Feng et al. *Boundary Value Problems* 2014, **2014**:225
http://www.boundaryvalueproblems.com/content/2014/1/225

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Positive solutions to fractional boundary value problems with nonlinear boundary conditions

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Abstract

We consider the existence of at least one positive solution of the problem $-D_{0+}^{\alpha}u(t) = f(t, u(t))$, $0 < t < 1$, under the circumstances that $u(0) = 0$, $u(1) = H_1(\varphi(u)) + \int_E H_2(s, u(s))\,ds$, where $1 < \alpha < 2$, D_{0+}^{α} is the Riemann-Liouville fractional derivative, and $u(1) = H_1(\varphi(u)) + \int_E H_2(s, u(s))\,ds$ represents a nonlinear nonlocal boundary condition. By imposing some relatively mild structural conditions on f, H_1, H_2, and φ, one positive solution to the problem is ensured. Our results generalize the existing results and an example is given as well.

MSC: 34A08; 34B18

Keywords: fractional differential equation; nonlinear boundary condition; Krasnosel'skiĭ's fixed point theorem; positive solution

1 Introduction

In this paper we consider the existence of at least one positive solution of the fractional differential equation

$$-D_{0+}^{\alpha}u(t) = f\bigl(t, u(t)\bigr), \quad 0 < t < 1, \tag{1}$$

subject to the boundary conditions

$$u(0) = 0, \qquad u(1) = H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s, u(s)\bigr)\,ds, \tag{2}$$

here $E \Subset (0,1)$ is some measurable set, $1 < \alpha < 2$, D_{0+}^{α} is the Riemann-Liouville fractional derivative and φ is a linear functional having the form

$$\varphi(u) := \int_0^1 u(t)\,d\theta(t), \tag{3}$$

where the integral appearing in (3) is taken in the Lebesgue-Stieltjes sense, θ is a function of bounded variation.

Let us review briefly some recent results on such problems in order to see our problem (1)-(2) in a more appropriate context.

So far, in view of their various applications in science and engineering, such as fluid mechanics, control system, viscoelasticity, porous media, edge detection, optical systems, electromagnetism and so forth, see [1–15], fractional differential equations have attracted great attention of mathematicians.

There are a great number of works on the existence of solutions of various classes of ordinary differential equations and fractional differential equations; readers may refer to [16–32].

Some of them discussed two-point boundary value problems. For example, Bai and Lü [6] studied the following two-point boundary value problem of fractional differential equations:

$$\begin{aligned} &D_{0+}^{\alpha}u(t)+f\big(t,u(t)\big)=0, \quad 0<t<1, \\ &u(0)=u(1)=0, \end{aligned} \tag{4}$$

where $1<\alpha\leq 2$, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. By means of Guo-Krasnosel'skiĭ's fixed point theorem and the Leggett-Williams fixed point theorem they obtained the existence of positive solutions.

Some authors discussed multi-point boundary value problems, for instance, by using fixed point index theory, the Krein-Rutman theorem and some other methods, Jiang [7] studied the eigenvalue interval of the multi-point boundary value problem

$$\begin{aligned} &D^{\alpha}u(t)-Mu(t)=\lambda f\big(t,u(t)\big), \quad t\in[0,1], \\ &u(0)=\sum_{i=1}^{n}\beta_i u(\xi_i), \end{aligned} \tag{5}$$

where $0<\alpha<1$, D^{α} is the Caputo derivative, $M\geq 0$, $0<\xi_1<\xi_2<\cdots<\xi_n\leq 1$.

There are also results on fractional boundary value problem with integral boundary conditions, let us refer to Vong [8]. He investigated positive solutions of the nonlocal boundary value problem for a class of singular fractional differential equations with an integral boundary condition,

$$\begin{aligned} &{}^{c}D_{0^+}^{\alpha}u(t)+f\big(t,u(t)\big)=0, \\ &u'(0)=\cdots=u^{(n-1)}(0)=0, \qquad u(1)=\int_0^1 u(s)\,d\mu(s), \end{aligned} \tag{6}$$

where $n\geq 2$, $\alpha\in(n-1,n)$ and μ is a function of bounded variation.

To proceed, Goodrich [9] considered the existence of at least one positive solution of the ordinary differential equation

$$\begin{aligned} &-y''(t)=f\big(t,y(t)\big), \quad 0<t<1, \\ &y(0)=H_1\big(\varphi(y)\big)+\int_E H_2\big(s,y(s)\big)\,ds, \qquad y(1)=0, \end{aligned} \tag{7}$$

in which the boundary condition is more general.

Motivated by the above works, we decided to consider the problem (1)-(2). As to the novel contributions of this work, we hold in the first place that the problem discussed in [9] is an ordinary differential equation, while we take a look into the fractional differential equation under the same boundary conditions. Secondly, the boundary conditions are more flexible and general than often. Let us take the condition $u(0) = H_1(\varphi(u)) + \int_E H_2(s,u(s))\,ds$ into consideration, where H_1, H_2, $\varphi(u)$ are defined in the sequel. If $H_1(\varphi(u)) + \int_E H_2(s,u(s))\,ds = 0$, the conditions are the standard Dirichlet boundary conditions. Readers might refer to Bai and Lü [6]. As far as we are concerned, $\varphi(u)$ varies among many sorts of functionals. If $H_1(\varphi(u)) = \int_F u(t)\,dt$ (where $F \subset E \Subset (0,1)$ is defined in the sequel) or $H_1(\varphi(u)) = \int_{[0,1]} u(t)\,d\theta(t)$, our conditions reduce to integral boundary conditions, while if $H_1(\varphi(u)) = \sum_{i=1}^{n} |a_i|u(\xi_i)$, we have multi-point boundary conditions. Thirdly, compared to Goodrich [9], we make an adjustment to the Green function and define a function $r(\cdot)$ instead of a constant which affects the defined cone.

This paper is organized as follows. In Section 2, we review some preliminaries and lemmas. In Section 3, a theorem and five corollaries about the existence of at least one positive solution of problem (1)-(2) are obtained. Lastly, we give an example to illustrate the obtained theorem.

2 Preliminaries and lemmas

For the convenience of the readers, we give some background materials from fractional calculus theory to facilitate the analysis of the boundary value problem (1)-(2).

Definition 2.1 ([2]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds,$$

provided the right side is pointwise defined on $(0,+\infty)$.

Definition 2.2 ([2]) The Riemann-Liouville typed fractional derivative of order α ($\alpha > 0$) of a continuous function $f : (0,+\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha+1} f(s)\,ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0,+\infty)$.

Lemma 2.1 ([2]) *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0^+}^{\alpha} u(t) = 0$$

has $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, as a unique solution, where n is the smallest integer greater than or equal to α.

Lemma 2.2 ([2]) *Let $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order α ($\alpha > 0$) that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad \textit{for some } c_i \in \mathbb{R}, i = 1, 2, \ldots, n,$$

where n is the smallest integer greater than or equal to α.

Remark 2.1 ([2]) The Riemann-Liouville type fractional derivative and integral of order α ($\alpha > 0$) have the following properties:

$$D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t), \qquad I_{0^+}^{\alpha} I_{0^+}^{\beta} u(t) = I_{0^+}^{\alpha+\beta} u(t), \quad \alpha, \beta > 0, u \in L(0,1).$$

Lemma 2.3 *Let $u \in C[0,1]$ and $1 < \alpha < 2$. Then the fractional differential equation boundary value problem*

$$\begin{aligned} &D_{0^+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ &u(0) = 0, \qquad u(1) = H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s, u(s)\bigr)\, ds, \end{aligned}$$

has a unique solution,

$$u(t) = t^{\alpha-1}\left[H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s, u(s)\bigr)\, ds\right] + \int_0^1 G(t,s) y(s)\, ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases} \tag{8}$$

Proof We may apply Lemma 2.2 to reduce (1) to an equivalent integral equation,

$$u(t) = -I_{0+}^{\alpha} y(s) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \quad c_1, c_2 \in \mathbb{R}.$$

Consequently, the general solution of (1) is

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \quad c_1, c_2 \in \mathbb{R}.$$

By (2), we have

$$\begin{aligned} &c_1 = H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s, u(s)\bigr)\, ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds, \\ &c_2 = 0. \end{aligned}$$

Therefore, the unique solution of problem (1) and (2) is

$$\begin{aligned} u(t) &= t^{\alpha-1}\left[H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s, u(s)\bigr)\, ds\right] \\ &\quad - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + t^{\alpha-1} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds \\ &= t^{\alpha-1}\left[H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s, u(s)\bigr)\, ds\right] + \int_0^1 G(t,s) y(s)\, ds. \end{aligned}$$

The proof is complete. □

Lemma 2.4 ([6]) *Let $(a,b) \subset (0,1)$ be an arbitrary but fixed interval. Then the function $G(t,s)$ defined by* (8) *satisfies the following conditions*:

(1) *$G(t,s) > 0$, for $t,s \in (0,1)$;*

(2) *there exists a positive function $\gamma(\cdot) \in C(0,1)$ such that*

$$\min_{a\le t\le b} G(t,s) \ge \gamma(s) \max_{0\le t\le 1} G(t,s) = \gamma(s)G(s,s), \quad \text{for each } 0 \le s \le 1. \tag{9}$$

Lemma 2.5 ([4]) *Let B be a Banach space, and let $K \subset B$ be a cone. Assume Ω_1, Ω_2 are open and bounded subsets of B with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that*

(i) *$\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$; or*

(ii) *$\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.*

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

In order to get the main results, we first need some structure on H_1, H_2, φ, and f appearing in problem (1)-(2).

Let B be the Banach space on $C([0,1])$ equipped with the usual supremum norm $\|\cdot\|$. Then define the cone $K \subseteq B$ by

$$K = \Bigl\{u \in B \mid u(t) \ge 0, \min_{a\le t\le b} u(t) \ge \gamma^*\|u\|, \varphi_1(u), \varphi_2(u) \ge 0\Bigr\},$$

where $\gamma^* = \min\{\min_{t\in[a,b]} t^{\alpha-1}, \min_{s\in[a,b]} \gamma(s)\}$.

Define the operator $T : C[0,1] \to C[0,1]$ by

$$Tu(t) = t^{\alpha-1}\biggl[H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s,u(s)\bigr)\,ds\biggr] + \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds. \tag{10}$$

Here we come to the nine significant assumptions.

(H_1) Let $H_1 : [0,+\infty) \to [0,+\infty)$ and $H_2 : [0,1] \times [0,+\infty) \to [0,+\infty)$ be real-valued continuous functions.

(H_2) The functional $\varphi(u) := \int_{[0,1]} u(t)\,d\theta(t)$ can be written in the form

$$\varphi(u) = \varphi_1(u) + \varphi_2(u) := \int_{[0,1]} u(t)\,d\theta_1(t) + \int_{[0,1]} u(t)\,d\theta_2(t), \tag{11}$$

where $\theta, \theta_1, \theta_2 : [0,1] \to \mathbb{R}$ satisfy $\theta, \theta_1, \theta_2 \in BV([0,1])$, and φ_1, φ_2 are continuous linear functionals.

(H_3) There is a constant $C_1 \in [0,1)$ such that the functional φ in (11) satisfies the inequality

$$\bigl|\varphi(u)\bigr| \le C_1\|u\| \tag{12}$$

for all $u \in C([0,1])$. Furthermore, there is a constant $C_2 > 0$ such that the functional φ_2 in (11) satisfies $|\varphi_2(u)| \ge C_2\|u\|$ whenever $u \in K$.

(H_4) For each given $\varepsilon > 0$, there are $C_3 > 0$ and $M_\varepsilon > 0$ whenever $z \ge M_\varepsilon$ and we have

$$\bigl|H_1(z) - C_3 z\bigr| < \varepsilon C_3 z. \tag{13}$$

(H_5) There exists a function $F:[0,+\infty)\to[0,+\infty)$ satisfying the growth condition

$$F(u) < C_4 u \tag{14}$$

for some $C_4 \geq 0$, having the property that for each given $\varepsilon > 0$, there is $M_\varepsilon > 0$ such that

$$\left|H_2(x,u) - F(u)\right| < \varepsilon F(u) \tag{15}$$

for all $x \in [0,1]$, whenever $u \geq M_\varepsilon$.

(H_6) Assume that the nonlinearity $f(t,u)$ splits in the sense that $f(t,u) = a(t)g(u)$, for continuous functions $a:[0,1]\to[0,+\infty)$ and $g:\mathbb{R}\to[0,+\infty)$ such that a is not identically zero on any subinterval of $[0,1]$.

(H_7) Suppose $\lim_{u\to 0^+}\frac{g(u)}{u} = +\infty$.

(H_8) Suppose $\lim_{u\to +\infty}\frac{g(u)}{u} = 0$.

(H_9) For each $i = 1,2$ both

$$\int_{[0,1]} t^{\alpha-1}\,d\theta_i(t) \geq 0 \tag{16}$$

and

$$\int_{[0,1]} G(t,s)\,d\theta_i(t) \geq 0 \tag{17}$$

hold, where (17) holds for each $s \in [0,1]$.

3 Main results

In this section we state and prove the existence theorem of problem (1)-(2).

Lemma 3.1 *Let T be the operator defined in* (10). *Assume conditions* (H_1)-(H_9) *hold. Then $T:K\to K$, and the operator T is completely continuous.*

Proof Now we divide the proof into two steps; in the first step we prove that $T:K\to K$, then in the next, the conclusion that the operator T is completely continuous is treated.

Step 1. Here we are going to show that $T:K\to K$. In fact, since H_1, H_2, a, and g are all nonnegative, it is easy to find that whenever $u\in K$, it follows that $(Tu)(t)\geq 0$, for each $t\in[0,1]$, where we use the fact $\varphi(u)\geq 0$, following $H_1(\varphi(u))\geq 0$.

On the other hand, provided $u\in K$ we get

$$\begin{aligned}
\min_{t\in[a,b]} Tu(t) &\geq \min_{t\in[a,b]} t^{\alpha-1}\left[H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s,u(s)\bigr)\,ds\right] + \min_{t\in[a,b]}\int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds \\
&\geq \gamma_0\left[H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s,u(s)\bigr)\,ds\right] + \max_{t\in[0,1]}\int_0^1 \gamma(s)G(t,s)f\bigl(s,u(s)\bigr)\,ds \\
&\geq \gamma_0\left[H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s,u(s)\bigr)\,ds\right] + \max_{t\in[0,1]}\int_a^b \gamma(s)G(t,s)f\bigl(s,u(s)\bigr)\,ds \\
&\geq \gamma_0\left[H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s,u(s)\bigr)\,ds\right]
\end{aligned}$$

$$
\begin{aligned}
&+ \min_{t\in[a,b]} \gamma(t) \max_{t\in[0,1]} \int_a^b G(t,s)f\bigl(s,u(s)\bigr)\,ds \\
&\geq \gamma^* \|Tu\|, \qquad (18)
\end{aligned}
$$

where $\gamma_0 = \min_{t\in[a,b]} t^{\alpha-1}$.

Consequently, for each $i = 1,2$ we deduce that

$$
\begin{aligned}
\varphi_i(Tu) &= \int_{[0,1]} t^{\alpha-1}\biggl[H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s,u(s)\bigr)\,ds\biggr]\,d\theta_i(t) \\
&\quad + \int_{[0,1]} \biggl(\int_0^1 G(t,s)a(s)g\bigl(u(s)\bigr)\,ds\biggr)\,d\theta_i(t) \\
&= \biggl[H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s,u(s)\bigr)\,ds\biggr]\int_{[0,1]} t^{\alpha-1}\,d\theta_i(t) \\
&\quad + \int_0^1 \biggl[\int_{[0,1]} G(t,s)\,d\theta_i(t)\biggr]a(s)g\bigl(u(s)\bigr)\,ds \\
&\geq 0, \qquad (19)
\end{aligned}
$$

where the inequality follows from assumption (H_9). Thus, $\varphi_i(Tu) \geq 0$, therefore as desired we conclude $T(K) \subset K$.

Step 2. In this part we turn to the proof that $T(\Omega)$ in the sequel is bounded as well as equicontinuous with the help of the Arzelà-Ascoli theorem.

With the continuity of H_1, H_2, a, and g, it is easy to find T is continuous.

Let $\Omega \subset K$ be bounded, namely, there exists a number $M > 0$, such that for each $u \in \Omega$ we have $\|u\| \leq M$. By the continuity of H_1, H_2, and φ, we find H_1, H_2 are bounded, so there exist constants $P > 0$ and $Q > 0$ such that $|H_1(\varphi(\Omega))| \leq P$ and $|H_2(t,u(t))| \leq Q$. $E \Subset (0,1)$ is a measurable set, take $L = \max_{t\in[0,1],u\in[0,M]} |f(t,u)| + 1$, then

$$
\begin{aligned}
\bigl|(Tu)(t)\bigr| &= \biggl|t^{\alpha-1}\biggl[H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s,u(s)\bigr)\,ds\biggr] + \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds\biggr| \\
&\leq \bigl|H_1\bigl(\varphi(u)\bigr)\bigr| + \int_E \bigl|H_2\bigl(s,u(s)\bigr)\bigr|\,ds + \int_0^1 G(t,s)\bigl|f\bigl(s,u(s)\bigr)\bigr|\,ds \\
&\leq P + Qm(E) + L\int_0^1 G(t,s)\,ds \\
&\leq P + Qm(E) + L\int_0^1 G(s,s)\,ds \\
&= P + Qm(E) + \frac{L}{\Gamma(\alpha+2)}. \qquad (20)
\end{aligned}
$$

Hence $T(\Omega)$ is bounded.

For each $u \in \Omega$, $t_1, t_2 \in [0,1]$, $t_1 < t_2$, and assuming there is a $\delta > 0$, such that $t_2 - t_1 < \delta$, for each $\varepsilon > 0$ put

$$
\delta = \min\biggl\{\varepsilon, \frac{1}{2}\biggl(\frac{\varepsilon}{2N}\biggr)^{\frac{1}{\alpha-1}}, \biggl[\frac{\varepsilon}{2(\alpha-1)N}\biggr]^{\frac{1}{\alpha-1}}, \biggl(\frac{\varepsilon}{2N\alpha}\biggr)^{\frac{1}{\alpha}}\biggr\}, \qquad (21)
$$

where $N = P + Qm(E) + \frac{L}{\Gamma(\alpha+1)}$, we find $|(Tu)(t_2) - (Tu)(t_1)| < \varepsilon$, so that $T(\Omega)$ is equicontinuous.

Indeed,

$$\begin{aligned}
&\bigl|(Tu)(t_2) - (Tu)(t_1)\bigr| \\
&\quad = \biggl| t_2^{\alpha-1}\biggl[H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s,u(s)\bigr)\,ds\biggr] + \int_0^1 G(t_2,s)f\bigl(s,u(s)\bigr)\,ds \\
&\qquad - t_1^{\alpha-1}\biggl[H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s,u(s)\bigr)\,ds\biggr] - \int_0^1 G(t_1,s)f\bigl(s,u(s)\bigr)\,ds\biggr| \\
&\quad = \biggl|\bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr)\biggl[H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s,u(s)\bigr)\,ds\biggr] + \int_0^1 \bigl(G(t_2,s) - G(t_1,s)\bigr)f\bigl(s,u(s)\bigr)\,ds\biggr| \\
&\quad \le \bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr)\biggl[\bigl|H_1\bigl(\varphi(u)\bigr)\bigr| + \int_E \bigl|H_2\bigl(s,u(s)\bigr)\bigr|\,ds\biggr] \\
&\qquad + \biggl|\int_0^1 \bigl(G(t_2,s) - G(t_1,s)\bigr)f\bigl(s,u(s)\bigr)\,ds\biggr|.
\end{aligned} \tag{22}$$

By the continuity of H_1, H_2, and φ, we find H_1, H_2 are bounded, so there exist constants $P > 0$ and $Q > 0$ such that $|H_1(\varphi(\Omega))| \le P$ and $|H_2(t,u(t))| \le Q$. $E \Subset (0,1)$ is a measurable set, take $L = \max_{t\in[0,1],u\in[0,M]} |f(t,u)| + 1$, then

$$\begin{aligned}
&\bigl|(Tu)(t_2) - (Tu)(t_1)\bigr| \\
&\quad \le \bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr)\bigl[P + Qm(E)\bigr] + \int_0^1 \bigl|\bigl(G(t_2,s) - G(t_1,s)\bigr)f\bigl(s,u(s)\bigr)\bigr|\,ds \\
&\quad \le \bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr)\bigl[P + Qm(E)\bigr] \\
&\qquad + \frac{L}{\Gamma(\alpha)}\biggl[\int_0^{t_1} \bigl|t_2^{\alpha-1}(1-s)^{\alpha-1} - (t_2-s)^{\alpha-1} - t_1^{\alpha-1}(1-s)^{\alpha-1} + (t_1-s)^{\alpha-1}\bigr|\,ds \\
&\qquad + \int_{t_1}^{t_2} \bigl|t_2^{\alpha-1}(1-s)^{\alpha-1} - (t_2-s)^{\alpha-1} - t_1^{\alpha-1}(1-s)^{\alpha-1}\bigr|\,ds \\
&\qquad + \int_{t_2}^{1} \bigl|t_2^{\alpha-1}(1-s)^{\alpha-1} - t_1^{\alpha-1}(1-s)^{\alpha-1}\bigr|\,ds\biggr] \\
&\quad \le \bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr)\bigl[P + Qm(E)\bigr] + \frac{L}{\Gamma(\alpha)}\biggl\{\bigl|t_2^{\alpha-1} - t_1^{\alpha-1}\bigr|\int_0^{t_1}(1-s)^{\alpha-1}\,ds \\
&\qquad + \int_0^{t_1}\bigl[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\bigr]\,ds + \bigl|t_2^{\alpha-1} - t_1^{\alpha-1}\bigr|\int_{t_1}^{t_2}(1-s)^{\alpha-1}\,ds \\
&\qquad + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds + \bigl|t_2^{\alpha-1} - t_1^{\alpha-1}\bigr|\int_{t_2}^{1}(1-s)^{\alpha-1}\,ds\biggr\} \\
&\quad = \bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr)\biggl[P + Qm(E) + \frac{L}{\Gamma(\alpha+1)}\biggr] + \frac{L}{\Gamma(\alpha+1)}\bigl(t_2^{\alpha} - t_1^{\alpha}\bigr).
\end{aligned} \tag{23}$$

Next we discuss the following three cases.

Case 1. $\delta \le t_1 < t_2 < 1$.

$$\begin{aligned}
&t_2^{\alpha-1} - t_1^{\alpha-1} \le \frac{\alpha-1}{\delta^{2-\alpha}}(t_2 - t_1) \le (\alpha-1)\delta^{\alpha-1} \le \frac{\varepsilon}{2N}, \\
&t_2^{\alpha} - t_1^{\alpha} \le \frac{\alpha}{\delta^{1-\alpha}}(t_2 - t_1) \le \alpha\delta^{\alpha} \le \frac{\varepsilon}{2N}.
\end{aligned} \tag{24}$$

Case 2. $0 \le t_1 < \delta < t_2 < 2\delta$.

$$t_2^{\alpha-1} - t_1^{\alpha-1} \le t_2^{\alpha-1} \le (2\delta)^{\alpha-1} \le \frac{\varepsilon}{2N}, \qquad (25)$$
$$t_2^{\alpha} - t_1^{\alpha} \le t_2^{\alpha} \le (2\delta)^{\alpha} \le \frac{\varepsilon}{2N}.$$

Case 3. $0 \le t_1 < t_2 < \delta$.

$$t_2^{\alpha-1} - t_1^{\alpha-1} \le t_2^{\alpha-1} \le \delta^{\alpha-1} \le \frac{\varepsilon}{2N}, \qquad (26)$$
$$t_2^{\alpha} - t_1^{\alpha} \le t_2^{\alpha} \le \delta^{\alpha} \le \frac{\varepsilon}{2N}.$$

Hence $|(Tu)(t_2) - (Tu)(t_1)| \le \varepsilon$.

From the Arzelà-Ascoli theorem, T is completely continuous. The proof is complete. □

With Lemma 3.1 in hand, we are now ready to present the first existence theorem for problem (1) and (2).

Theorem 3.1 *Assume that conditions* (H_1)-(H_9) *hold. Suppose that*

$$C_1C_3 + C_4 m(E) < 1 \qquad (27)$$

and that $E \Subset (0,1)$. *Then problem* (1)-(2) *has at least one positive solution.*

Proof Begin by selecting a number η_1 such that

$$\eta_1 \int_a^b \gamma^* G\left(\frac{1}{2}, s\right) a(s)\, ds \ge 1. \qquad (28)$$

From (H_7), there exists a number $r_1 > 0$ such that $g(u) \ge \eta_1 u$ whenever $0 < u \le r_1$. Then set

$$\Omega_{r_1} := \{u \in B : \|u\| < r_1\}, \qquad (29)$$

for $u \in K \cap \partial\Omega_{r_1}$ we find

$$\begin{aligned}(Tu)\left(\frac{1}{2}\right) &\ge \int_0^1 G\left(\frac{1}{2}, s\right) a(s) g\big(u(s)\big)\, ds \ge \|u\| \eta_1 \int_a^b \gamma^* G\left(\frac{1}{2}, s\right) a(s)\, ds \\ &\ge \|u\|, \end{aligned} \qquad (30)$$

from the definition of norm $\|\cdot\|$, we get $\|Tu\| \ge \|u\|$, and so, T is a cone expansion on $K \cap \partial\Omega_{r_1}$.

From condition (27), we choose a positive number ε small enough, so that we may assume

$$C_1C_3\varepsilon + C_1C_3 + \varepsilon + (C_4 + C_4\varepsilon)m(E) \le 1. \tag{31}$$

Since $\varphi_1(u) \ge 0$, it follows that $\varphi(u) \ge \varphi_2(u) \ge C_2\|u\|$, if $\|u\| \ge \frac{M_\varepsilon}{C_2}$, then $\varphi(u) > M_\varepsilon$, and then by condition (H_4), we have

$$\bigl|H_1\bigl(\varphi(u)\bigr) - C_3\varphi(u)\bigr| < \varepsilon C_3\varphi(u). \tag{32}$$

For $E \Subset (0,1)$, we may select $0 < a < b < 1$ such that $E \subseteq (a,b)$. By the definition of the cone, we have

$$\min_{t\in E} u(t) \ge \min_{t\in[a,b]} u(t) \ge r^*\|u\|. \tag{33}$$

If $\|u\| \ge \frac{M_\varepsilon}{r^*}$, then

$$\min_{t\in E} u(t) \ge \min_{t\in[a,b]} u(t) \ge r^*\|u\| \ge r^*\frac{M_\varepsilon}{r^*} = M_\varepsilon. \tag{34}$$

Hence, by condition (H_5) we get

$$\bigl|H_2\bigl(x,u(s)\bigr) - F\bigl(u(s)\bigr)\bigr| < \varepsilon F\bigl(u(s)\bigr). \tag{35}$$

Provided that

$$\|u\| \ge \max\left\{\frac{M_\varepsilon}{C_2}, \frac{M_\varepsilon}{r^*}\right\}, \tag{36}$$

both (32) and (35) hold.

Next we are going to discuss these two cases: g is bounded and unbounded on $[0,+\infty)$, respectively.

Now if g is bounded on $[0,+\infty)$, then there exists $r_2 > 0$ sufficiently large such that

$$g(u) \le r_2, \quad \text{for any } u \ge 0. \tag{37}$$

Indeed, we might assume without loss of generality that

$$g(u) \le \frac{r_2\varepsilon}{\int_0^1 G(s,s)a(s)\,ds}, \tag{38}$$

where ε is selected sufficiently small such that both (32) and (35) hold. We define a number

$$r_2^* := \max\left\{\frac{2r_1}{r^*}, r_2, \frac{M_\varepsilon}{C_2}, \frac{M_\varepsilon}{r^*}\right\}.$$

Set

$$\Omega_{r_2^*} := \bigl\{u \in B : \|u\| < r_2^*\bigr\}. \tag{39}$$

Then for each $u \in K \cap \partial\Omega_{r_2^*}$ we find that

$$\begin{aligned}
\|Tu\| &\le H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s,u(s)\bigr)\,ds + \int_0^1 G(s,s)a(s)g\bigl(u(s)\bigr)\,ds \\
&\le \bigl|H_1\bigl(\varphi(u)\bigr) - C_3\varphi(u)\bigr| + C_3\bigl|\varphi(u)\bigr| + \int_E \bigl|H_2\bigl(s,u(s)\bigr) - F\bigl(u(s)\bigr)\bigr|\,ds \\
&\quad + \int_E \bigl|F\bigl(u(s)\bigr)\bigr|\,ds + \int_0^1 r_2\varepsilon ds \\
&\le \varepsilon C_1C_3\|u\| + C_1C_3\|u\| + m(E)C_4(1+\varepsilon)\|u\| + \varepsilon\|u\| \\
&= \bigl[C_1C_3\varepsilon + C_1C_3 + m(E)C_4(1+\varepsilon) + \varepsilon\bigr]\|u\| \\
&\le \|u\|,
\end{aligned} \tag{40}$$

whence T is a cone compression on $K \cap \partial\Omega_{r_2^*}$.

On the other hand, assume g is unbounded on $[0,+\infty)$. From condition (H_8) there is a number $r_3 > 0$ such that $g(u) \le \eta_2 u$ whenever $u > r_3$, where η_2 is picked with

$$\eta_2 \int_0^1 G(s,s)a(s)\,ds \le \varepsilon. \tag{41}$$

Since g is unbounded on $[0,+\infty)$, we can find a number r_3^* satisfying

$$r_3^* > \max\left\{\frac{2r_1}{r^*}, r_3, \frac{M_\varepsilon}{C_2}, \frac{M_\varepsilon}{r^*}\right\},$$

such that $g(u) \le g(r_3^*)$ for any $u \in [0, r_3^*]$.

Take

$$\Omega_{r_3^*} := \bigl\{u \in B : \|u\| < r_3^*\bigr\}. \tag{42}$$

Then for each $u \in K \cap \partial\Omega_{r_3^*}$ we find that

$$\begin{aligned}
\|Tu\| &\le H_1\bigl(\varphi(u)\bigr) + \int_E H_2\bigl(s,u(s)\bigr)\,ds + \int_0^1 G(s,s)a(s)g\bigl(u(s)\bigr)\,ds \\
&\le \bigl|H_1\bigl(\varphi(u)\bigr) - C_3\varphi(u)\bigr| + C_3\bigl|\varphi(u)\bigr| + \int_E \bigl|H_2\bigl(s,u(s)\bigr) - F\bigl(u(s)\bigr)\bigr|\,ds \\
&\quad + \int_E \bigl|F\bigl(u(s)\bigr)\bigr|\,ds + \int_0^1 G(s,s)a(s)g\bigl(r_3^*\bigr)\,ds \\
&\le C_3\varepsilon\varphi(u) + C_1C_3\|u\| + \int_E (1+\varepsilon)F\bigl(u(s)\bigr)\,ds + \int_0^1 G(s,s)a(s)g\bigl(r_3^*\bigr)\,ds \\
&\le \varepsilon C_1C_3\|u\| + C_1C_3\|u\| + m(E)C_4(1+\varepsilon)\|u\| + \varepsilon\|u\| \\
&= \bigl[C_1C_3\varepsilon + C_1C_3 + m(E)C_4(1+\varepsilon) + \varepsilon\bigr]\|u\| \\
&\le \|u\|,
\end{aligned} \tag{43}$$

whence T is a cone compression on $K \cap \partial\Omega_{r_3^*}$.

Therefore, in either case, define $r_4^* = \max\{r_2^*, r_3^*\}$, we find $\|Tu\| \leq \|u\|$ whenever $u \in K \cap \partial\Omega_{r_4^*}$. From Lemma 2.5, we claim that problem (1)-(2) has at least one positive solution; the proof is complete. □

Next we are going to give some corollaries since $\varphi(u)$ admits a wide variety of functionals. First we assume $H_1(\varphi(u)) = 0$, then, respectively, that $H_1(\varphi(u)) = \int_F u(t)\,dt$, $H_1(\varphi(u)) = \sum_{i=1}^n |a_i| u(\xi_i)$ and $H_1(\varphi(u)) = \int_{[0,1]} u(t)\,d\theta(t)$, where $F \subset E \Subset (0,1)$ is not Lebesgue null. In addition, we know $u(0) = 0$ is also well defined and if $u(0) = 0$ the problem as well as the boundary conditions are similar to [6].

Corollary 3.1 *Assume that conditions* (H_1), (H_6)-(H_9) *hold, then the problem with Dirichlet conditions*

$$\begin{aligned} &-D_{0+}^{\alpha} u(t) = f\bigl(t, u(t)\bigr), \quad 0 < t < 1, \\ &u(0) = 0, \qquad u(1) = 0, \end{aligned} \tag{44}$$

has at least one positive solution.

Corollary 3.2 *Assume that conditions* (H_1)-(H_9) *hold. Suppose, in addition,*

$$C_1 C_3 + C_4 m(E) < 1,$$

and $F \subset E \Subset (0,1)$ *is not Lebesgue null, and* $m(F) \leq C_1$, *then the problem with integral conditions*

$$\begin{aligned} &-D_{0+}^{\alpha} u(t) = f\bigl(t, u(t)\bigr), \quad 0 < t < 1, \\ &u(1) = \int_F u(t)\,dt + \int_E H_2\bigl(s, u(s)\bigr)\,ds, \qquad u(0) = 0, \end{aligned} \tag{45}$$

has at least one positive solution.

Corollary 3.3 *Assume that conditions* (H_1)-(H_9) *hold. Suppose, in addition,*

$$C_1 C_3 + C_4 m(E) < 1,$$

and $E \Subset (0,1)$ *is not Lebesgue null, and* $\sum_{i=1}^n |a_i| \leq C_1$, *then the problem with multi-point conditions*

$$\begin{aligned} &-D_{0+}^{\alpha} u(t) = f\bigl(t, u(t)\bigr), \quad 0 < t < 1, \\ &u(1) = \sum_{i=1}^{n} |a_i| u(\xi_i) + \int_E H_2\bigl(s, u(s)\bigr)\,ds, \qquad u(0) = 0, \end{aligned} \tag{46}$$

has at least one positive solution.

Corollary 3.4 *Assume that conditions* (H_1)-(H_9) *hold. Suppose, in addition,*

$$C_1 C_3 + C_4 m(E) < 1,$$

and $E \Subset (0,1)$ is not Lebesgue null, and the total variation of θ over $[0,1]$ satisfies $V_{[0,1]}(\theta) \leq C_1$, then the problem with the Lebesgue-Stieltjes integral conditions

$$\begin{aligned} &-D_{0+}^{\alpha}u(t) = f\big(t,u(t)\big), \quad 0 < t < 1, \\ &u(1) = \int_{[0,1]} u(t)\,d\theta(t) + \int_E H_2\big(s,u(s)\big)\,ds, \qquad u(0) = 0, \end{aligned} \tag{47}$$

has at least one positive solution.

Specially, take $\theta(t) = t^2 - t$, for any $x, y \in [0,1]$ we have

$$\big|\theta(x) - \theta(y)\big| = \big|(x-y)(x+y-1)\big| \leq |x-y|,$$

then

$$V_{[0,1]}(\theta) = \sum_{i=1}^{n}\big|\theta(x_i) - \theta(x_{i-1})\big| \leq \sum_{i=1}^{n}|x_i - x_{i-1}| = \sum_{i=1}^{n}(x_i - x_{i-1}) = 1,$$

thus we have the following corollary.

Corollary 3.5 *Assume that conditions* (H_1)-(H_4) *and* (H_6)-(H_9) *hold. Suppose, in addition,*

$$C_1C_3 < 1,$$

then the problem

$$\begin{aligned} &-D_{0+}^{\alpha}u(t) = f\big(t,u(t)\big), \quad 0 < t < 1, \\ &u(1) = \int_{[0,1]} u(t)\,d\big(t^2 - t\big), \qquad u(0) = 0, \end{aligned} \tag{48}$$

has at least one positive solution.

4 Example

In this part we give an example of Theorem 3.1.

Define $\varphi(u)$ in the first place by

$$\varphi(u) = \frac{1}{5}u\left(\frac{3}{4}\right) - \frac{1}{4}u\left(\frac{2}{5}\right) + \int_{[\frac{3}{10},\frac{2}{5}]} u(t)\,dt, \tag{49}$$

where

$$\varphi_1(u) = \frac{1}{5}u\left(\frac{3}{4}\right) - \frac{1}{4}u\left(\frac{2}{5}\right), \qquad \varphi_2(u) = \int_{[\frac{3}{10},\frac{2}{5}]} u(t)\,dt. \tag{50}$$

Then define H_1, H_2 by

$$H_1(z) = \ln(z+1) + z \tag{51}$$

and

$$H_2(x,u) = 3u + 2x^2 + e^x\sqrt{u}. \tag{52}$$

It is clear that

$$\lim_{z\to\infty}\left|\big(\ln(z+1)+z\big)-z\right| = 0, \tag{53}$$

and moreover

$$\lim_{u\to\infty}\frac{|(3u+2x^2+e^x\sqrt{u})-3u|}{3u} = 0, \tag{54}$$

so conditions (H_4) and (H_5) hold, and we see $C_3 = 1$, $C_4 = 3$, and $F(u) = 3u$.

Now we consider the boundary value problem

$$\begin{aligned} &-D_{0+}^{\alpha}u(t) = f\big(t,u(t)\big), \quad 0 < t < 1, \\ &u(1) = \ln\big(\varphi(u)+1\big) + \varphi(u) + \int_{\frac{7}{20}}^{\frac{2}{5}}\big[3u(s)+2s^2+e^s\sqrt{u(s)}\big]\,ds, \qquad u(0) = 0, \end{aligned} \tag{55}$$

where $f(t,u(t))$ is a given function with conditions (H_7) and (H_8) satisfied. Here $E = [\frac{7}{20},\frac{2}{5}]$ is chosen such that $m(E) = \frac{1}{20}$ and $E \subset (0,1)$.

What is more, for each $u \in K$

$$\left|\varphi(u)\right| \le \frac{1}{5}\|u\| + \frac{1}{4}\|u\| + \left(\frac{2}{5}-\frac{3}{10}\right)\|u\| = \frac{11}{20}\|u\| \tag{56}$$

and

$$\left|\varphi_2(u)\right| \ge \frac{1}{10}r^*\|u\|. \tag{57}$$

Then we find that $C_1 = \frac{11}{20} \in [0,1]$ and $C_2 = \frac{1}{10}r^* > 0$, so condition ($H_3$) is met as well. Finally, after straightforward numerical calculations, condition (H_9) can also be achieved, since

$$C_1C_3 + C_4m(E) = \frac{11}{20}\cdot 1 + 3\cdot\frac{1}{20} < 1.$$

As a consequence, each of conditions (H_1)-(H_9) is satisfied. From Theorem 3.1, problem (55) has at least one positive solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
The authors sincerely thank the reviewers for their valuable suggestions and useful comments, which have led to the present improved version of the original manuscript. This research is supported by the Natural Science Foundation of China (61374074), Natural Science Outstanding Youth Foundation of Shandong Province (JQ201119) and supported by Shandong Provincial Natural Science Foundation (ZR2012AM009, ZR2013AL003), also supported by Graduate Innovation Foundation of University of Jinan (YCX13013).

Received: 22 January 2014 Accepted: 20 June 2014 Published online: 30 September 2014

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doi:10.1186/s13661-014-0225-0
Cite this article as: Feng et al.: **Positive solutions to fractional boundary value problems with nonlinear boundary conditions.** *Boundary Value Problems* 2014 **2014**:225.