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Eigenvalues of higher order Sturm-Liouville boundary value problems with derivatives in nonlinear terms

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Abstract

We shall consider the Sturm-Liouville boundary value problem
$y^{(m)}(t) + \lambda F(t, y(t), y'(t), \dots, y^{(q)}(t)) = 0$, $t \in (0, 1)$, $y^{(k)}(0) = 0$, $0 \le k \le m - 3$,
$\zeta y^{(m-2)}(0) - \theta y^{(m-1)}(0) = 0$, $\rho y^{(m-2)}(1) + \delta y^{(m-1)}(1) = 0$ where $m \ge 3$, $1 \le q \le m - 2$, and
$\lambda > 0$. It is noted that the boundary value problem considered has a
derivative-dependent nonlinear term, which makes the investigation much more
challenging. In this paper we shall develop a *new* technique to characterize the
eigenvalues λ so that the boundary value problem has a positive solution. Explicit
eigenvalue intervals are also established. Some examples are included to dwell upon
the usefulness of the results obtained.

MSC: 34B15

Keywords: eigenvalues; positive solutions; Sturm-Liouville boundary value
problems; derivative-dependent

1 Introduction

In this paper we shall consider the higher order Sturm-Liouville boundary value problem

$$\begin{cases} y^{(m)}(t) + \lambda F(t, y(t), y'(t), \dots, y^{(q)}(t)) = 0, & t \in (0, 1), \\ y^{(k)}(0) = 0, & 0 \le k \le m - 3, \\ \zeta y^{(m-2)}(0) - \theta y^{(m-1)}(0) = 0, & \rho y^{(m-2)}(1) + \delta y^{(m-1)}(1) = 0, \end{cases} \quad (1.1)$$

where $m \ge 3$, $1 \le q \le m - 2$, $\lambda > 0$, and F is continuous at least in the domain of interest.
The constants ζ, θ, ρ, and δ are such that

$$\theta \ge 0, \qquad \delta \ge 0, \qquad \theta + \zeta > 0, \qquad \delta + \rho > 0, \qquad \kappa \equiv \zeta\rho + \zeta\delta + \theta\rho > 0. \quad (1.2)$$

These assumptions allow ζ and ρ to be negative.

A vast amount of research has been done on the existence of positive solutions of
Sturm-Liouville boundary value problems. The general interest in (1.1) may stem from the
fact that the boundary value problem models a wide spectrum of nonlinear phenomena,
such as gas diffusion through porous media, nonlinear diffusion generated by nonlinear
sources, thermal self ignition of a chemically active mixture of gases in a vessel, catalysis
theory, chemically reacting systems, adiabatic tubular reactor processes, as well as con-
centration in chemical or biological problems; see [1–7]. For the special case $\lambda = 1$, (1.1)

and its particular and related cases have been the subject matter of many publications on singular boundary value problems, *e.g.*, see [8–14]. For details of recent development in (1.1) as well as in other types of boundary value problems, the reader is referred to the monographs [15, 16] and the hundreds of references cited therein. It is noted that in most of this research the nonlinear terms considered do *not* involve derivatives of the dependent variable; only a comparatively small number of papers tackle nonlinear terms that involve derivatives, and we mention some below.

Fink [17] has discussed the radial symmetric form of the semilinear elliptic equation $\Delta y + \lambda q(|x|)f(y) = 0$ in $\mathbb{R}^N$, namely,

$$\begin{cases} y'' + \frac{N-1}{t}y' + \lambda q(t)f(y) = 0, & t \in (0,1), \\ y'(0) = y(1) = 0, \end{cases}$$

which is actually a second order Sturm-Liouville eigenvalue problem that has y' in the nonlinear term. Later, Wong [18] has considered (1.1) when $\lambda = 1$ and $q = m - 2$, the existence of a solution (not necessarily positive) is obtained by assuming that (1.1) has lower and upper solutions v and w such that $v^{(m-2)}(t) \leq w^{(m-2)}(t)$ on $[0,1]$, and

$$F\big(t, v(t), \ldots, v^{(m-3)}(t), u_{m-1}\big) \leq F\big(t, u_1, \ldots, u_{m-2}, u_{m-1}\big) \leq F\big(t, w(t), \ldots, w^{(m-3)}(t), u_{m-1}\big)$$

for $t \in [0,1]$ and $(v(t), \ldots, v^{(m-3)}(t)) \leq (u_1, \ldots, u_{m-2}) \leq (w(t), \ldots, w^{(m-3)}(t))$. A few years later, Grossinho and Minhós [19] established the existence of a solution to a related problem of (1.1) when $\lambda = 1$ and $q = m - 1$; their method requires again the existence of lower and upper solutions and F must satisfy a *Nagumo-type condition* on some set $A \subset [0,1] \times \mathbb{R}^m$, *viz.*,

$$\begin{cases} \text{there exists a continuous function } h : [0,\infty) \to (0,\infty) \text{ such that} \\ |F(t, u_1, \ldots, u_m)| \leq h(|u_m|), \quad (t, u_1, \ldots, u_m) \in A; \\ \int_0^\infty \frac{s}{h(s)}\, ds = \infty. \end{cases}$$

The comparatively small number of papers on problems involving derivative-dependent nonlinear terms shows that problems of this type are more difficult to tackle analytically, we note, however, that numerical methods are more developed for this type of problems, see for example [20–25].

We also mention some problems related to (1.1). Recently, Pei and Chang [26] have studied a fourth order problem with focal-Sturm-Liouville type boundary conditions

$$\begin{cases} y^{(4)}(t) = F(t, y(t), y'(t), y''(t), y^{(3)}(t)), & t \in [0,1], \\ y(0) = y'(1) = a_0 y''(0) - b_0 y^{(3)}(0) = a_1 y''(1) - b_1 y^{(3)}(1) = 0. \end{cases}$$

Here, once again F should satisfy a Nagumo-type condition and also F is monotone in certain arguments. For multi-point problems, Zhang *et al.* [27, 28] have discussed the following using the Avery-Peterson fixed point theorem:

$$\begin{cases} y''(t) + h(t)f(t, y(t), y'(t)) = 0, & t \in (0,1), \\ ay(0) - by'(0) = \sum_{i=1}^{n-2} a_i y(\xi_i), & cy(1) + dy'(1) = \sum_{i=1}^{n-2} b_i y(\xi_i). \end{cases}$$

For infinite interval problems, Lian *et al.* [29, 30] have investigated the following:

$$
\begin{cases}
-y^{(m)}(t) = h(t)f(t, y(t), y'(t), \ldots, y^{(m-1)}(t)), & t \in (0, \infty), \\
y^{(k)}(0) = A_k, & 0 \le k \le m - 3, \\
y^{(m-2)}(0) - ay^{(m-1)}(0) = B, & y^{(m-1)}(\infty) = C.
\end{cases}
$$

Here, once again the lower and upper solutions method is used and the Nagumo-type condition plays an important role in handling the derivatives in the nonlinear term.

Motivated by the above research, in the present work we shall employ a different (and new) technique to tackle the eigenvalue problem (1.1). It is noted that our technique *neither* requires the existence of lower and upper solutions *nor* the assumption of a Nagumo-type condition, both of these conditions are not easy to check in practical applications.

To specify some terminology used: if, for a particular λ, the boundary value problem (1.1) has a positive solution y, then λ is called an *eigenvalue* and y a corresponding *eigenfunction* of (1.1). We let E be the set of eigenvalues of (1.1), *i.e.*,

$$
E = \left\{ \lambda > 0 \mid (1.1) \text{ has a positive solution} \right\}.
$$

Here, by a *positive solution* y of (1.1), we mean a nontrivial $y \in C^{(m)}(0, 1) \cap C^{(m-1)}[0, 1]$ satisfying (1.1), y is nonnegative on $[0, 1]$ and is positive on some subinterval of $[0, 1]$. The first focus of this paper is the characterization of the set of eigenvalues E, specifically we shall establish criteria for E to contain an interval, for E to be an interval, and for E to be an open finite or half-closed finite or infinite interval. Our second focus is to derive explicit subintervals of E. Due to the presence of derivatives in the nonlinear term, our current work naturally generalizes and extends the known theorems for Sturm-Liouville eigenvalue problems [17, 31–37] as well as complements the work of many authors [18, 19, 38–47]. We remark that our conditions/assumptions, which do not involve lower and upper solutions and a Nagumo-type condition, are comparatively easy to check - this practical usefulness will be illustrated by examples with known eigenvalues and eigenfunctions.

The plan of the paper is as follows. In Section 2 we shall state a fixed point theorem and present some properties of a certain Green's function which are needed later. The set E is characterized in Section 3, while the eigenvalue subintervals are derived in Section 4.

2 Preliminaries

We shall state the *Krasnosel'skii fixed point theorem in a cone* which is used later and also the properties of a certain Green's function related to the boundary value problem (1.1).

Theorem 2.1 (Krasnosel'skii fixed point theorem in a cone) [48] *Let B be a Banach space, and let $C \subset B$ be a cone in B. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$
S : C \cap (\bar{\Omega}_2 \setminus \Omega_1) \to C
$$

be a completely continuous operator such that either

(a) $\|Sy\| \le \|y\|$, $y \in C \cap \partial\Omega_1$, *and* $\|Sy\| \ge \|y\|$, $y \in C \cap \partial\Omega_2$, *or*

(b) $\|Sy\| \geq \|y\|$, $y \in C \cap \partial\Omega_1$, *and* $\|Sy\| \leq \|y\|$, $y \in C \cap \partial\Omega_2$.
Then S has a fixed point in $C \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Let $G(t,s)$ be the Green's function of the *second order* Sturm-Liouville boundary value problem

$$
\begin{cases}
-w''(t) = 0, & t \in (0,1), \\
\zeta w(0) - \theta w'(0) = 0, & \rho w(1) + \delta w'(1) = 0.
\end{cases}
\tag{2.1}
$$

It is well known that [35–37]

$$
G(t,s) = \frac{1}{\kappa}
\begin{cases}
(\theta + \zeta s)[\delta + \rho(1-t)], & 0 \leq s \leq t \leq 1, \\
(\theta + \zeta t)[\delta + \rho(1-s)], & 0 \leq t \leq s \leq 1.
\end{cases}
\tag{2.2}
$$

Lemma 2.2 [35–37] *The Green's function $G(t,s)$ has the following properties*:
　(a) $G(t,s) \geq 0$ *for* $(t,s) \in [0,1] \times [0,1]$ *and* $G(t,s) > 0$ *for* $(t,s) \in (0,1) \times (0,1)$.
　(b) $G(t,s) \leq LG(s,s)$ *for* $(t,s) \in [0,1] \times [0,1]$ *where*

$$
L = \max\left\{1, \frac{\theta}{\theta + \zeta}, \frac{\delta}{\delta + \rho}\right\}.
$$

　(c) $G(t,s) \geq K_\eta G(s,s)$ *for* $(t,s) \in [\eta, 1-\eta] \times [0,1]$, *where* $\eta \in (0, \frac{1}{2})$ *is fixed and*

$$
K_\eta = \min\left\{\frac{\delta + \rho\eta}{\delta + \rho}, \frac{\delta + \rho(1-\eta)}{\delta + \rho\eta}, \frac{\theta + \zeta\eta}{\theta + \zeta}, \frac{\theta + \zeta(1-\eta)}{\theta + \zeta\eta}\right\}.
$$

　(d) $g_n(t,s)$, *defined by the relation* $\frac{\partial^{n-2}}{\partial t^{n-2}} g_n(t,s) = G(t,s)$, *is the Green's function of the nth order Sturm-Liouville boundary value problem*

$$
\begin{cases}
-w^{(n)}(t) = 0, & t \in (0,1), \\
w^{(k)}(0) = 0, & 0 \leq k \leq n-3, \\
\zeta w^{(n-2)}(0) - \theta w^{(n-1)}(0) = 0, & \rho w^{(n-2)}(1) + \delta w^{(n-1)}(1) = 0.
\end{cases}
\tag{2.3$_n$}
$$

　(e) $0 \leq g_n(t,s) \leq \frac{L}{(n-2)!} G(s,s)$ *for* $(t,s) \in [0,1] \times [0,1]$.

3 Eigenvalue characterization

Recall that $E = \{\lambda > 0 \mid (1.1)$ has a positive solution$\}$. In this section, we shall establish criteria for E to contain an interval (Theorem 3.5), and for E to be an interval (Corollary 3.7), which may either be bounded or unbounded (Theorem 3.9).

To begin, we consider the initial value problem

$$
\begin{cases}
y^{(q)}(t) = x(t), & t \in (0,1), \\
y(0) = y'(0) = y''(0) = \cdots = y^{(q-1)}(0) = 0.
\end{cases}
\tag{3.1}
$$

Noting the initial conditions in (3.1), we have by repeated integration

$$
y^{(k)}(t) = \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{q-k-1}} x(s_{q-k})\, ds_{q-k} \cdots ds_1, \quad 0 \leq k \leq q-1.
\tag{3.2}
$$

Denote the integral

$$J^k x(t) = \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{k-1}} x(s_k)\, ds_k \cdots ds_1, \quad k \geq 1. \tag{3.3}$$

Then (3.2) is simply

$$y^{(k)}(t) = J^{q-k} x(t), \quad 0 \leq k \leq q - 1. \tag{3.4}$$

In view of (3.1) and (3.4), we can rewrite (1.1) as the following $(m-q)th\ order$ Sturm-Liouville boundary value problem:

$$\begin{cases} x^{(m-q)}(t) + \lambda F(t, \tilde{J}x(t)) = 0, & t \in (0,1), \\ x^{(k)}(0) = 0, & 0 \leq k \leq m - q - 3, \\ \zeta x^{(m-q-2)}(0) - \theta x^{(m-q-1)}(0) = 0, & \rho x^{(m-q-2)}(1) + \delta x^{(m-q-1)}(1) = 0, \end{cases} \tag{3.5}$$

where we denote

$$\tilde{J}x(t) = \left(J^q x(t), J^{q-1} x(t), \ldots, J x(t), x(t) \right).$$

If (3.5) has a solution x^*, then the boundary value problem (1.1) has a solution y^* given by

$$y^*(t) = J^q x^*(t) = \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{q-1}} x^*(s_q)\, ds_q \cdots ds_1. \tag{3.6}$$

Hence, the existence of a solution of (1.1) follows from the existence of a solution of (3.5). Further, it is obvious from (3.6) that y^* is positive if x^* is. An eigenvalue of (3.5) is thus also an eigenvalue of (1.1), *i.e.*,

$$E = \left\{ \lambda > 0 \mid (1.1) \text{ has a positive solution} \right\} = \left\{ \lambda > 0 \mid (3.5) \text{ has a positive solution} \right\}.$$

We shall study the eigenvalue problem (1.1) via (3.5) and a new technique will be developed to handle the nonlinear term F.

For easy reference, the conditions that will be mentioned later are listed below.

(C1) There exist continuous functions $f : (0, \infty)^{q+1} \to (0, \infty)$ and $a, b : (0,1) \to [0, \infty)$ such that for $t \in (0,1)$ and $u_j \in (0, \infty)$, $1 \leq j \leq q + 1$,

$$a(t)f(u_1, \ldots, u_{q+1}) \leq F(t, u_1, \ldots, u_{q+1}) \leq b(t)f(u_1, \ldots, u_{q+1}).$$

(C2) $a(t)$ is not identically zero on any nondegenerate subinterval of $(0,1)$ and there exists $r \in (0,1]$ such that $a(t) \geq r b(t)$ for all $t \in (0,1)$.

(C3) $0 < \int_0^1 (\theta + \zeta t)[\delta + \rho(1 - t)] b(t)\, dt < \infty$.

(C4) f is nondecreasing in each of its arguments, *i.e.*, for $u_j, v, w \in (0, \infty)$, $1 \leq j \leq q + 1$ with $v \leq w$, we have

$$f(u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_{q+1}) \leq f(u_1, \ldots, u_{i-1}, w, u_{i+1}, \ldots, u_{q+1}), \quad 1 \leq i \leq q + 1.$$

(C5) For $t \in (0,1)$ and $u_j, v, w \in (0, \infty)$, $1 \le j \le q+1$ with $v \le w$, we have

$$F(t, u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_{q+1}) \le F(t, u_1, \ldots, u_{i-1}, w, u_{i+1}, \ldots, u_{q+1}),$$

$$1 \le i \le q+1.$$

Let the Banach space

$$B = \left\{ x \in C^{(m-q)}(0,1) \cap C^{(m-q-1)}[0,1] \mid x^{(k)}(0) = 0, 0 \le k \le m-q-3 \right\}$$

be equipped with the norm

$$\|x\| = \sup_{t \in [0,1]} \left| x^{(m-q-2)}(t) \right|.$$

Throughout the paper, let $\eta \in (0, \frac{1}{2})$ be fixed. Define the cone C in B by

$$C = \left\{ x \in B \mid x^{(m-q-2)}(t) \ge 0, t \in [0,1]; \min_{t \in [\eta, 1-\eta]} x^{(m-q-2)}(t) \ge \gamma \|x\| \right\}, \tag{3.7}$$

where $\gamma = rK_\eta / L$. For a constant $M > 0$, let

$$C(M) = \left\{ x \in C \mid \|x\| \le M \right\}.$$

Lemma 3.1 [36, 37] *Let $x \in B$. For $0 \le i \le m-q-2$, we have*

$$\left| x^{(i)}(t) \right| \le \frac{t^{m-q-2-i}}{(m-q-2-i)!} \|x\|, \quad t \in [0,1]. \tag{3.8}$$

In particular,

$$\left| x(t) \right| \le \frac{1}{(m-q-2)!} \|x\|, \quad t \in [0,1]. \tag{3.9}$$

Lemma 3.2 [36, 37] *Let $x \in C$. For $0 \le i \le m-q-2$, we have*

$$x^{(i)}(t) \ge 0, \quad t \in [0,1], \tag{3.10}$$

and

$$x^{(i)}(t) \ge (t-\eta)^{m-q-2-i} \frac{\gamma}{(m-q-2-i)!} \|x\|, \quad t \in [\eta, 1-\eta]. \tag{3.11}$$

In particular, we have, for fixed $z \in (\eta, 1-\eta)$,

$$x(t) \ge (z-\eta)^{m-q-2} \frac{\gamma}{(m-q-2)!} \|x\|, \quad t \in [z, 1-\eta]. \tag{3.12}$$

Remark 3.1 If $x^* \in C$ is a nontrivial solution of (3.5), then (3.10) and (3.12) imply that x^* is a positive solution of (3.5). As noted earlier a positive solution y^* of (1.1) can be obtained via (3.6).

The next result is useful in handling the nonlinear term F.

Lemma 3.3 *Let $x \in C$ and let $z \in (\eta, 1 - \eta)$ be fixed. Then we have, for $1 \le k \le q$,*

$$J^k x(t) \le \frac{1}{(m - q - 2)!} \|x\|, \quad t \in [0, 1], \tag{3.13}$$

and

$$J^k x(t) \ge (z - \eta)^{m-q-2+k} \frac{\gamma}{(m - q - 2 + k)!} \|x\|, \quad t \in [z, 1 - \eta]. \tag{3.14}$$

Proof Since $x \in C \subset B$, using (3.9) we obtain for $1 \le k \le q$ and $t \in [0, 1]$,

$$
\begin{aligned}
J^k x(t) &= \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{k-1}} x(s_k) \, ds_k \cdots ds_1 \\
&\le \int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1 \frac{\|x\|}{(m - q - 2)!} \, ds_k \cdots ds_1 = \frac{\|x\|}{(m - q - 2)!}.
\end{aligned}
$$

Next, since $x \in C$, it follows from (3.11) that

$$x(t) \ge (t - \eta)^{m-q-2} \frac{\gamma \|x\|}{(m - q - 2)!}, \quad t \in [\eta, 1 - \eta]. \tag{3.15}$$

Let $t \in [z, 1 - \eta]$. Using (3.15) we find, for $1 \le k \le q$,

$$
\begin{aligned}
J^k x(t) &= \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{k-1}} x(s_k) \, ds_k \cdots ds_1 \ge \int_\eta^z \int_\eta^{s_1} \int_\eta^{s_2} \cdots \int_\eta^{s_{k-1}} x(s_k) \, ds_k \cdots ds_1 \\
&\ge \int_\eta^z \int_\eta^{s_1} \int_\eta^{s_2} \cdots \int_\eta^{s_{k-1}} (s_k - \eta)^{m-q-2} \frac{\gamma \|x\|}{(m - q - 2)!} \, ds_k \cdots ds_1 \\
&= (z - \eta)^{m-q-2+k} \frac{\gamma \|x\|}{(m - q - 2 + k)!}. \qquad \Box
\end{aligned}
$$

In view of Remark 3.1, to obtain a positive solution of (3.5), we shall seek a fixed point of the operator S in the cone C, where $S : C \to B$ is defined by

$$Sx(t) = \lambda \int_0^1 g_{m-q}(t, s) F\big(s, \tilde{J}x(s)\big) \, ds, \quad t \in [0, 1]. \tag{3.16}$$

Recall that $g_{m-q}(t, s)$ (see Lemma 2.2(d)) is the Green's function of $((2.3)_n)_{m-q}$, thus (3.16) is equivalent to

$$(Sx)^{(m-q-2)}(t) = \lambda \int_0^1 G(t, s) F\big(s, \tilde{J}x(s)\big) \, ds, \quad t \in [0, 1], \tag{3.17}$$

where $G(t, s)$ is the Green's function of (2.1).

We further define the operators $U, V : C \to B$ by

$$Ux(t) = \lambda \int_0^1 g_{m-q}(t, s) a(s) f\big(\tilde{J}x(s)\big) \, ds \quad \text{and} \quad Vx(t) = \lambda \int_0^1 g_{m-q}(t, s) b(s) f\big(\tilde{J}x(s)\big) \, ds.$$

As in (3.17), differentiating gives

$$(Ux)^{(m-q-2)}(t) = \lambda \int_0^1 G(t, s) a(s) f\big(\tilde{J}x(s)\big) \, ds$$

and

$$(Vx)^{(m-q-2)}(t) = \lambda \int_0^1 G(t,s)b(s)f\big(\tilde{J}x(s)\big)\,ds.$$

If (C1) is satisfied, then it is clear that

$$Ux(t) \le Sx(t) \le Vx(t), \quad t \in [0,1], \tag{3.18}$$

and

$$(Ux)^{(m-q-2)}(t) \le (Sx)^{(m-q-2)}(t) \le (Vx)^{(m-q-2)}(t), \quad t \in [0,1]. \tag{3.19}$$

Lemma 3.4 *Let* (C1)-(C4) *hold. Then the operator S is compact on the cone C.*

Proof Let us consider the case when $a(t)$ is unbounded in a deleted right neighborhood of 0 and also in a deleted left neighborhood of 1. Clearly, $b(t)$ is also unbounded near 0 and 1. For $n \in \{1,2,3,\ldots\}$, let $a_n, b_n : [0,1] \to [0,\infty)$ be defined by

$$a_n(t) = \begin{cases} a(\frac{1}{n+1}), & 0 \le t \le \frac{1}{n+1}, \\ a(t), & \frac{1}{n+1} \le t \le \frac{n}{n+1}, \\ a(\frac{n}{n+1}), & \frac{n}{n+1} \le t \le 1, \end{cases}$$

and

$$b_n(t) = \begin{cases} b(\frac{1}{n+1}), & 0 \le t \le \frac{1}{n+1}, \\ b(t), & \frac{1}{n+1} \le t \le \frac{n}{n+1}, \\ b(\frac{n}{n+1}), & \frac{n}{n+1} \le t \le 1. \end{cases}$$

Also, we define the operators $U_n, V_n : C \to B$ by

$$U_n x(t) = \lambda \int_0^1 g_{m-q}(t,s)a_n(s)f\big(\tilde{J}x(s)\big)\,ds$$

and

$$V_n x(t) = \lambda \int_0^1 g_{m-q}(t,s)b_n(s)f\big(\tilde{J}x(s)\big)\,ds.$$

It is standard that, for each n, both U_n and V_n are compact operators on C. Let $M > 0$ and $x \in C(M)$. For $t \in [0,1]$, we obtain

$$\begin{aligned} \big|V_n x(t) - Vx(t)\big| &\le \lambda \int_0^1 g_{m-q}(t,s)\big|b_n(s) - b(s)\big|f\big(\tilde{J}x(s)\big)\,ds \\ &= \lambda \int_0^{\frac{1}{n+1}} g_{m-q}(t,s)\big|b_n(s) - b(s)\big|f\big(\tilde{J}x(s)\big)\,ds \\ &\quad + \lambda \int_{\frac{n}{n+1}}^1 g_{m-q}(t,s)\big|b_n(s) - b(s)\big|f\big(\tilde{J}x(s)\big)\,ds \end{aligned}$$

$$= \lambda \int_0^{\frac{1}{n+1}} g_{m-q}(t,s) \left| b_n\left(\frac{1}{n+1}\right) - b(s) \right| f\left(\tilde{J}x(s)\right) ds$$

$$+ \lambda \int_{\frac{n}{n+1}}^1 g_{m-q}(t,s) \left| b_n\left(\frac{n}{n+1}\right) - b(s) \right| f\left(\tilde{J}x(s)\right) ds.$$

Since $\|x\| \le M$, from (3.13) we get

$$J^k x(s) \le \frac{M}{(m-q-2)!}, \quad s \in [0,1], 1 \le k \le q.$$

Hence, together with (3.9), it follows from the monotonicity of f (condition (C4)) that

$$f\left(\tilde{J}x(s)\right) = f\left(J^q x(s), J^{q-1}x(s), \ldots, Jx(s), x(s)\right)$$

$$\le f\left(\frac{M}{(m-q-2)!}, \frac{M}{(m-q-2)!}, \ldots, \frac{M}{(m-q-2)!}\right) \equiv \bar{f}_M. \tag{3.20}$$

Applying (3.20) and Lemma 2.2(e), we obtain

$$\left| V_n x(t) - Vx(t) \right| \le \lambda \bar{f}_M \left[\int_0^{\frac{1}{n+1}} \frac{LG(s,s)}{(m-q-2)!} \left| b\left(\frac{1}{n+1}\right) - b(s) \right| ds \right.$$

$$\left. + \int_{\frac{n}{n+1}}^1 \frac{LG(s,s)}{(m-q-2)!} \left| b\left(\frac{n}{n+1}\right) - b(s) \right| ds \right].$$

The integrability of $G(s,s)b(s)$ (which is simply (C3)) ensures that V_n converges uniformly to V on $C(M)$. Hence, V is compact on C. By a similar argument, we see that U_n converges uniformly to U on $C(M)$ and therefore U is also compact on C. It follows immediately from inequality (3.18) that the operator S is compact on C. □

Remark 3.2 From the proof of Lemma 3.4, we see that if the functions a and b are continuous on the close interval $[0,1]$, then the conditions (C3) and (C4) are *not* needed in Lemma 3.4.

The first main result shows that E contains an interval.

Theorem 3.5 *Let* (C1)-(C4) *hold. Then there exists* $\ell > 0$ *such that the interval* $(0, \ell] \subseteq E$.

Proof For a given $M > 0$, we define

$$\ell = M\left[\bar{f}_M L \int_0^1 G(s,s)b(s)\, ds\right]^{-1}. \tag{3.21}$$

Let $\lambda \in (0, \ell]$. We shall prove that $S(C(M)) \subseteq C(M)$. For this, let $x \in C(M)$. First, we shall show that $Sx \in C$. From (3.19), it is clear that

$$(Sx)^{(m-q-2)}(t) \ge \lambda \int_0^1 G(t,s)a(s)f\left(\tilde{J}x(s)\right) ds \ge 0, \quad t \in [0,1]. \tag{3.22}$$

Also, (3.19) and Lemma 2.2(b) provide

$$(Sx)^{(m-q-2)}(t) \le \lambda \int_0^1 G(t,s)b(s)f\left(\tilde{J}x(s)\right) ds \le \lambda \int_0^1 LG(s,s)b(s)f\left(\tilde{J}x(s)\right) ds, \quad t \in [0,1],$$

which immediately implies

$$\|Sx\| \le \lambda \int_0^1 LG(s,s)b(s)f\big(\tilde{J}x(s)\big)\,ds. \tag{3.23}$$

Further, using (3.19), Lemma 2.2(c), (C2), and (3.23) successively, we find, for $t \in [\eta, 1-\eta]$,

$$
\begin{aligned}
(Sx)^{(m-q-2)}(t) &\ge \lambda \int_0^1 G(t,s)a(s)f\big(\tilde{J}x(s)\big)\,ds \\
&\ge \lambda \int_0^1 K_\eta G(s,s)a(s)f\big(\tilde{J}x(s)\big)\,ds \\
&\ge \lambda \int_0^1 K_\eta G(s,s)rb(s)f\big(\tilde{J}x(s)\big)\,ds \\
&\ge \frac{K_\eta}{L}r\|Sx\| = \gamma\|Sx\|.
\end{aligned}
$$

Hence,

$$\min_{t\in[\eta,1-\eta]}(Sx)^{(m-q-2)}(t) \ge \gamma\|Sx\|. \tag{3.24}$$

Inequalities (3.22) and (3.24) show that $Sx \in C$.

Next, we shall prove that $\|Sx\| \le M$. Noting that $\|x\| \le M$, we use (3.20), Lemma 2.2(b), and (3.21) to get

$$
\begin{aligned}
(Sx)^{(m-q-2)}(t) &\le \ell \int_0^1 G(t,s)b(s)f\big(\tilde{J}x(s)\big)\,ds \\
&\le \ell\bar{f}_M \int_0^1 LG(s,s)b(s)\,ds = M, \quad t \in [0,1],
\end{aligned}
$$

or equivalently

$$\|Sx\| \le M.$$

Hence, $S(C(M)) \subseteq C(M)$. Also, the standard arguments yield that S is completely continuous. By Schauder's fixed point theorem, S has a fixed point in $C(M)$. Clearly, this fixed point is a positive solution of (3.5) and therefore λ is an eigenvalue of (3.5). Noting that $\lambda \in (0,\ell]$ is arbitrary, it follows immediately that the interval $(0,\ell] \subseteq E$. □

Remark 3.3 From the proof of Theorem 3.5, it is clear that conditions (C1) and (C2) ensure that $S : C \to C$.

Theorem 3.6 *Let* (C1)-(C5) *hold. Suppose that* $\lambda^* \in E$. *For any* $\lambda \in (0,\lambda^*)$, *we have* $\lambda \in E$, *i.e.,* $(0,\lambda^*] \subseteq E$.

Proof Let x^* be the eigenfunction corresponding to the eigenvalue λ^*. Then we have

$$x^*(t) = Sx^*(t) = \lambda^* \int_0^1 g_{m-q}(t,s)F\big(s,\tilde{J}x^*(s)\big)\,ds, \quad t \in [0,1]. \tag{3.25}$$

Define

$$A = \left\{ x \in B \mid 0 \le x(t) \le x^*(t), t \in [0,1] \right\}.$$

Let $\lambda \in (0, \lambda^*)$ and $x \in A$. It is obvious from definition (3.3) that

$$J^k x(t) \le J^k x^*(t), \quad t \in [0,1], 1 \le k \le q.$$

Now, applying (C5) and noting (3.25), we obtain

$$0 \le Sx(t) = \lambda \int_0^1 g_{m-q}(t,s) F\big(s, \tilde{J}x(s)\big) \, ds$$

$$\le \lambda^* \int_0^1 g_{m-q}(t,s) F\big(s, \tilde{J}x^*(s)\big) \, ds$$

$$= Sx^*(t), \quad t \in [0,1].$$

This shows that the operator S maps A into A. Moreover, the operator S is continuous and completely continuous. Schauder's fixed point theorem guarantees that S has a fixed point in A which is a positive solution of (3.5). Hence, λ is an eigenvalue of (3.5), *i.e.*, $\lambda \in E$. □

The next result states that E is itself an interval.

Corollary 3.7 *Let* (C1)-(C5) *hold. If* $E \ne \emptyset$, *then* E *is an interval.*

Proof Suppose E is not an interval. Then there exist $\lambda_0, \lambda_1 \in E$ ($\lambda_0 < \lambda_1$), and $\tau \in (\lambda_0, \lambda_1)$ with $\tau \notin E$. However, this is not possible as Theorem 3.6 guarantees that $\tau \in E$. Hence, E is an interval. □

The next result gives upper and lower bounds of an eigenvalue.

Theorem 3.8 *Let* (C1)-(C4) *hold. Let* λ *be an eigenvalue of* (3.5) *and* $x \in C$ *be a corresponding eigenfunction. Further, let* $\|x\| = p$ *and* $z \in (\eta, 1 - \eta)$ *be fixed. Then*

$$\lambda \ge \frac{p}{f\big(\frac{p}{(m-q-2)!}, \frac{p}{(m-q-2)!}, \dots, \frac{p}{(m-q-2)!}\big)} \left[\int_0^1 LG(s,s)b(s) \, ds \right]^{-1}, \tag{3.26}$$

and

$$\lambda \le \frac{p}{f\big(\frac{\gamma p}{(m-2)!}(z-\eta)^{m-2}, \frac{\gamma p}{(m-3)!}(z-\eta)^{m-3}, \dots, \frac{\gamma p}{(m-q-1)!}(z-\eta)^{m-q-1}, \frac{\gamma p}{(m-q-2)!}(z-\eta)^{m-q-2}\big)}$$

$$\times \left[\int_z^{1-\eta} G(t_0,s)a(s) \, ds \right]^{-1}, \tag{3.27}$$

where t_0 *is any number in* $(0,1)$ *such that* $x^{(m-q-2)}(t_0) \ne 0$.

Proof Let $t_1 \in [0,1]$ be such that

$$p = \|x\| = x^{(m-q-2)}(t_1).$$

Then, applying (3.19), Lemma 2.2(b), (3.20), and (C4), we find

$$p = x^{(m-q-2)}(t_1) = (Sx)^{(m-q-2)}(t_1)$$

$$\leq \lambda \int_0^1 G(t_1, s) b(s) f\big(\tilde{J}x(s)\big) \, ds$$

$$\leq \lambda \int_0^1 L G(s, s) b(s) \bar{f}_p \, ds,$$

which gives (3.26) immediately.

Next, noting (3.19), (3.14), (3.12), and (C4), we get

$$p \geq x^{(m-q-2)}(t_0)$$

$$\geq \lambda \int_0^1 G(t_0, s) a(s) f\big(\tilde{J}x(s)\big) \, ds$$

$$\geq \lambda \int_z^{1-\eta} G(t_0, s) a(s) f\big(\tilde{J}x(s)\big) \, ds$$

$$\geq \lambda \int_z^{1-\eta} G(t_0, s) a(s) \, ds$$

$$\times f\bigg(\frac{\gamma p}{(m-2)!}(z-\eta)^{m-2}, \frac{\gamma p}{(m-3)!}(z-\eta)^{m-3}, \ldots, \frac{\gamma p}{(m-q-1)!}(z-\eta)^{m-q-1},$$

$$\frac{\gamma p}{(m-q-2)!}(z-\eta)^{m-q-2} \bigg),$$

from which (3.27) is immediate. $\qquad\square$

The next result gives the criteria for E to be a bounded/unbounded interval.

Theorem 3.9 *Define*

$$P_B = \bigg\{ f \ \bigg| \ \frac{u}{f(u, u, \ldots, u)} \text{ is bounded for } u \in (0, \infty) \bigg\},$$

$$P_0 = \bigg\{ f \ \bigg| \ \lim_{u \to \infty} \frac{u}{f(u, u, \ldots, u)} = 0 \bigg\},$$

$$P_\infty = \bigg\{ f \ \bigg| \ \lim_{u \to \infty} \frac{u}{f(u, u, \ldots, u)} = \infty \bigg\}.$$

(a) *Let* (C1)-(C5) *hold. If* $f \in P_B$, *then* $E = (0, \ell)$ *or* $(0, \ell]$ *for some* $\ell \in (0, \infty)$.

(b) *Let* (C1)-(C5) *hold. If* $f \in P_0$, *then* $E = (0, \ell]$ *for some* $\ell \in (0, \infty)$.

(c) *Let* (C1)-(C4) *hold. If* $f \in P_\infty$, *then* $E = (0, \infty)$.

Proof (a) This follows from (3.27) and Corollary 3.7.

(b) Since $P_0 \subseteq P_B$, we have from Case (a) that $E = (0, \ell)$ or $(0, \ell]$ for some $\ell \in (0, \infty)$. In particular,

$$\ell = \sup E.$$

Let $\{\lambda_n\}_{n=1}^{\infty}$ be a monotonically increasing sequence in E which converges to ℓ, and let $\{x_n\}_{n=1}^{\infty}$ be a corresponding sequence of eigenfunctions in the context of (3.5). Further, let $p_n = \|x_n\|$. Then (3.27) together with $f \in P_0$ implies that no subsequence of $\{p_n\}_{n=1}^{\infty}$ can diverge to infinity. Thus, there exists $M > 0$ such that $p_n \le M$ for all n. So $\{x_n\}_{n=1}^{\infty}$ is uniformly bounded. This implies that there is a subsequence of $\{x_n\}_{n=1}^{\infty}$, relabeled as the original sequence, which converges uniformly to some x, where $x(t) \ge 0$ for $t \in [0,1]$. Clearly, we have $Sx_n = x_n$, i.e.,

$$x_n(t) = \lambda_n \int_0^1 g_{m-q}(t,s)F\big(s,\tilde{J}x_n(s)\big)\,ds, \quad t \in [0,1]. \tag{3.28}$$

Since x_n converges to x and λ_n converges to ℓ, letting $n \to \infty$ in (3.28) leads to

$$x(t) = \ell \int_0^1 g_{m-q}(t,s)F\big(s,\tilde{J}x(s)\big)\,ds, \quad t \in [0,1].$$

Hence, ℓ is an eigenvalue with corresponding eigenfunction x, i.e., $\ell = \sup E \in E$. This completes the proof for Case (b).

(c) Let $\lambda > 0$ be fixed. Choose $\epsilon > 0$ so that

$$\lambda \frac{L}{(m-q-2)!} \int_0^1 G(s,s)b(s)\,ds \le \frac{1}{\epsilon}. \tag{3.29}$$

If $f \in P_\infty$, then there exists $M = M(\epsilon) > 0$ such that

$$f(u,u,\dots,u) < \epsilon u, \quad u \ge \frac{M}{(m-q-2)!}. \tag{3.30}$$

We shall show that $S(C(M)) \subseteq C(M)$. Let $x \in C(M)$. From the proof of Theorem 3.5, we have (3.22) and (3.24) and so $Sx \in C$. It remains to show that $\|Sx\| \le M$. Applying (3.19), Lemma 2.2(b), (3.20), (3.30), and (3.29), we find, for $t \in [0,1]$,

$$
\begin{aligned}
(Sx)^{(m-q-2)}(t) &\le \lambda \int_0^1 G(t,s)b(s)f\big(\tilde{J}x(s)\big)\,ds \\
&\le \lambda \int_0^1 LG(s,s)b(s)f\left(\frac{M}{(m-q-2)!}, \frac{M}{(m-q-2)!}, \dots, \frac{M}{(m-q-2)!}\right)ds \\
&\le \lambda\epsilon \frac{M}{(m-q-2)!} \int_0^1 LG(s,s)b(s)\,ds \le M.
\end{aligned}
$$

It follows that $\|Sx\| \le M$ and hence $S(C(M)) \subseteq C(M)$. Also, S is continuous and completely continuous. Schauder's fixed point theorem guarantees that S has a fixed point in $C(M)$. Clearly, this fixed point is a positive solution of (3.5) and therefore λ is an eigenvalue of (3.5). Since $\lambda > 0$ is arbitrary, it shows that $E = (0,\infty)$. □

Example 3.1 Consider the Sturm-Liouville boundary value problem

$$
\begin{cases}
y^{(5)}(t) + \lambda F(t,y(t),y'(t),y''(t),y'''(t)) = 0, \quad t \in (0,1), \\
y(0) = y'(0) = y''(0) = 0, \qquad 2y^{(3)}(0) - y^{(4)}(0) = 0, \qquad -y^{(3)}(1) + 3y^{(4)}(1) = 0,
\end{cases} \tag{3.31}
$$

where $\lambda > 0$ and

$$F(t, y, y', y'', y''') = \left[\frac{300 + 900t + 270t^2 + 90t^3 + 10t^4 - 3t^5}{200} + 2\right]^{-1/2}$$
$$\times \left(\frac{y + y' + y'' + y'''}{4} + 2\right)^{1/2}.$$

Here, $m = 5$, $q = 3$, $\zeta = 2$, $\theta = 1$, $\rho = -1$, and $\delta = 3$. Clearly, (C1)-(C5) are satisfied with

$$a(t) = b(t) = \left[\frac{300 + 900t + 270t^2 + 90t^3 + 10t^4 - 3t^5}{200} + 2\right]^{-1/2}$$

and

$$f(u_1, u_2, u_3, u_4) = \left(\frac{u_1 + u_2 + u_3 + u_4}{4} + 2\right)^{1/2}.$$

It is obvious that $f \in P_\infty$. Hence, by Theorem 3.9(c) we have $E = (0, \infty)$. In fact, when $\lambda = \frac{36}{5} \in (0, \infty)$, (3.31) has a positive solution $y(t) = t^3 + \frac{1}{2}t^4 - \frac{3}{50}t^5$.

4 Explicit eigenvalue intervals

In this section, the functions a and b appearing in (C1)-(C3) are assumed to be continuous on the closed interval $[0, 1]$. Hence, noting Remark 3.2, we shall not require conditions (C3) and (C4) to show the compactness of the operator S. With respect to the function f in (C1), we define

$$\bar{f}_0 = \limsup_{u_i \to 0, 1 \le i \le q+1} \frac{f(u_1, u_2, \ldots, u_{q+1})}{u_{q+1}}, \qquad \underline{f}_0 = \liminf_{u_i \to 0, 1 \le i \le q+1} \frac{f(u_1, u_2, \ldots, u_{q+1})}{u_{q+1}},$$

$$\bar{f}_\infty = \limsup_{u_i \to \infty, 1 \le i \le q+1} \frac{f(u_1, u_2, \ldots, u_{q+1})}{u_{q+1}}, \qquad \underline{f}_\infty = \liminf_{u_i \to \infty, 1 \le i \le q+1} \frac{f(u_1, u_2, \ldots, u_{q+1})}{u_{q+1}}.$$

Our main tool in this section is the *Krasnosel'skii fixed point theorem in a cone* (Theorem 2.1) which we shall apply with the operator S and the cone C defined in (3.16) and (3.7), respectively. Recall that $\eta \in (0, \frac{1}{2})$ is fixed. Throughout this section, we further let $z \in [\eta, 1 - \eta]$ be fixed. Define $t_z, t^* \in [0, 1]$ by

$$\int_z^{1-\eta} G(t_z, s)a(s)(s - \eta)^{m-q-2}\, ds = \sup_{t \in [0,1]} \int_z^{1-\eta} G(t, s)a(s)(s - \eta)^{m-q-2}\, ds,$$

$$\int_\eta^{1-\eta} G(t^*, s)a(s)(s - \eta)^{m-q-2}\, ds = \sup_{t \in [0,1]} \int_\eta^{1-\eta} G(t, s)a(s)(s - \eta)^{m-q-2}\, ds.$$

$$(4.1)$$

Theorem 4.1 *Let (C1)-(C3) hold. Then $\lambda \in E$ if λ satisfies*

$$\frac{1}{\underline{f}_\infty}\left[\frac{\gamma}{(m - q - 2)!}\int_z^{1-\eta} G(t_z, s)a(s)(s - \eta)^{m-q-2}\, ds\right]^{-1}$$

$$< \lambda < \frac{1}{\bar{f}_0}\left[\frac{L}{(m - q - 2)!}\int_0^1 G(s, s)b(s)s^{m-q-2}\, ds\right]^{-1}.$$

$$(4.2)$$

Proof Let λ satisfy (4.2) and let $\epsilon > 0$ be such that

$$
\frac{1}{\underline{f}_{-\infty} - \epsilon} \left[\frac{\gamma}{(m-q-2)!} \int_z^{1-\eta} G(t_z, s) a(s)(s-\eta)^{m-q-2} \, ds \right]^{-1}
$$

$$
< \lambda < \frac{1}{\overline{f}_0 + \epsilon} \left[\frac{L}{(m-q-2)!} \int_0^1 G(s,s) b(s) s^{m-q-2} \, ds \right]^{-1}. \tag{4.3}
$$

First, we choose $w > 0$ so that

$$
f(u_1, u_2, \ldots, u_{q+1}) \leq (\overline{f}_0 + \epsilon) u_{q+1}, \quad 0 < u_i \leq w, \ 1 \leq i \leq q+1. \tag{4.4}
$$

Let $x \in C$ be such that $\|x\| = w(m-q-2)!$. By (3.13) and (3.9), we get

$$
\begin{cases} x(s) \leq \frac{\|x\|}{(m-q-2)!} = w, & s \in [0,1], \\ J^k x(s) \leq \frac{\|x\|}{(m-q-2)!} = w, & s \in [0,1], \ 1 \leq k \leq q. \end{cases} \tag{4.5}
$$

Thus, together with (3.19), Lemma 2.2(b), (4.5), (4.4), (3.8), and (4.3), we find, for $t \in [0,1]$,

$$
(Sx)^{(m-q-2)}(t) \leq \lambda \int_0^1 L G(s,s) b(s) f(\tilde{J} x(s)) \, ds
$$

$$
\leq \lambda \int_0^1 L G(s,s) b(s)(\overline{f}_0 + \epsilon) x(s) \, ds
$$

$$
\leq \lambda \int_0^1 L G(s,s) b(s)(\overline{f}_0 + \epsilon) \frac{s^{m-q-2}}{(m-q-2)!} \|x\| \, ds \leq \|x\|.
$$

It follows that

$$
\|Sx\| \leq \|x\|. \tag{4.6}
$$

If we set $\Omega_1 = \{x \in B \mid \|x\| < w(m-q-2)!\}$, then (4.6) holds for $x \in C \cap \partial \Omega_1$.

Next, let $d > 0$ be such that

$$
f(u_1, u_2, \ldots, u_{q+1}) \geq (\underline{f}_{-\infty} - \epsilon) u_{q+1}, \quad u_i \geq d, \ 1 \leq i \leq q+1. \tag{4.7}
$$

Let $x \in C$ be such that

$$
\|x\| = \max \left\{ w(m-q-2)! + 1, \frac{d(m-q-2)!}{\gamma(z-\eta)^{m-q-2}}, \frac{d(m-q-2+k)!}{\gamma(z-\eta)^{m-q-2+k}}, 1 \leq k \leq q \right\}
$$

$$
= \max \left\{ w(m-q-2)! + 1, \frac{d(m-2)!}{\gamma(z-\eta)^{m-2}} \right\} \equiv D. \tag{4.8}
$$

Using (3.12), (3.14), and (4.8), we have

$$
\begin{cases} x(s) \geq \frac{(z-\eta)^{m-q-2}}{(m-q-2)!} \gamma \|x\| \geq d, & s \in [z, 1-\eta], \\ J^k x(s) \geq \frac{(z-\eta)^{m-q-2+k}}{(m-q-2+k)!} \gamma \|x\| \geq d, & s \in [z, 1-\eta], \ 1 \leq k \leq q. \end{cases} \tag{4.9}
$$

Now, applying (3.19), (4.9), (4.7), and (3.11), we find, for $t \in [0,1]$,

$$(Sx)^{(m-q-2)}(t) \geq \lambda \int_z^{1-\eta} G(t,s)a(s)f\big(\tilde{J}x(s)\big)\,ds$$

$$\geq \lambda \int_z^{1-\eta} G(t,s)a(s)(\underline{f}_\infty - \epsilon)x(s)\,ds$$

$$\geq \lambda \int_z^{1-\eta} G(t,s)a(s)(\underline{f}_\infty - \epsilon)(s-\eta)^{m-q-2}\frac{\gamma \|x\|}{(m-q-2)!}\,ds.$$

Taking the supremum on both sides leads to

$$\|Sx\| \geq \lambda \int_z^{1-\eta} G(t_z,s)a(s)(s-\eta)^{m-q-2}\,ds \cdot (\underline{f}_\infty - \epsilon)\frac{\gamma \|x\|}{(m-q-2)!} \geq \|x\|,$$

where t_z is defined in (4.1), and (4.3) has been used in the last inequality. If we set $\Omega_2 = \{x \in B \mid \|x\| < D\}$, then for $x \in C \cap \partial\Omega_2$ we have

$$\|Sx\| \geq \|x\|. \tag{4.10}$$

With (4.6) and (4.10) established and also noting that S maps C into C (Remark 3.3), by Theorem 2.1 S has a fixed point $x \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $w(m-q-2)! \leq \|x\| \leq D$. Obviously, this x is a positive solution of (3.5) and hence $\lambda \in E$. $\qquad\square$

Theorem 4.2 *Let* (C1)-(C3) *hold. Then* $\lambda \in E$ *if* λ *satisfies*

$$\frac{1}{\underline{f}_0}\left[\frac{\gamma}{(m-q-2)!}\int_\eta^{1-\eta} G(t^*,s)a(s)(s-\eta)^{m-q-2}\,ds\right]^{-1}$$

$$< \lambda < \frac{1}{\overline{f}_\infty}\left[\frac{L}{(m-q-2)!}\int_0^1 G(s,s)b(s)\,ds\right]^{-1}. \tag{4.11}$$

Proof Let λ satisfy (4.11) and let $\epsilon > 0$ be such that

$$\frac{1}{\underline{f}_0 - \epsilon}\left[\frac{\gamma}{(m-q-2)!}\int_\eta^{1-\eta} G(t^*,s)a(s)(s-\eta)^{m-q-2}\,ds\right]^{-1}$$

$$< \lambda < \frac{1}{\overline{f}_\infty + \epsilon}\left[\frac{L}{(m-q-2)!}\int_0^1 G(s,s)b(s)\,ds\right]^{-1}. \tag{4.12}$$

First, we pick $w > 0$ so that

$$f(u_1, u_2, \ldots, u_{q+1}) \geq (\underline{f}_0 - \epsilon)u_{q+1}, \quad 0 < u_i \leq w, \; 1 \leq i \leq q+1. \tag{4.13}$$

Let $x \in C$ be such that $\|x\| = w(m-q-2)!$. As in (4.5), we have $x(s) \leq w$ and $J^k x(s) \leq w$, $1 \leq k \leq q$ for $s \in [0,1]$. Hence, applying (3.19), (4.13), and (3.11), we find, for $t \in [0,1]$,

$$(Sx)^{(m-q-2)}(t) \geq \lambda \int_0^1 G(t,s)a(s)f\big(\tilde{J}x(s)\big)\,ds \geq \lambda \int_0^1 G(t,s)a(s)(\underline{f}_0 - \epsilon)x(s)\,ds$$

$$\geq \lambda \int_\eta^{1-\eta} G(t,s)a(s)(\underline{f}_0 - \epsilon)(s-\eta)^{m-q-2}\frac{\gamma \|x\|}{(m-q-2)!}\,ds.$$

Taking the supremum on both sides and using (4.12) lead to

$$\|Sx\| \geq \lambda \int_\eta^{1-\eta} G(t^*,s)a(s)(s-\eta)^{m-q-2}\,ds \cdot (\underline{f}_0 - \epsilon)\frac{\gamma\|x\|}{(m-q-2)!} \geq \|x\|,$$

where t^* is defined in (4.1). Hence, if we set $\Omega_1 = \{x \in B \mid \|x\| < w(m-q-2)!\}$, then $\|Sx\| \geq \|x\|$ holds for $x \in C \cap \partial\Omega_1$.

Next, let $d > 0$ be such that

$$f(u_1, u_2, \ldots, u_{q+1}) \leq (\overline{f}_\infty + \epsilon)u_{q+1}, \quad u_i \geq d,\ 1 \leq i \leq q+1. \tag{4.14}$$

We shall consider two cases - when f is bounded and when f is unbounded.

Case 1. Suppose that f is bounded. Then there exists a positive constant M such that

$$f(u_1, u_2, \ldots, u_{q+1}) \leq M, \quad u_i \in (0, \infty),\ 1 \leq i \leq q+1. \tag{4.15}$$

Let

$$D = \max\left\{ w+1, \frac{\lambda M}{(m-q-2)!}\int_0^1 LG(s,s)b(s)\,ds \right\}$$

and let $x \in C$ be such that $\|x\| = D(m-q-2)!$. Using (3.19), Lemma 2.2(b), and (4.15), we have, for $t \in [0,1]$,

$$(Sx)^{(m-q-2)}(t) \leq \lambda \int_0^1 LG(s,s)b(s)f\big(\tilde{J}x(s)\big)\,ds$$

$$\leq \lambda \int_0^1 LG(s,s)b(s)M\,ds \leq D(m-q-2)! = \|x\|.$$

Hence, $\|Sx\| \leq \|x\|$ holds.

Case 2. Suppose that f is unbounded. Then there exists $D > \max\{w+1, d\}$ such that

$$f(u_1, u_2, \ldots, u_{q+1}) \leq f(D, D, \ldots, D), \quad 0 < u_i \leq D,\ 1 \leq i \leq q+1. \tag{4.16}$$

Let $x \in C$ be such that $\|x\| = D(m-q-2)!$. As in (4.5), we have $x(s) \leq D$ and $J^k x(s) \leq D$, $1 \leq k \leq q$ for $s \in [0,1]$. Now, using (3.19), Lemma 2.2(b), (4.16), (4.14), and (4.12), we get, for $t \in [0,1]$,

$$(Sx)^{(m-q-2)}(t) \leq \lambda \int_0^1 LG(s,s)b(s)f\big(\tilde{J}x(s)\big)\,ds$$

$$\leq \lambda \int_0^1 LG(s,s)b(s)f(D,D,\ldots,D)\,ds$$

$$\leq \lambda \int_0^1 LG(s,s)b(s)(\overline{f}_\infty + \epsilon)D\,ds$$

$$= \lambda \int_0^1 LG(s,s)b(s)(\overline{f}_\infty + \epsilon)\frac{\|x\|}{(m-q-2)!}\,ds \leq \|x\|.$$

Hence, immediately we have $\|Sx\| \leq \|x\|$.

In both Case 1 and 2, if we set $\Omega_2 = \{x \in B \mid \|x\| < D(m - q - 2)!\}$, then $\|Sx\| \le \|x\|$ holds for $x \in C \cap \partial\Omega_2$.

With (4.6) and (4.10) established and also noting that S maps C into C (Remark 3.3), it follows from Theorem 2.1 that S has a fixed point $x \in C \cap (\bar{\Omega}_2 \setminus \Omega_1)$ such that $w(m - q - 2)! \le \|x\| \le D(m - q - 2)!$. Clearly, this x is a positive solution of (3.5) and hence $\lambda \in E$. □

Theorems 4.1 and 4.2 provide explicit eigenvalue intervals as follows.

Corollary 4.3 *Let* (C1)-(C3) *hold. Then*

$$
E \supseteq \left(\frac{1}{\underline{f}_\infty} \left[\frac{\gamma}{(m - q - 2)!} \int_z^{1-\eta} G(t_z, s) a(s)(s - \eta)^{m-q-2} \, ds \right]^{-1}, \right.
$$

$$
\left. \frac{1}{\overline{f}_0} \left[\frac{L}{(m - q - 2)!} \int_0^1 G(s, s) b(s) s^{m-q-2} \, ds \right]^{-1} \right)
$$

$$
\supseteq \left(\frac{1}{\underline{f}_\infty} \left[\frac{\gamma}{(m - q - 2)!} \int_z^{1-\eta} K_\eta G(s, s) a(s)(s - \eta)^{m-q-2} \, ds \right]^{-1}, \right.
$$

$$
\left. \frac{1}{\overline{f}_0} \left[\frac{L}{(m - q - 2)!} \int_0^1 G(s, s) b(s) s^{m-q-2} \, ds \right]^{-1} \right)
$$

and

$$
E \supseteq \left(\frac{1}{\underline{f}_0} \left[\frac{\gamma}{(m - q - 2)!} \int_\eta^{1-\eta} G(t^*, s) a(s)(s - \eta)^{m-q-2} \, ds \right]^{-1}, \right.
$$

$$
\left. \frac{1}{\overline{f}_\infty} \left[\frac{L}{(m - q - 2)!} \int_0^1 G(s, s) b(s) \, ds \right]^{-1} \right)
$$

$$
\supseteq \left(\frac{1}{\underline{f}_0} \left[\frac{\gamma}{(m - q - 2)!} \int_\eta^{1-\eta} K_\eta G(s, s) a(s)(s - \eta)^{m-q-2} \, ds \right]^{-1}, \right.
$$

$$
\left. \frac{1}{\overline{f}_\infty} \left[\frac{L}{(m - q - 2)!} \int_0^1 G(s, s) b(s) \, ds \right]^{-1} \right).
$$

Proof We apply Theorems 4.1 and 4.2. Moreover, using Lemma 2.2(c) we have

$$
\int_z^{1-\eta} G(t_z, s) a(s)(s - \eta)^{m-q-2} \, ds \ge \sup_{t \in [\eta, 1-\eta]} \int_z^{1-\eta} G(t, s) a(s)(s - \eta)^{m-q-2} \, ds
$$

$$
\ge \int_z^{1-\eta} K_\eta G(s, s) a(s)(s - \eta)^{m-q-2} \, ds
$$

and

$$
\int_\eta^{1-\eta} G(t^*, s) a(s)(s - \eta)^{m-q-2} \, ds \ge \sup_{t \in [\eta, 1-\eta]} \int_\eta^{1-\eta} G(t, s) a(s)(s - \eta)^{m-q-2} \, ds
$$

$$
\ge \int_\eta^{1-\eta} K_\eta G(s, s) a(s)(s - \eta)^{m-q-2} \, ds,
$$

from which we are able to avoid the calculations of t_z and t^*, amid getting smaller intervals. □

The function f is said to be *superlinear* if $\overline{f}_0 = 0$ and $\underline{f}_\infty = \infty$; and f is said to be *sublinear* if $\underline{f}_0 = \infty$ and $\overline{f}_\infty = 0$. The next result is immediate from Corollary 4.3.

Corollary 4.4 *Let* (C1)-(C3) *hold. If f is superlinear or sublinear, then $E = (0, \infty)$, i.e., the boundary value problem* (3.5) *(or* (1.1)*) has a positive solution for any $\lambda > 0$.*

Example 4.1 Consider the Sturm-Liouville boundary value problem

$$\begin{cases} y^{(4)}(t) + \lambda F(t, y(t), y'(t), y''(t)) = 0, & t \in (0, 1), \\ y(0) = y'(0) = 0, \qquad y''(1) = 0, \qquad -y''(0) - 3y^{(3)}(0) = 0, \end{cases} \tag{4.17}$$

where $\lambda > 0$. Here, $m = 4$, $q = 2$, $\zeta = -1$, $\theta = 3$, $\rho = 1$, and $\delta = 0$. Let $\eta = \frac{1}{4}$. By direct computation, we have

$$L = \frac{3}{2}, \qquad K_{\frac{1}{4}} = \frac{1}{4}, \qquad \gamma = \frac{1}{6}, \qquad G(s, s) = \frac{1}{2}(3 - s)(1 - s).$$

Below we shall consider three different F's.

Case 1.

$$F(t, y, y', y'') = \left\{ \frac{90 - 30t - 51t^2 - t^3 - t^4}{100} + 1 - \exp\left[-100\left(18t - 3t^2 - 4t^3\right)\right] \right\}^{-1}$$
$$\times \left(\frac{y}{100} + \frac{y''}{20} + 1 - e^{-100y'} \right). \tag{4.18}$$

Clearly, (C1)-(C3) are satisfied with

$$a(t) = b(t) = \left\{ \frac{90 - 30t - 51t^2 - t^3 - t^4}{100} + 1 - \exp\left[-100\left(18t - 3t^2 - 4t^3\right)\right] \right\}^{-1}$$

and

$$f(u_1, u_2, u_3) = \frac{u_1}{100} + \frac{u_3}{20} + 1 - e^{-100u_2}.$$

By direct computation, we have

$$\underline{f}_0 = \frac{5,003}{50}, \qquad \overline{f}_\infty = \frac{3}{50}.$$

It follows from Corollary 4.3 that

$$E \supseteq \left(\frac{1}{\underline{f}_0} \left[\gamma K_{\frac{1}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G(s, s) a(s) \, ds \right]^{-1}, \frac{1}{\overline{f}_\infty} \left[L \int_0^1 G(s, s) b(s) \, ds \right]^{-1} \right) = (1.24, 28.48).$$

In fact, we note that when $\lambda = 24 \in (1.24, 28.48)$, the problem (4.17), (4.18) has a positive solution given by $y(t) = 9t^2 - t^3 - t^4$.

Case 2.

$$F(t, y, y', y'') = \left(\frac{126 + 6t - 18t^2 - 5t^3 - t^4}{10} \right)^{-1} \left(\frac{y + y' + 2y''}{10} + 9 \right). \tag{4.19}$$

Here, (C1)-(C3) are satisfied with

$$a(t) = b(t) = \left(\frac{126 + 6t - 18t^2 - 5t^3 - t^4}{10} \right)^{-1}$$

and

$$f(u_1, u_2, u_3) = \frac{u_1 + u_2 + 2u_3}{10} + 9.$$

Direct computation also provides

$$\underline{f}_0 = \infty, \qquad \overline{f}_\infty = \frac{2}{5}.$$

By Corollary 4.3, we have

$$E \supseteq \left(0, \frac{1}{\overline{f}_\infty} \left[L \int_0^1 G(s,s) b(s)\, ds \right]^{-1} \right) = (0, 31.15).$$

Indeed, when $\lambda = 24 \in (0, 31.15)$, the problem (4.17), (4.19) has a positive solution given by $y(t) = 9t^2 - t^3 - t^4$.

Case 3.

$$F(t, y, y', y'') = \left[\exp\left(-18 - 12t + 6t^2 + 5t^3 + t^4 \right) + 2 \right]^{-1} \left[\exp\left(-y - y' - y'' \right) + 2 \right]. \qquad (4.20)$$

In this case, (C1)-(C3) are satisfied with

$$a(t) = b(t) = \left[\exp\left(-18 - 12t + 6t^2 + 5t^3 + t^4 \right) + 2 \right]^{-1}$$

and

$$f(u_1, u_2, u_3) = \exp(-u_1 - u_2 - u_3) + 2.$$

We check that f is sublinear, *i.e.*, $\underline{f}_0 = \infty$ and $\overline{f}_\infty = 0$. By Corollary 4.4, the set $E = (0, \infty)$. As an example, when $\lambda = 24 \in (0, \infty)$, the problem (4.17), (4.20) has a positive solution given by $y(t) = 9t^2 - t^3 - t^4$.

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