phi(t) \geq \Phi(0)e^{2Mt}$ and $\Phi(t)$ is increasing for all $t > 0$. So $t^* = +\infty$.

The proof of Theorem 3.1 is completed. □

4 Lower bound estimation of t^*

In this section, under the assumptions that $\Omega \subset R^3$ is a bounded star-sharped domain and convex in two orthogonal directions, we establish a lower bound for the blow-up time t^*. Now we state the result as follows.

Theorem 4.1 *Let $u(\boldsymbol{x},t)$ be the nonnegative solution of problem* (1.1), *and $u(\boldsymbol{x},t)$ blow up at time t^*; moreover, the nonnegative functions f and g satisfy the following conditions*:

$$f(s) \geq k_1 s^{\alpha}, \quad s \geq 0, \qquad g(s) \leq k_2 s^{2\beta}, \quad s \geq 0,$$

for some constants $k_i > 0$ and $\beta > \frac{1}{2}$. Define

$$\Phi(t) := \int_{\Omega} u^{\sigma(2\beta-1)}\, d\boldsymbol{x} \quad \textit{with } \sigma \geq \max\left\{4, \frac{2}{2\beta-1}\right\}.$$

Then $\Phi(t)$ satisfies the inequality

$$\Phi'(t) \leq \sum_{i=1}^{4} \gamma_i \Phi^{\theta_i} := \Psi(\Phi),$$

where γ_i, θ_i are computable (nonnegative) constants. It follows that t^* is bounded below by

$$t^* \geq \int_{\Phi(0)}^{\infty} \frac{d\eta}{\Psi(\eta)}.$$

In order to prove Theorem 4.1, we list the following lemmas.

Lemma 4.1 *Let $\Omega \subset R^3$ be a bounded star-sharped domain and convex in two orthogonal directions. Then, for any nonnegative C^1 function w and $\sigma \geq 1$, we have*

$$\int_\Omega w^{\frac{3\sigma}{2}}\, d\boldsymbol{x} \leq \left\{ \frac{3}{2\rho_0} \int_\Omega w^\sigma\, d\boldsymbol{x} + \frac{\sigma}{2}\left(1 + \frac{d}{\rho_0}\right) \int_\Omega w^{\sigma-1} |\nabla w|\, d\boldsymbol{x} \right\}^{\frac{3}{2}},$$

where

$$\rho_0 := \min_{\boldsymbol{x}\in\partial\Omega} (\boldsymbol{x}\cdot\boldsymbol{n}), \qquad d := \max_{\boldsymbol{x}\in\partial\Omega} |\boldsymbol{x}|.$$

Proof The proof can be found in [19]. □

Lemma 4.2 *For all $a \geq 0$ and $b \geq 0$, we have*

$$(a+b)^{\frac{3}{2}} \leq \sqrt{2}\big(a^{\frac{3}{2}} + b^{\frac{3}{2}}\big).$$

Proof Let $f(x) = x^{\frac{3}{2}}$. Since $f''(x) = \frac{3}{4\sqrt{x}} > 0$ for all $x > 0$, we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}.$$

This completes the proof. □

Proof of Theorem 4.1 Differentiating $\Phi(t) := \int_\Omega u^{\sigma(2\beta-1)}\, d\boldsymbol{x}$, we obtain

$$\begin{aligned}
\Phi'(t) &= \sigma(2\beta-1) \int_\Omega u^{\sigma(2\beta-1)-1} u_t\, d\boldsymbol{x} \\
&= \sigma(2\beta-1) \int_\Omega u^{\sigma(2\beta-1)-1} \big[\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) - f(u)\big]\, d\boldsymbol{x} \\
&= \sigma(2\beta-1) \int_{\partial\Omega} u^{\sigma(2\beta-1)-1} g(u)\, dS \\
&\quad - \sigma(2\beta-1)\big[\sigma(2\beta-1)-1\big] \int_\Omega u^{\sigma(2\beta-1)-2} |\nabla u|^p\, d\boldsymbol{x} \\
&\quad - \sigma k_1(2\beta-1) \int_\Omega u^{\sigma(2\beta-1)+\alpha-1}\, d\boldsymbol{x}.
\end{aligned} \tag{4.1}$$

By Lemma 2.1, we have

$$\begin{aligned}
&\int_{\partial\Omega} u^{\sigma(2\beta-1)-1} g(u)\,dS \\
&\quad\le k_2 \int_{\partial\Omega} u^{(\sigma+1)(2\beta-1)}\,dS \\
&\quad\le \frac{3k_2}{\rho_0}\int_\Omega u^{(\sigma+1)(2\beta-1)}\,d\boldsymbol{x} + \frac{(\sigma+1)(2\beta-1)k_2 d}{\rho_0}\int_\Omega u^{(\sigma+1)(2\beta-1)-1}|\nabla u|\,d\boldsymbol{x}.
\end{aligned} \tag{4.2}$$

Combining (4.1) with (4.2), we obtain

$$\begin{aligned}
\Phi'(t) \le{}& \frac{\sigma(\sigma+1)(2\beta-1)^2 k_2 d}{\rho_0} I_1 + \frac{3\sigma k_2(2\beta-1)}{\rho_0} I_2 \\
&- \sigma(2\beta-1)\big[\sigma(2\beta-1)-1\big] I - \sigma k_1(2\beta-1)\int_\Omega u^{\sigma(2\beta-1)+\alpha-1}\,d\boldsymbol{x},
\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
I_1 &= \int_\Omega u^{(\sigma+1)(2\beta-1)-1}|\nabla u|\,d\boldsymbol{x}, \\
I_2 &= \int_\Omega u^{(\sigma+1)(2\beta-1)}\,d\boldsymbol{x}, \\
I &= \int_\Omega u^{\sigma(2\beta-1)-2}|\nabla u|^r\,d\boldsymbol{x}.
\end{aligned}$$

Firstly, we give the estimation of I_2. Application of Lemma 4.1 leads to

$$\begin{aligned}
I_2 &= \int_\Omega u^{(\sigma+1)(2\beta-1)}\,d\boldsymbol{x} \\
&\le \left\{ \frac{3}{2\rho_0}\int_\Omega u^{\frac{2}{3}(\sigma+1)(2\beta-1)}\,d\boldsymbol{x} + \frac{(\sigma+1)(2\beta-1)}{3}\left(1+\frac{d}{\rho_0}\right)\int_\Omega u^{\frac{2(\sigma+1)(2\beta-1)}{3}-1}|\nabla u|\,d\boldsymbol{x} \right\}^{\frac{3}{2}} \\
&\le \left\{ \frac{3}{\rho_0}\Phi^{\frac{2(\sigma+1)}{3\sigma}}|\Omega|^{1-\frac{2(\sigma+1)}{3\sigma}} + \frac{(\sigma+1)(2\beta-1)}{3}\left(1+\frac{d}{\rho_0}\right)\int_\Omega u^{\frac{2(\sigma+1)(2\beta-1)}{3}-1}|\nabla u|\,d\boldsymbol{x} \right\}^{\frac{3}{2}}.
\end{aligned} \tag{4.4}$$

Using Hölder's inequality, we have

$$\begin{aligned}
\int_\Omega u^{\frac{2(\sigma+1)(2\beta-1)}{3}-1}|\nabla u|\,d\boldsymbol{x} &= \int_\Omega u^{\frac{2(\sigma+1)(2\beta-1)}{3}-1-\frac{\sigma(2\beta-1)-2}{p}}\left\{u^{\frac{\sigma(2\beta-1)-2}{p}}|\nabla u|\right\}d\boldsymbol{x} \\
&\le \left\{\int_\Omega u^{\frac{2(\sigma+1)(2\beta-1)(1-\delta_1)}{3}}\,d\boldsymbol{x}\right\}^{\frac{p-1}{p}} I^{\frac{1}{p}},
\end{aligned} \tag{4.5}$$

with

$$\delta_1 = \frac{(\sigma-2)(2\beta-1)+3p-6}{2(\sigma+1)(2\beta-1)(p-1)} < 1 \quad \left(\text{by } \sigma \ge \max\left\{4, \frac{2}{2\beta-1}\right\}\right).$$

By Hölder's inequality, we have

$$\int_\Omega u^{\frac{2(\sigma+1)(2\beta-1)(1-\delta_1)}{3}}\,d\boldsymbol{x} \le \Phi^{\frac{2(\sigma+1)(1-\delta_1)}{3\sigma}}|\Omega|^{1-\frac{2(\sigma+1)(1-\delta_1)}{3\sigma}}. \tag{4.6}$$

Inserting (4.5) and (4.6) into (4.4) and using Lemma 4.2, we obtain

$$I_2 \le \left\{c_1\Phi^{\frac{2(\sigma+1)}{3\sigma}} + c_2\Phi^{\frac{2(\sigma+1)(1-\delta_1)(p-1)}{3\sigma p}} I^{\frac{1}{p}}\right\}^{\frac{3}{2}} \le c_1\Phi^{\frac{(\sigma+1)}{\sigma}} + c_2\Phi^{\frac{(\sigma+1)(1-\delta_1)(p-1)}{\sigma p}} I^{\frac{3}{2p}} \tag{4.7}$$

for some positive constants c_1, c_2.

Secondly, we estimate I_1. Using Hölder's inequality, we have

$$\begin{aligned} I_1 &= \int_\Omega u^{(\sigma+1)(2\beta-1)-1}|\nabla u|\,d\boldsymbol{x} = \int_\Omega u^{(\sigma+1)(2\beta-1)-1-\frac{\sigma(2\beta-1)-2}{p}} u^{\frac{\sigma(2\beta-1)-2}{p}}|\nabla u|\,d\boldsymbol{x} \\ &\le \left\{\int_\Omega u^{(\sigma+2)(2\beta-1)(1-\delta_2)}\,d\boldsymbol{x}\right\}^{\frac{p-1}{p}} I^{\frac{1}{p}}, \end{aligned} \tag{4.8}$$

with

$$\delta_2 = \frac{2\beta(p-2)}{(\sigma+2)(2\beta-1)(p-1)} < 1 \quad \left(\text{by } \sigma \ge \max\left\{4, \frac{2}{2\beta-1}\right\}\right).$$

Application of Hölder's inequality leads to

$$\int_\Omega u^{(\sigma+2)(2\beta-1)(1-\delta_2)}\,d\boldsymbol{x} \le \left\{\int_\Omega u^{(\sigma+2)(2\beta-1)}\,d\boldsymbol{x}\right\}^{1-\delta_2}|\Omega|^{\delta_2}. \tag{4.9}$$

By Hölder's inequality and Lemma 4.1, we get

$$\begin{aligned} &\int_\Omega u^{(\sigma+2)(2\beta-1)}\,d\boldsymbol{x} \\ &\quad\le \left\{\frac{3}{2\rho_0}\int_\Omega u^{\frac{2(\sigma+2)(2\beta-1)}{3}}\,d\boldsymbol{x} + \frac{(\sigma+2)(2\beta-1)}{3}\left(1+\frac{d}{\rho_0}\right)\int_\Omega u^{\frac{2(\sigma+2)(2\beta-1)}{3}-1}|\nabla u|\,d\boldsymbol{x}\right\}^{\frac{3}{2}} \\ &\quad\le \left\{\frac{3}{\rho_0}\Phi^{\frac{2(\sigma+2)}{3\sigma}}|\Omega|^{1-\frac{2(\sigma+2)}{3\sigma}} + \frac{(\sigma+2)(2\beta-1)}{3}\left(1+\frac{d}{\rho_0}\right)\int_\Omega u^{\frac{2(\sigma+2)(2\beta-1)}{3}-1}|\nabla u|\,d\boldsymbol{x}\right\}^{\frac{3}{2}}. \end{aligned} \tag{4.10}$$

Using Hölder's inequality, we obtain

$$\begin{aligned} \int_\Omega u^{\frac{2(\sigma+2)(2\beta-1)}{3}-1}|\nabla u|\,d\boldsymbol{x} &= \int_\Omega u^{\frac{2(\sigma+2)(2\beta-1)}{3}-1-\frac{\sigma(2\beta-1)-2}{p}} u^{\frac{\sigma(2\beta-1)-2}{p}}|\nabla u|\,d\boldsymbol{x} \\ &\le \left\{\int_\Omega u^{\frac{2(\sigma+2)(2\beta-1)(1-\delta_3)}{3}}\,d\boldsymbol{x}\right\}^{\frac{p-1}{p}} I^{\frac{1}{p}}, \end{aligned} \tag{4.11}$$

with

$$\delta_3 = \frac{(\sigma-4)(2\beta-1)+3p-6}{2(\sigma+1)(2\beta-1)(p-1)} < \delta_1 < 1,$$

and

$$\int_\Omega u^{\frac{2(\sigma+2)(2\beta-1)(1-\delta_3)}{3}}\,d\boldsymbol{x} \le \Phi^{\frac{2(\sigma+2)(1-\delta_3)}{3\sigma}}|\Omega|^{1-\frac{2(\sigma+2)(1-\delta_3)}{3\sigma}}. \tag{4.12}$$

Inserting (4.12) into (4.11), we get

$$\int_\Omega u^{\frac{2(\sigma+2)(2\beta-1)}{3}-1}|\nabla u|\,d\boldsymbol{x} \le c\Phi^{\frac{2(\sigma+2)(1-\delta_3)(p-1)}{3\sigma p}} I^{\frac{1}{p}}. \tag{4.13}$$

Inserting (4.13) into (4.10), we have

$$\int_\Omega u^{(\sigma+2)(2\beta-1)}\,d\boldsymbol{x} \le \left\{c_3\Phi^{\frac{2(\sigma+2)}{3\sigma}} + c_4\Phi^{\frac{2(\sigma+2)(1-\delta_3)(p-1)}{3\sigma p}} I^{\frac{1}{p}}\right\}^{\frac{3}{2}}. \tag{4.14}$$

Inserting (4.9) and (4.14) into (4.8), we find

$$I_1(t) \le c_3\Phi^{\frac{(\sigma+2)\lambda(1-\delta_2)}{\sigma}} I^{\frac{1}{p}} + c_4\Phi^{\frac{\lambda^2(\sigma+2)(1-\delta_2)(1-\delta_3)}{\sigma}} I^{\mu}, \tag{4.15}$$

with

$$\lambda = \frac{p-1}{p}, \qquad \mu = \frac{1}{p} + \frac{3\lambda(1-\delta_2)}{2p} < 1.$$

Combining (4.7), (4.15) with (4.3), we have

$$\begin{aligned}\Phi'(t) \le{}& k_1\Phi^{\frac{\sigma+1}{\sigma}} + \tilde{k}_2\Phi^{\frac{(\sigma+1)\lambda(1-\delta_1)}{\sigma}} I^{\frac{3}{2p}} + \tilde{k}_3\Phi^{\frac{(\sigma+2)\lambda(1-\delta_2)}{\sigma}} I^{\frac{1}{p}} + \tilde{k}_4\Phi^{\frac{\lambda^2(\sigma+2)(1-\delta_2)(1-\delta_3)}{\sigma}} I^{\mu} \\ & - \sigma(2\beta-1)\big[\sigma(2\beta-1)-1\big]I - \sigma k_1(2\beta-1)\int_\Omega u^{\sigma(2\beta-1)+\alpha-1}\,d\boldsymbol{x},\end{aligned} \tag{4.16}$$

where k_i and $\tilde{k}_i$ are computable positive constants.

Using Young's inequality, we obtain

$$\begin{aligned}\Phi^{\tau_1}I^{\tau_2} &= (kI)^{\tau_2}\left(\frac{\Phi^{\frac{\tau_1}{1-\tau_2}}}{k^{\frac{\tau_2}{1-\tau_2}}}\right)^{1-\tau_2} \\ &\le \tau_2 kI + (1-\tau_2)k^{\frac{\tau_2}{\tau_2-1}}\Phi^{\frac{\tau_1}{1-\tau_2}}\end{aligned}$$

for $\tau_2 \in (0,1)$. So we have the following inequalities:

$$\tilde{k}_2\Phi^{\frac{(\sigma+1)\lambda(1-\delta_1)}{\sigma}} I^{\frac{3}{2p}} \le \alpha_1 I + \gamma_2\Phi^{\frac{2p(\sigma+1)\lambda(1-\delta_1)}{\sigma(2p-3)}}, \tag{4.17}$$

$$\tilde{k}_3\Phi^{\frac{(\sigma+2)\lambda(1-\delta_2)}{\sigma}} I^{\frac{1}{p}} \le \alpha_2 I + \gamma_3\Phi^{\frac{p(\sigma+2)\lambda(1-\delta_2)}{\sigma(p-1)}}, \tag{4.18}$$

$$\tilde{k}_4\Phi^{\frac{\lambda^2(\sigma+2)(1-\delta_2)(1-\delta_3)}{\sigma}} I^{\mu} \le \alpha_3 I + \gamma_4\Phi^{\frac{\lambda^2(\sigma+2)(1-\delta_2)(1-\delta_3)}{\sigma(1-\mu)}}. \tag{4.19}$$

Choose α_i satisfying

$$\alpha_1 + \alpha_2 + \alpha_3 - \sigma(2\beta-1)\big[\sigma(2\beta-1)-1\big] = 0. \tag{4.20}$$

From (4.16)-(4.20), we conclude

$$\begin{aligned}\Phi'(t) &\le \gamma_1 \Phi^{\frac{\sigma+1}{\sigma}} + \gamma_2 \Phi^{\frac{2p(\sigma+1)\lambda(1-\delta_1)}{\sigma(2p-3)}} + \gamma_3 \Phi^{\frac{p(\sigma+2)\lambda(1-\delta_2)}{\sigma(p-1)}} + \gamma_4 \Phi^{\frac{\lambda^2(\sigma+2)(1-\delta_2)(1-\delta_3)}{\sigma(1-\mu)}} \\ &=: \gamma_1 \Phi^{\theta_1} + \gamma_2 \Phi^{\theta_2} + \gamma_3 \Phi^{\theta_3} + \gamma_4 \Phi^{\theta_4} =: \Psi(\Phi),\end{aligned}$$

that is,

$$\left(\int_{\Phi(0)}^{\Phi(t)} \frac{d\eta}{\Psi(\eta)}\right)' = \frac{\Phi'(t)}{\Psi(\Phi)} \le 1. \tag{4.21}$$

Integrating (4.21) over $[0, t]$, we get

$$\int_{\Phi(0)}^{\Phi(t)} \frac{d\eta}{\Psi(\eta)} \le t.$$

As $u(\boldsymbol{x}, t)$ blows up, letting $\Phi(t) \to \infty$, we get the bound for t^* as follows:

$$t^* \ge \int_{\Phi(0)}^{\infty} \frac{d\eta}{\Psi(\eta)}.$$

The proof of Theorem 4.1 is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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