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Solvability of *n*th-order Lipschitz equations with nonlinear three-point boundary conditions

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Abstract

In this paper, we investigate the solvability of nth-order Lipschitz equations $y^{(n)} = f(x,y,y',\ldots,y^{(n-1)})$, $x_1 \le x \le x_3$, with nonlinear three-point boundary conditions of the form $k(y(x_2),y'(x_2),\ldots,y^{(n-1)}(x_2);y(x_1),y'(x_1),\ldots,y^{(n-1)}(x_1)) = 0$, $g_i(y^{(i)}(x_2),y^{(i+1)}(x_2),\ldots,y^{(n-1)}(x_2)) = 0$, $i = 0,1,\ldots,n-3$, $h(y(x_2),y'(x_2),\ldots,y^{(n-1)}(x_2);$ $y(x_3),y'(x_3),\ldots,y^{(n-1)}(x_3)) = 0$, where $n \ge 3$, $x_1 < x_2 < x_3$. By using the matching technique together with set-valued function theory, the existence and uniqueness of solutions for the problems are obtained. Meanwhile, as an application of our results, an example is given.

MSC: 34B10; 34B15

Keywords: *n*th-order Lipschitz equation; nonlinear three-point boundary value problem; matching method; existence; uniqueness

1 Introduction

As is well known, the differential equations with right hand sides satisfying the Lipschitz conditions (Lipschitz equations for short) are important, and thus their solvability has attracted much attention from many researchers. Among a substantial number of works dealing with higher order Lipschitz equations with three-point boundary conditions, we mention [1–14] and references therein. Most of these results are obtained via applying control theory methods (Pontryagin maximum principle), matching methods, and topological degree methods *etc.* To the best of our knowledge, most of the three-point boundary conditions in the above mentioned references are limited to simple boundary conditions.

In 1973, Barr and Sherman [2] showed by the matching technique that the third-order three-point boundary value problem

$$\begin{cases} y''' = f(x,y,y',y''), & x_1 \le x \le x_3, \\ y^{(\alpha)}(x_1) = y_1, \quad y(x_2) = y_2, \quad y^{(\beta)}(x_3) = y_3 \end{cases} \tag{$*$}$$

with $\alpha = \beta = 0$ has a unique solution, under the following four conditions:

(A) $f(x,y,y',y'')$ is continuous on $[x_1,x_3] \times \mathbb{R}^3$;

(B) $f(x,y,y',y'')$ satisfies the monotonicity conditions, *i.e.*, $y_1 \geq y_2$, $z_1 < z_2$ implies

$$f(x,y_1,z_1,w) < f(x,y_2,z_2,w) \quad \text{on } (x_1,x_2],$$

and $y_1 \leq y_2$, $z_1 < z_2$ implies

$$f(x,y_1,z_1,w) < f(x,y_2,z_2,w) \quad \text{on } [x_2,x_3);$$

(C) for any $(x,y_1,z_1,w_1), (x,y_2,z_2,w_2) \in [x_1,x_3] \times \mathbb{R}^3$,

$$\bigl|f(x,y_1,z_1,w_1) - f(x,y_2,z_2,w_2)\bigr| \leq L_0|y_1-y_2| + L_1|z_1-z_2| + L_2|w_1-w_2|,$$

where L_0, L_1, and L_2 are nonnegative constants;

(D_1) for each $i=1,2$,

$$\frac{\sqrt{3}}{27}L_0h_i^3 + \frac{1}{3}L_1h_i^2 + L_2h_i < 1,$$

where $h_i = x_{i+1} - x_i$, $i = 1,2$.

In 1978, Moorti and Garner [12] by using the matching technique showed that BVP $(*)$ with $\alpha, \beta \in \{0,1\}$ and $\alpha + \beta \neq 0$ has a unique solution, under the conditions (A), (B), (C), and

(D_2) for each $i=1,2$,

$$\frac{1}{3}L_0h_i^3 + \frac{1}{2}L_1h_i^2 + L_2h_i < 1.$$

Since then, many authors improved the condition (D_i), $i=1,2$. For example, in [4], Das and Lalli proved that BVP $(*)$ with $\alpha = \beta = 0$ has a unique solution, under the conditions of (A), (B), (C), and

(D_3) for each $i=1,2$,

$$\frac{1}{60}L_0h_i^3 + \frac{1}{6}L_1h_i^2 + \frac{2}{3}L_2h_i < 1.$$

In [1], Agarwal showed that BVP $(*)$ with $\alpha = \beta = 0$ has a unique solution, under the conditions of (A), (B), (C), and

(D_4) for each $i=1,2$,

$$\frac{3}{160}L_0h_i^3 + \frac{33}{320}L_1h_i^2 + \frac{3}{8}L_2h_i < 1.$$

In [14], Piao and Shi generalized the above results. They not only generalized the simple three boundary conditions to the nonlinear boundary conditions, but also they weakened the monotonicity condition (B) and removed the restriction (D_i) on the length of the interval.

Recently, Pei and Chang [13] generalized the results of Piao and Shi [14].

The purpose of this paper is to study the solvability of nth-order Lipschitz equations with more general nonlinear three-point boundary conditions of the form ($n \geq 3$)

$$y^{(n)} = f\bigl(x,y,y',\ldots,y^{(n-1)}\bigr), \quad x_1 \leq x \leq x_3, \tag{1.1}$$

$$\begin{cases} k(y(x_2), y'(x_2), \ldots, y^{(n-1)}(x_2); y(x_1), y'(x_1), \ldots, y^{(n-1)}(x_1)) = 0, \\ g_i(y^{(i)}(x_2), y^{(i+1)}(x_2), \ldots, y^{(n-1)}(x_2)) = 0, \quad i = 0, 1, \ldots, n-3, \\ h(y(x_2), y'(x_2), \ldots, y^{(n-1)}(x_2); y(x_3), y'(x_3), \ldots, y^{(n-1)}(x_3)) = 0, \end{cases} \tag{1.2}$$

where $-\infty < x_1 < x_2 < x_3 < +\infty$.

The paper is organized as follows. In Section 2, as a preliminary, we state some useful results as regards the solvability for the nth-order Lipschitz equation with the nonlinear two-point boundary conditions and a lemma of the differential inequality for nth-order differential equations. In Section 3, by using the matching technique together with set-valued function theory and nested interval theorem, we establish the existence and uniqueness theorems of solutions for BVP (1.1), (1.2). Our results improve and generalize widely the results of [1, 2, 4, 12–14].

We remark that the matching technique used in this paper is different from the classical one. In fact, by using the classical matching technique to obtain a matching solution of a three-point boundary value problem, it needs usually four two-point boundary value problems and among them two two-point boundary value problems need to have unique solutions, the other two two-point boundary value problems need to have at most one solution. However, our matching technique needs only two two-point boundary value problems and each of them needs to have at least one solution. For more about the three-point boundary value problems, we refer the readers to the references [15–19], with matching techniques, and to [20–35], with other techniques.

Throughout this paper, we make the following assumptions:

($\bar{\mathrm{H}}_1$) $f(x, y_0, y_1, \ldots, y_{n-1})$ is continuous on $[x_1, x_3] \times \mathbb{R}^n$;

($\bar{\mathrm{H}}_2$) If $x \in [x_2, x_3]$ and $y_i \le \bar{y}_i$, $i = 0, 1, \ldots, n-2$, then

$$f(x, y_0, y_1, \ldots, y_{n-2}, y_{n-1}) \le f(x, \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_{n-2}, y_{n-1}).$$

Also if $x \in [x_1, x_2]$ and $(-1)^{n+i} y_i \le (-1)^{n+i} \bar{y}_i$, $i = 0, 1, \ldots, n-2$, then

$$f(x, y_0, y_1, \ldots, y_{n-2}, y_{n-1}) \le f(x, \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_{n-2}, y_{n-1});$$

($\bar{\mathrm{H}}_3$) For any $(x, y_0, y_1, \ldots, y_{n-1}), (x, \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_{n-1}) \in [x_1, x_3] \times \mathbb{R}^n$,

$$\left| f(x, y_0, y_1, \ldots, y_{n-1}) - f(x, \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_{n-1}) \right| \le \sum_{i=0}^{n-1} L_i |y_i - \bar{y}_i|,$$

where L_i, $i = 0, 1, \ldots, n-1$, are nonnegative Lipschitz constants;

($\bar{\mathrm{H}}_4$) $g_i(y_i, y_{i+1}, \ldots, y_{n-1})$, $i = 0, 1, \ldots, n-3$, are continuously differentiable on $\mathbb{R}^{n-i}$, $\frac{\partial g_i}{\partial y_i} \ge \delta > 0$, $\frac{\partial g_i}{\partial y_j} \le 0$, $i = 0, 1, \ldots, n-3$, $j = i+1, i+2, \ldots, n-1$, on $\mathbb{R}^{n-i}$, and for any bounded set $D_i \subset \mathbb{R}^{n-i-1}$, $i = 0, 1, \ldots, n-3$, the functions $\frac{\partial g_i}{\partial y_j}$, $j = i+1, i+2, \ldots, n-1$, are bounded on $\mathbb{R} \times D_i$;

($\bar{\mathrm{H}}_5$) The functions $h(y_0, y_1, \ldots, y_{n-1}; z_0, z_1, \ldots, z_{n-1})$, $k(y_0, y_1, \ldots, y_{n-1}; z_0, z_1, \ldots, z_{n-1})$ are continuously differentiable on $\mathbb{R}^{2n}$, and for each $i = 0, 1, \ldots, n-1$, $\frac{\partial h}{\partial y_i} \ge 0$, $\frac{\partial h}{\partial z_i} \ge 0$, $(-1)^{n+i} \frac{\partial k}{\partial y_i} \ge 0$, $(-1)^{n+i} \frac{\partial k}{\partial z_i} \ge 0$ on $\mathbb{R}^{2n}$;

($\bar{\mathrm{H}}_6$) $\sum_{i=0}^{n-2} \frac{\partial h}{\partial z_i} \ge \delta > 0$, $\sum_{i=0}^{n-1} (-1)^{n+i} \frac{\partial k}{\partial z_i} \ge \delta > 0$ on $\mathbb{R}^{2n}$;

($\bar{\mathrm{H}}'_6$) $\sum_{i=0}^{n-1} \frac{\partial h}{\partial z_i} \ge \delta > 0$, $\sum_{i=0}^{n-2} (-1)^{n+i} \frac{\partial k}{\partial z_i} \ge \delta > 0$ on $\mathbb{R}^{2n}$;

($\bar{\mathrm{H}}_7$) $\frac{\partial h}{\partial y_{n-1}} + \sum_{i=0}^{n-1} \frac{\partial h}{\partial z_i} \geq \delta > 0$, $\frac{\partial h}{\partial y_{n-2}} + \sum_{i=0}^{n-2} \frac{\partial h}{\partial z_i} \geq \delta > 0$, $-\frac{\partial k}{\partial y_{n-1}} + \sum_{i=0}^{n-1} (-1)^{n+i} \frac{\partial k}{\partial z_i} \geq \delta > 0$ on $\mathbb{R}^{2n}$;

($\bar{\mathrm{H}}'_7$) $\frac{\partial h}{\partial y_{n-1}} + \sum_{i=0}^{n-1} \frac{\partial h}{\partial z_i} \geq \delta > 0$, $\frac{\partial k}{\partial y_{n-2}} + \sum_{i=0}^{n-2} (-1)^{n+i} \frac{\partial k}{\partial z_i} \geq \delta > 0$, $-\frac{\partial k}{\partial y_{n-1}} + \sum_{i=0}^{n-1} (-1)^{n+i} \frac{\partial k}{\partial z_i} \geq \delta > 0$ on $\mathbb{R}^{2n}$.

In the above conditions, δ denotes a constant.

2 Preliminary results

In this section, we introduce some lemmas which will be useful in the proof of our main results.

Consider the following nonlinear two-point boundary value problems for the nth-order differential equation ($n \geq 3$):

$$y^{(n)} = f\left(x, y, y', \dots, y^{(n-1)}\right), \quad a \leq x \leq b, \tag{2.1}$$

with nonlinear two-point boundary conditions

$$\begin{cases} g_i(y^{(i)}(a), y^{(i+1)}(a), \dots, y^{(n-1)}(a)) = 0, & i = 0, 1, \dots, n-2, \\ h(y(a), y'(a), \dots, y^{(n-1)}(a); y(b), y'(b), \dots, y^{(n-1)}(b)) = 0, \end{cases} \tag{2.2}$$

where $-\infty < a < b < +\infty$.

Let us list the following conditions for convenience.

(H_1) $f(x, y_0, y_1, \dots, y_{n-1})$ is continuous on $[a, b] \times \mathbb{R}^n$;

(H_2) for any $(x, y_0, \dots, y_{n-2}, y_{n-1}), (x, \bar{y}_0, \dots, \bar{y}_{n-2}, y_{n-1}) \in [a, b] \times \mathbb{R}^n$, if $y_i \leq \bar{y}_i$, $i = 0, 1, \dots, n-2$, then

$$f(x, y_0, \dots, y_{n-2}, y_{n-1}) \leq f(x, \bar{y}_0, \dots, \bar{y}_{n-2}, y_{n-1});$$

(H_3) for any $(x, y_0, y_1, \dots, y_{n-1}), (x, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_{n-1}) \in [a, b] \times \mathbb{R}^n$,

$$\left|f(x, y_0, y_1, \dots, y_{n-1}) - f(x, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_{n-1})\right| \leq \sum_{i=0}^{n-1} L_i |y_i - \bar{y}_i|,$$

where L_i, $i = 0, 1, \dots, n-1$, are nonnegative constants;

(H'_3) for any $(x, y_0, \dots, y_{n-2}, y_{n-1}), (x, y_0, \dots, y_{n-2}, \bar{y}_{n-1}) \in [a, b] \times \mathbb{R}^n$,

$$\left|f(x, y_0, \dots, y_{n-2}, y_{n-1}) - f(x, y_0, \dots, y_{n-2}, \bar{y}_{n-1})\right| \leq L_{n-1} |y_{n-1} - \bar{y}_{n-1}|,$$

where L_{n-1} is a nonnegative constant;

(H_4) $g_i(y_i, y_{i+1}, \dots, y_{n-1})$, $i = 0, 1, \dots, n-2$, are continuously differentiable on $\mathbb{R}^{n-i}$ and $h(y_0, y_1, \dots, y_{n-1}; z_0, z_1, \dots, z_{n-1})$ is continuously differentiable on $\mathbb{R}^{2n}$;

(H_5) $\frac{\partial g_i}{\partial y_i} \geq \delta > 0$, $i = 0, 1, \dots, n-2$ on $\mathbb{R}^{n-i}$, $\frac{\partial g_i}{\partial y_j} \leq 0$, $i = 0, 1, \dots, n-2$, $j = i+1, i+2, \dots, n-1$ on $\mathbb{R}^{n-i}$;

(H'_5) $\frac{\partial g_i}{\partial y_i} \geq \delta > 0$, $i = 0, 1, \dots, n-3$ on $\mathbb{R}^{n-i}$, $\frac{\partial g_{n-2}}{\partial y_{n-2}} \geq 0$ on $\mathbb{R}^2$, $\frac{\partial g_i}{\partial y_j} \leq 0$, $i = 0, 1, \dots, n-3$, $j = i+1, i+2, \dots, n-1$ on $\mathbb{R}^{n-i}$, $\frac{\partial g_{n-2}}{\partial y_{n-1}} \leq -\delta$ on $\mathbb{R}^2$;

(H_6) $\frac{\partial h}{\partial y_i} \geq 0$, $i = 0, 1, \dots, n-1$ on $\mathbb{R}^{2n}$;

(H_7) $\frac{\partial h}{\partial z_i} \geq 0$, $i = 0,1,\ldots,n-1$, $\sum_{i=0}^{n-1} \frac{\partial h}{\partial z_i} \geq \delta > 0$ on $\mathbb{R}^{2n}$;
(H_7') $\frac{\partial h}{\partial z_i} \geq 0$, $i = 0,1,\ldots,n-1$, $\sum_{i=0}^{n-2} \frac{\partial h}{\partial z_i} \geq \delta > 0$ on $\mathbb{R}^{2n}$;
(H_8) $\frac{\partial h}{\partial y_i} \geq 0$, $\frac{\partial h}{\partial z_i} \geq 0$, $i = 0,1,\ldots,n-1$, $\frac{\partial h}{\partial y_{n-1}} + \sum_{i=0}^{n-1} \frac{\partial h}{\partial z_i} \geq \delta > 0$ on $\mathbb{R}^{2n}$;
(H_8') $\frac{\partial h}{\partial y_i} \geq 0$, $\frac{\partial h}{\partial z_i} \geq 0$, $i = 0,1,\ldots,n-1$, $\frac{\partial h}{\partial y_{n-2}} + \sum_{i=0}^{n-2} \frac{\partial h}{\partial z_i} \geq \delta > 0$ on $\mathbb{R}^{2n}$.
In the above conditions, δ denotes a constant.

Now we recall the results [36] of the existence and uniqueness of solutions for BVP (2.1), (2.2) and a lemma for a differential inequality for differential equation (2.1) of the nth order.

Lemma 2.1 (See [36, Theorem 3.1]) *Assume that* (H_1), (H_2), (H_3), (H_4), (H_5), *and* (H_8) *hold*. *Then BVP* (2.1), (2.2) *has at least one solution*.

Lemma 2.2 (See [36, Theorem 3.2]) *Assume that* (H_1), (H_2), (H_3), (H_4), (H_5'), *and* (H_8') *hold*. *Then BVP* (2.1), (2.2) *has at least one solution*.

Lemma 2.3 (See [36, Theorem 3.3]) *Assume that* (H_1), (H_2), (H_3), (H_4), (H_5), (H_6), *and* (H_7) *hold*. *Then BVP* (2.1), (2.2) *has exactly one solution*.

Lemma 2.4 (See [36, Theorem 3.4]) *Assume that* (H_1), (H_2), (H_3), (H_4), (H_5'), (H_6), *and* (H_7') *hold*. *Then BVP* (2.1), (2.2) *has exactly one solution*.

Lemma 2.5 (See [36, Lemma 2.4]) *Assume that* (H_1), (H_2), *and* (H_3') *hold*. *Let* $\phi_1(x)$, $\phi_2(x)$ *be solutions of the differential equation* (2.1) *on some interval* $[a_1,b_1) \subset [a,b]$ *satisfying*

$$\phi_1^{(i)}(a_1) \leq \phi_2^{(i)}(a_1), \quad i = 0,1,\ldots,n-1,$$

and

$$\phi_1^{(n-2)}(a_1) + \phi_1^{(n-1)}(a_1) < \phi_2^{(n-2)}(a_1) + \phi_2^{(n-1)}(a_1).$$

Then $\phi_1^{(n-1)}(x) \leq \phi_2^{(n-1)}(x)$ *for* $x \in [a_1,b_1)$.

3 Main results

In order to obtain the existence and uniqueness of solutions for BVP (1.1), (1.2) by using the matching technique, we need first to discuss the existence and uniqueness of solutions for the nth-order Lipschitz equation (1.1) with one of the following sets of two-point boundary conditions:

$$\begin{cases} g_i(y^{(i)}(x_2), y^{(i+1)}(x_2), \ldots, y^{(n-1)}(x_2)) = 0, \quad i = 0,1,\ldots,n-3, \\ y^{(n-2)}(x_2) = \mu, \\ h(y(x_2), y'(x_2), \ldots, y^{(n-1)}(x_2); y(x_3), y'(x_3), \ldots, y^{(n-1)}(x_3)) = 0, \end{cases} \tag{3.1}$$

$$\begin{cases} k(y(x_2), y'(x_2), \ldots, y^{(n-1)}(x_2); y(x_1), y'(x_1), \ldots, y^{(n-1)}(x_1)) = 0, \\ g_i(y^{(i)}(x_2), y^{(i+1)}(x_2), \ldots, y^{(n-1)}(x_2)) = 0, \quad i = 0,1,\ldots,n-3, \\ y^{(n-2)}(x_2) = \mu, \end{cases} \tag{3.2}$$

$$\begin{cases} g_i(y^{(i)}(x_2), y^{(i+1)}(x_2), \ldots, y^{(n-1)}(x_2)) = 0, \quad i = 0,1,\ldots,n-3, \\ y^{(n-1)}(x_2) = \mu, \\ h(y(x_2), y'(x_2), \ldots, y^{(n-1)}(x_2); y(x_3), y'(x_3), \ldots, y^{(n-1)}(x_3)) = 0, \end{cases} \tag{3.3}$$

where $\mu \in \mathbb{R} = (-\infty, +\infty)$.

Let $x = -t$ and $y(x) = (-1)^n z(t)$. Then BVP (1.1), (3.2) becomes an equivalent boundary value problem:

$$z^{(n)} = F\big(t, z, z', \ldots, z^{(n-1)}\big), \tag{1.1'}$$

$$\begin{cases} G_i(z^{(i)}(-x_2), z^{(i+1)}(-x_2), \ldots, z^{(n-1)}(-x_2)) = 0, \quad i = 0, 1, \ldots, n-3, \\ z^{(n-2)}(-x_2) = \mu, \\ H(z(-x_2), \ldots, z^{(n-1)}(-x_2); z(-x_1), \ldots, z^{(n-1)}(-x_1)) = 0, \end{cases} \tag{3.2'}$$

where

$$\begin{aligned} &F(t, y_0, y_1, \ldots, y_{n-1}) = f\big(-t, (-1)^n y_0, (-1)^{n+1} y_1, \ldots, (-1)^{2n-1} y_{n-1}\big), \\ &G_i(y_i, y_{i+1}, \ldots, y_{n-1}) = g_i\big((-1)^{n+i} y_i, (-1)^{n+i+1} y_{i+1}, \ldots, (-1)^{2n-1} y_{n-1}\big), \\ &H(y_0, y_1, \ldots, y_{n-1}; z_0, z_1, \ldots, z_{n-1}) \\ &\quad = k\big((-1)^n y_0, (-1)^{n+1} y_1, \ldots, (-1)^{2n-1} y_{n-1}; (-1)^n z_0, (-1)^{n+1} z_1, \ldots, (-1)^{2n-1} z_{n-1}\big). \end{aligned}$$

This shows that BVP (1.1), (3.2) on the interval $[x_1, x_2]$ can be transformed to the same type as BVP (1.1), (3.1) on the interval $[-x_2, -x_1]$.

Lemma 3.1 *Suppose that* $(\bar{\mathrm{H}}_1)$, $(\bar{\mathrm{H}}_2)$, $(\bar{\mathrm{H}}_3)$, $(\bar{\mathrm{H}}_4)$, $(\bar{\mathrm{H}}_5)$, *and* $(\bar{\mathrm{H}}_7)$ *hold. Then each of BVP* (1.1), (3.1), *BVP* (1.1), (3.2), *and BVP* (1.1), (3.3) *has at least one solution.*

Proof It is easy to check that conditions $(\bar{\mathrm{H}}_1)$, $(\bar{\mathrm{H}}_2)$, $(\bar{\mathrm{H}}_3)$, $(\bar{\mathrm{H}}_4)$, $(\bar{\mathrm{H}}_5)$, and $(\bar{\mathrm{H}}_7)$ imply conditions (H_1), (H_2), (H_3), (H_4), (H_5), and (H_8) for BVP (1.1), (3.1) as well as conditions (H_1), (H_2), (H_3), (H_4), (H_5'), and (H_8') for BVP (1.1), (3.3), respectively. Hence by Lemma 2.1 and 2.2, each of BVP (1.1), (3.1) and BVP (1.1), (3.3) has at least one solution.

Similarly, by Lemma 2.1 BVP (1.1′), (3.2′) has at least one solution. Hence BVP (1.1), (3.2) has at least one solution. □

Lemma 3.2 *Suppose that* $(\bar{\mathrm{H}}_1)$, $(\bar{\mathrm{H}}_2)$, $(\bar{\mathrm{H}}_3)$, $(\bar{\mathrm{H}}_4)$, $(\bar{\mathrm{H}}_5)$, *and* $(\bar{\mathrm{H}}_6)$ *hold. Then each of BVP* (1.1), (3.1), *BVP* (1.1), (3.2), *and BVP* (1.1), (3.3) *has exactly one solution.*

Proof Similarly to the proof of Lemma 3.1 by Lemma 2.3 and 2.4, the lemma follows. □

In order to prove our main results, we introduce some concepts as follows.

Definition 3.1 A set-valued function $T : \mathbb{R} \to 2^{\mathbb{R}}$ is said to be upper semi-continuous at $\mu_0 \in \mathbb{R}$ if for any open set U with $T(\mu_0) \subset U$, there exists a neighborhood V of μ_0 such that $\bigcup_{\mu \in V} T(\mu) \subset U$.

Definition 3.2 Let I_1 and I_2 be subsets of $\mathbb{R}$.

(1) If for any $t_1 \in I_1$ and $t_2 \in I_2$, $t_1 \leq t_2$ holds, then we denote $I_1 \leq I_2$ and say that I_1 is not greater than I_2.

(2) If for any $t_1 \in I_1$ and $t_2 \in I_2$, $t_1 < t_2$ holds, then we denote $I_1 < I_2$ and say that I_1 is less than I_2.

Definition 3.3

(1) Define a set-valued function $T_1 : \mathbb{R} \to 2^{\mathbb{R}}$ by

$$T_1(\mu) = S_\mu \quad \text{for any } \mu \in \mathbb{R},$$

where $S_\mu = \{y^{(n-1)}(x_2, \mu) : y(x, \mu) \text{ are solutions of BVP (1.1), (3.1)}\}$;

(2) Define a set-valued function $T_2 : \mathbb{R} \to 2^{\mathbb{R}}$ by

$$T_2(\mu) = J_\mu \quad \text{for any } \mu \in \mathbb{R},$$

where $J_\mu = \{y^{(n-1)}(x_2, \mu) : y(x, \mu) \text{ are solutions of BVP (1.1), (3.2)}\}$.

Lemma 3.3

(1) *Suppose that* $(\bar{\mathrm{H}}_1)$, $(\bar{\mathrm{H}}_2)$, $(\bar{\mathrm{H}}_3)$, $(\bar{\mathrm{H}}_4)$, $(\bar{\mathrm{H}}_5)$ *and* $(\bar{\mathrm{H}}_7)$ *hold. If* $\mu_1 < \mu_2$, *then*

$$T_1(\mu_1) \geq T_1(\mu_2), \qquad T_2(\mu_1) \leq T_2(\mu_2).$$

(2) *Suppose that* $(\bar{\mathrm{H}}_1)$, $(\bar{\mathrm{H}}_2)$, $(\bar{\mathrm{H}}_3)$, $(\bar{\mathrm{H}}_4)$, $(\bar{\mathrm{H}}_5)$ *and* $(\bar{\mathrm{H}}_6)$ *hold. If* $\mu_1 < \mu_2$, *then*

$$T_1(\mu_1) > T_1(\mu_2).$$

Proof (1) Let us show first the inequality with respect to T_1. To do this, we take any $y_1^{(n-1)}(x_2, \mu_1) \in S_{\mu_1}$, $y_2^{(n-1)}(x_2, \mu_2) \in S_{\mu_2}$. Suppose that $y_1^{(n-1)}(x_2, \mu_1) \geq y_2^{(n-1)}(x_2, \mu_2)$ is false, *i.e.*, $y_1^{(n-1)}(x_2, \mu_1) < y_2^{(n-1)}(x_2, \mu_2)$. Then, for each $i = 0, 1, \dots, n-3$, from (3.1) we have by the mean value theorem

$$\begin{aligned} 0 &= g_i\bigl(y_2^{(i)}(x_2, \mu_2), \dots, y_2^{(n-1)}(x_2, \mu_2)\bigr) - g_i\bigl(y_1^{(i)}(x_2, \mu_1), \dots, y_1^{(n-1)}(x_2, \mu_1)\bigr) \\ &= \frac{\partial g_i}{\partial y_i} \cdot \bigl(y_2^{(i)}(x_2, \mu_2) - y_1^{(i)}(x_2, \mu_1)\bigr) + \sum_{j=i+1}^{n-1} \frac{\partial g_i}{\partial y_j} \cdot \bigl(y_2^{(j)}(x_2, \mu_2) - y_1^{(j)}(x_2, \mu_1)\bigr), \end{aligned}$$

and $y_1^{(n-2)}(x_2, \mu_1) = \mu_1 < \mu_2 = y_2^{(n-2)}(x_2, \mu_2)$. By $(\bar{\mathrm{H}}_4)$ we can inductively show that, for each $i = n-3, \dots, 1, 0$, $y_1^{(i)}(x_2, \mu_1) \leq y_2^{(i)}(x_2, \mu_2)$. Consequently by Lemma 2.5 we have $y_1^{(n-1)}(x, \mu_1) \leq y_2^{(n-1)}(x, \mu_2)$ for $x_2 \leq x \leq x_3$. Furthermore one can inductively get for each $i = n-2, \dots, 1, 0$ the result $y_1^{(i)}(x, \mu_1) < y_2^{(i)}(x, \mu_2)$ for $x_2 < x \leq x_3$. Now by $(\bar{\mathrm{H}}_5)$ and $(\bar{\mathrm{H}}_7)$ we get

$$\begin{aligned} &h\bigl(y_2(x_2, \mu_2), \dots, y_2^{(n-1)}(x_2, \mu_2); y_2(x_3, \mu_2), \dots, y_2^{(n-1)}(x_3, \mu_2)\bigr) \\ &\quad - h\bigl(y_1(x_2, \mu_1), \dots, y_1^{(n-1)}(x_2, \mu_1); y_1(x_3, \mu_1), \dots, y_1^{(n-1)}(x_3, \mu_1)\bigr) \\ &= \sum_{i=0}^{n-1} \frac{\partial h}{\partial y_i} \cdot \bigl(y_2^{(i)}(x_2, \mu_2) - y_1^{(i)}(x_2, \mu_1)\bigr) + \sum_{i=0}^{n-1} \frac{\partial h}{\partial z_i} \cdot \bigl(y_2^{(i)}(x_3, \mu_2) - y_1^{(i)}(x_3, \mu_1)\bigr) \\ &> 0. \end{aligned}$$

This is a contradiction to (3.1). Thus we conclude that

$$y_1^{(n-1)}(x_2, \mu_1) \geq y_2^{(n-1)}(x_2, \mu_2),$$

i.e., $T_1(\mu_1) \geq T_1(\mu_2)$ for $\mu_1 < \mu_2$.

By similar arguments, we can show the inequality for T_2.

(2) Since ($\bar{\mathrm{H}}_5$) and ($\bar{\mathrm{H}}_6$) imply ($\bar{\mathrm{H}}_7$), for any $y_1^{(n-1)}(x_2,\mu_1)\in S_{\mu_1}$ and $y_2^{(n-1)}(x_2,\mu_2)\in S_{\mu_2}$, we have by (1), $y_1^{(n-1)}(x_2,\mu_1)\ge y_2^{(n-1)}(x_2,\mu_2)$. Suppose $y_1^{(n-1)}(x_2,\mu_1)=y_2^{(n-1)}(x_2,\mu_2)$. Then both $y_1(x,\mu_1)$ and $y_2(x,\mu_2)$ are solutions of BVP (1.1), (3.3) with $\mu=y_1^{(n-1)}(x_2,\mu_1)=y_2^{(n-1)}(x_2,\mu_2)$. By Lemma 3.2 of the uniqueness, we conclude $y_1(x,\mu_1)=y_2(x,\mu_2)$ for $x_2\le x\le x_3$, which implies

$$\mu_1=y_1^{(n-2)}(x_2,\mu_1)=y_2^{(n-2)}(x_2,\mu_2)=\mu_2.$$

This is a contradiction. Thus $y_1^{(n-1)}(x_2,\mu_1)>y_2^{(n-1)}(x_2,\mu_2)$, *i.e.*, $T_1(\mu_1)>T_1(\mu_2)$ for $\mu_1<\mu_2$. □

Lemma 3.4 *Suppose that* ($\bar{\mathrm{H}}_1$), ($\bar{\mathrm{H}}_2$), ($\bar{\mathrm{H}}_3$), ($\bar{\mathrm{H}}_4$), ($\bar{\mathrm{H}}_5$) *and* ($\bar{\mathrm{H}}_7$) (*or* ($\bar{\mathrm{H}}_6$)) *hold. Then, for any* $\mu\in\mathbb{R}$, *both* S_μ *and* J_μ *are compact and connected subsets of* $\mathbb{R}$.

Proof If ($\bar{\mathrm{H}}_i$), $i=1,2,3,4,5,6$ hold, then by Lemma 3.2, each of BVP (1.1), (3.1) and BVP (1.1), (3.2) has exactly one solution. Consequently both S_μ and J_μ are single-point sets. Hence the theorem holds.

Now let ($\bar{\mathrm{H}}_i$), $i=1,2,3,4,5,7$ hold. First, we prove that S_μ is an interval. To do this, let us take any $y_1^{(n-1)}(x_2,\mu),y_2^{(n-1)}(x_2,\mu)\in S_\mu$ with $y_1^{(n-1)}(x_2,\mu)<y_2^{(n-1)}(x_2,\mu)$. We need to show that if $y_1^{(n-1)}(x_2,\mu)<y_0^{(n-1)}<y_2^{(n-1)}(x_2,\mu)$, then $y_0^{(n-1)}\in S_\mu$. By ($\bar{\mathrm{H}}_4$), it is easy to see inductively that $y_1^{(i)}(x_2,\mu)\le y_2^{(i)}(x_2,\mu)$, $i=n-3,\dots,1,0$, and for any fixed $y_0^{(n-1)}\in(y_1^{(n-1)}(x_2,\mu),y_2^{(n-1)}(x_2,\mu))$ there exist unique $y_0^{(i)}\in[y_1^{(i)}(x_2,\mu),y_2^{(i)}(x_2,\mu)]$, $i=n-3,\dots,1,0$, such that

$$g_i\bigl(y_0^{(i)},y_0^{(i+1)},\dots,y_0^{(n-3)},\mu,y_0^{(n-1)}\bigr)=0,\quad i=0,1,\dots,n-3.$$

Now let $y_0(x)$ be the unique solution of (1.1) which satisfies the initial conditions $y_0^{(i)}(x_2)=y_0^{(i)}$, $i=0,1,\dots,n-1$, where $y_0^{(n-2)}=\mu$. Then by Lemma 2.5, $y_1^{(n-1)}(x,\mu)\le y_0^{(n-1)}(x)$ for $x_2\le x\le x_3$. Furthermore we have $y_1^{(i)}(x,\mu)\le y_0^{(i)}(x)$ for $x_2\le x\le x_3$, $i=0,1,\dots,n-2$. Similarly we can show that $y_0^{(i)}(x)\le y_2^{(i)}(x,\mu)$ for $x_2\le x\le x_3$, $i=0,1,\dots,n-1$. Hence by ($\bar{\mathrm{H}}_5$), we have

$$\begin{aligned}&h\bigl(y_0(x_2),y_0'(x_2),\dots,y_0^{(n-1)}(x_2);y_0(x_3),y_0'(x_3),\dots,y_0^{(n-1)}(x_3)\bigr)\\&\quad\ge h\bigl(y_1(x_2,\mu),y_1'(x_2,\mu),\dots,y_1^{(n-1)}(x_2,\mu);y_1(x_3,\mu),y_1'(x_3,\mu),\dots,y_1^{(n-1)}(x_3,\mu)\bigr)\\&\quad=0\end{aligned}$$

and

$$\begin{aligned}&h\bigl(y_0(x_2),y_0'(x_2),\dots,y_0^{(n-1)}(x_2);y_0(x_3),y_0'(x_3),\dots,y_0^{(n-1)}(x_3)\bigr)\\&\quad\le h\bigl(y_2(x_2,\mu),y_2'(x_2,\mu),\dots,y_2^{(n-1)}(x_2,\mu);y_2(x_3,\mu),y_2'(x_3,\mu),\dots,y_2^{(n-1)}(x_3,\mu)\bigr)\\&\quad=0.\end{aligned}$$

Thus

$$h\bigl(y_0(x_2),y_0'(x_2),\dots,y_0^{(n-1)}(x_2);y_0(x_3),y_0'(x_3),\dots,y_0^{(n-1)}(x_3)\bigr)=0.$$

Hence $y_0(x)$ satisfies the boundary condition (3.1), which implies that $y_0(x)$ is the solution of BVP (1.1), (3.1), and then $y_0^{(n-1)} = y_0^{(n-1)}(x_2) \in S_\mu$.

Next, we show that S_μ is closed. To do this, for any sequence $\{y_m^{(n-1)}\}_{m=1}^\infty$ in S_μ with $y_m^{(n-1)} \to y_0^{(n-1)}$ as $m \to \infty$, we need to show $y_0^{(n-1)} \in S_\mu$. By the definition of S_μ, corresponding to $\{y_m^{(n-1)}\}_{m=1}^\infty$ there exists a sequence $\{y_m(x,\mu)\}_{m=1}^\infty$ of solutions of BVP (1.1), (3.1) such that $y_m^{(n-1)} = y_m^{(n-1)}(x_2,\mu)$. By $(\bar{\mathrm{H}}_4)$, it is easy to see that, for each $y_m^{(n-1)}$, there exist $y_m^{(i)}$, $i = 0,1,\dots,n-3$, such that

$$g_i\bigl(y_m^{(i)}, y_m^{(i+1)}, \dots, y_m^{(n-3)}, \mu, y_m^{(n-1)}\bigr) = 0, \quad i = 0,1,\dots,n-3.$$

Furthermore we have, by $(\bar{\mathrm{H}}_4)$,

$$y_m^{(i)} = y_m^{(i)}(x_2,\mu), \quad i = 0,1,\dots,n-3, m = 1,2,\dots.$$

Now let us show that the sequences $\{y_m^{(i)}\}_{m=1}^\infty$, $i = 0,1,\dots,n-3$, are convergent. In fact, when $i = n-3$, for any positive integers $m,p \in \mathbb{N}$ we have

$$\begin{aligned} 0 &= g_{n-3}\bigl(y_m^{(n-3)}, \mu, y_m^{(n-1)}\bigr) - g_{n-3}\bigl(y_{m+p}^{(n-3)}, \mu, y_{m+p}^{(n-1)}\bigr) \\ &= \frac{\partial g_{n-3}}{\partial y_{n-3}} \cdot \bigl(y_m^{(n-3)} - y_{m+p}^{(n-3)}\bigr) + \frac{\partial g_{n-3}}{\partial y_{n-1}} \cdot \bigl(y_m^{(n-1)} - y_{m+p}^{(n-1)}\bigr). \end{aligned}$$

Consequently by $(\bar{\mathrm{H}}_4)$, we get

$$\bigl|y_m^{(n-3)} - y_{m+p}^{(n-3)}\bigr| \le \delta^{-1}\left|\frac{\partial g_{n-3}}{\partial y_{n-1}}\right|\bigl|y_m^{(n-1)} - y_{m+p}^{(n-1)}\bigr|.$$

Since $\{y_m^{(n-1)}\}_{m=1}^\infty$ is a Cauchy sequence, so is the sequence $\{y_m^{(n-3)}\}_{m=1}^\infty$. Hence $\{y_m^{(n-3)}\}_{m=1}^\infty$ converges to a number $y_0^{(n-3)}$. Similarly we can show inductively that, for each $i = n-4,\dots,1,0$, the sequence $\{y_m^{(i)}\}_{m=1}^\infty$ converges to a number $y_0^{(i)}$.

We note that $y_m^{(n-2)} = y_0^{(n-2)} = \mu$, $m = 1,2,\dots$. Then by Kamke's standard convergence theorem [37], there exists a solution $y = \hat{y}(x)$ of (1.1) defined on $[x_2,x_3]$ satisfying initial conditions $\hat{y}^{(i)}(x_2) = y_0^{(i)}$, $i = 0,1,\dots,n-1$, and there exists a subsequence $\{y_{m_j}(x,\mu)\}_{j=1}^\infty$ of $\{y_m(x,\mu)\}_{m=1}^\infty$ such that, for each $i = 0,1,\dots,n-1$, the sequence $\{y_{m_j}^{(i)}(x,\mu)\}_{j=1}^\infty$ uniformly converges to $\hat{y}^{(i)}(x)$ on $[x_2,x_3]$. It is easy to see that $y = \hat{y}(x)$ is the solution of BVP (1.1), (3.1). Hence $y_0^{(n-1)} = \hat{y}^{(n-1)}(x_2) \in S_\mu$.

Finally, we show that S_μ is bounded. To do this, we take $\mu_1,\mu_2 \in \mathbb{R}$ with $\mu_1 < \mu < \mu_2$. Then from Lemma 3.3, we have

$$S_{\mu_2} \le S_\mu \le S_{\mu_1}.$$

This implies the boundedness of S_μ.

By a similar argument for BVP (1.1′), (3.2′), we can show that J_μ is also a compact and connected subset of $\mathbb{R}$. □

Lemma 3.5 *Suppose that* $(\bar{\mathrm{H}}_1)$, $(\bar{\mathrm{H}}_2)$, $(\bar{\mathrm{H}}_3)$, $(\bar{\mathrm{H}}_4)$, $(\bar{\mathrm{H}}_5)$, *and* $(\bar{\mathrm{H}}_7)$ *hold. Then there exist sequences* $\{y_m(x,\mu_m)\}_{m=1}^\infty$ *and* $\{y_m(x,\nu_m)\}_{m=1}^\infty$ *of solutions of BVP* (1.1), (3.1) *with* $\mu = \mu_m$

and of BVP (1.1), (3.1) *with* $\mu = \nu_m$, *respectively, for which*

$$\lim_{m\to\infty} y_m^{(n-1)}(x_2, \mu_m) = \infty, \qquad \lim_{m\to\infty} y_m^{(n-1)}(x_2, \nu_m) = -\infty.$$

Proof Let us take a sequence $\{y_m^{(n-1)}\}_{m=1}^{\infty}$ with $\lim_{m\to\infty} y_m^{(n-1)} = \infty$. Then, by Lemma 3.1, BVP (1.1), (3.3) with $\mu = y_m^{(n-1)}$ has a solution, denoted by $y_m(x)$. It is easy to see that $y_m(x)$ is the solution of BVP (1.1), (3.1) with $\mu = y_m^{(n-2)}(x_2)$. Let $\mu_m = y_m^{(n-2)}(x_2)$ and let $y_m(x, \mu_m) = y_m(x)$. Then $y_m^{(n-1)}(x_2, \mu_m) \in S_{\mu_m}$ and

$$\lim_{m\to\infty} y_m^{(n-1)}(x_2, \mu_m) = \lim_{m\to\infty} y_m^{(n-1)} = \infty.$$

Similarly one can show that there exists $y_m^{(n-1)}(x_2, \nu_m) \in S_{\nu_m}$, for which

$$\lim_{m\to\infty} y_m^{(n-1)}(x_2, \nu_m) = -\infty. \qquad \square$$

Lemma 3.6 *Suppose that* $(\bar{\mathrm{H}}_1)$, $(\bar{\mathrm{H}}_2)$, $(\bar{\mathrm{H}}_3)$, $(\bar{\mathrm{H}}_4)$, $(\bar{\mathrm{H}}_5)$ *and* $(\bar{\mathrm{H}}_7)$ *hold. Then*

(1) *for any* $\mu_0 \in \mathbb{R}$ *and* $\varepsilon > 0$, *there exists* $\rho > 0$ *such that if* $|\mu - \mu_0| < \rho$, *then, for any* $y^{(n-1)}(x_2, \mu) \in S_\mu$, *there exists* $y^{(n-1)}(x_2, \mu_0) \in S_{\mu_0}$ *satisfying* $|y^{(n-1)}(x_2, \mu) - y^{(n-1)}(x_2, \mu_0)| < \varepsilon$;

(2) *for any* $\mu_0 \in \mathbb{R}$ *and* $\varepsilon > 0$, *there exists* $\rho > 0$ *such that if* $|\mu - \mu_0| < \rho$, *then, for any* $z^{(n-1)}(x_2, \mu) \in J_\mu$, *there exists* $z^{(n-1)}(x_2, \mu_0) \in J_{\mu_0}$ *satisfying* $|z^{(n-1)}(x_2, \mu) - z^{(n-1)}(x_2, \mu_0)| < \varepsilon$.

Proof Let us prove only (1), since (2) can be shown similarly.

Suppose the conclusion (1) is false. Then there exist $\mu_0 \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that, for each $\rho = \frac{1}{m}$, $m = 1, 2, \ldots$, there exist $\mu_m \in (\mu_0 - \frac{1}{m}, \mu_0 + \frac{1}{m})$ and $y^{(n-1)}(x_2, \mu_m) \in S_{\mu_m}$ such that, for any $y^{(n-1)}(x_2, \mu_0) \in S_{\mu_0}$,

$$\left|y^{(n-1)}(x_2, \mu_m) - y^{(n-1)}(x_2, \mu_0)\right| \geq \varepsilon_0.$$

Since $\mu_0 - \frac{1}{m} < \mu_m < \mu_0 + \frac{1}{m}$, $m = 1, 2, \ldots$, we have by Lemma 3.3

$$T_1(\mu_0 + 1) \leq T_1\left(\mu_0 + \frac{1}{m}\right) \leq \left\{y^{(n-1)}(x_2, \mu_m)\right\} \leq T_1\left(\mu_0 - \frac{1}{m}\right) \leq T_1(\mu_0 - 1).$$

Thus $\{y^{(n-1)}(x_2, \mu_m)\}_{m=1}^{\infty}$ is bounded. Without loss of generality, we may assume that $y^{(n-1)}(x_2, \mu_m) \to y_0^{(n-1)}$ as $m \to \infty$. For any positive integers $m, p \in \mathbb{N}$, we have, for each $i = 0, 1, \ldots, n-3$,

$$\begin{aligned}
0 &= g_i\big(y^{(i)}(x_2, \mu_m), \ldots, y^{(n-1)}(x_2, \mu_m)\big) \\
&\quad - g_i\big(y^{(i)}(x_2, \mu_{m+p}), \ldots, y^{(n-1)}(x_2, \mu_{m+p})\big) \\
&= \frac{\partial g_i}{\partial y_i} \cdot \big(y^{(i)}(x_2, \mu_m) - y^{(i)}(x_2, \mu_{m+p})\big) \\
&\quad + \sum_{j=i+1}^{n-1} \frac{\partial g_i}{\partial y_j} \cdot \big(y^{(j)}(x_2, \mu_m) - y^{(j)}(x_2, \mu_{m+p})\big).
\end{aligned}$$

Hence, for each $i = 0, 1, \dots, n-3$, by $(\bar{\mathrm{H}}_4)$ we have

$$\left|y^{(i)}(x_2, \mu_m) - y^{(i)}(x_2, \mu_{m+p})\right| \leq \delta^{-1} \sum_{j=i+1}^{n-1} \left|\frac{\partial g_i}{\partial y_j}\right| \left|y^{(j)}(x_2, \mu_m) - y^{(j)}(x_2, \mu_{m+p})\right|.$$

Since $\{y^{(n-1)}(x_2, \mu_m)\}_{m=1}^{\infty}$ and $\{y^{(n-2)}(x_2, \mu_m)\}_{m=1}^{\infty} = \{\mu_m\}_{m=1}^{\infty}$ are convergent, $\{y^{(n-3)}(x_2, \mu_m)\}_{m=1}^{\infty}$ is a Cauchy sequence, and thus $\{y^{(n-3)}(x_2, \mu_m)\}_{m=1}^{\infty}$ is convergent. Similarly one can show inductively that, for each $i = n-4, \dots, 1, 0$, $\{y^{(i)}(x_2, \mu_m)\}_{m=1}^{\infty}$ is also convergent. Set $\lim_{m\to\infty} y^{(i)}(x_2, \mu_m) = y_0^{(i)}$, $i = 0, 1, \dots, n-1$, where $y_0^{(n-2)} = \mu_0$. Then by Kamke's convergence theorem, there exists a solution $y = \hat{y}(x)$ of (1.1) defined on $[x_2, x_3]$ satisfying the initial conditions $\hat{y}^{(i)}(x_2) = y_0^{(i)}$, $i = 0, 1, \dots, n-1$ and there exists a subsequence $\{y(x, \mu_{m_j})\}_{j=1}^{\infty}$ of $\{y(x, \mu_m)\}_{m=1}^{\infty}$ such that, for each $i = 0, 1, \dots, n-1$, the sequence $\{y^{(i)}(x, \mu_{m_j})\}_{j=1}^{\infty}$ uniformly converges to $\hat{y}^{(i)}(x)$ on $[x_2, x_3]$. It is easy to see that $\hat{y}(x)$ is the solution of BVP (1.1), (3.1) with $\mu = \mu_0$. Consequently $y_0^{(n-1)} = \hat{y}^{(n-1)}(x_2) \in S_{\mu_0}$, and hence

$$\left|y^{(n-1)}(x_2, \mu_m) - y_0^{(n-1)}\right| \geq \varepsilon_0,$$

which is a contradiction to $\lim_{m\to\infty} y^{(n-1)}(x_2, \mu_m) = y_0^{(n-1)}$. Thus (1) holds. □

Lemma 3.7 *Suppose that* $(\bar{\mathrm{H}}_1)$, $(\bar{\mathrm{H}}_2)$, $(\bar{\mathrm{H}}_3)$, $(\bar{\mathrm{H}}_4)$, $(\bar{\mathrm{H}}_5)$, *and* $(\bar{\mathrm{H}}_7)$ *hold. Then both* T_1 *and* T_2 *are upper semi-continuous on* $\mathbb{R}$.

Proof For any $\mu_0 \in \mathbb{R}$, let us show T_1 is upper semi-continuous at $\mu = \mu_0$.

From Lemma 3.4, $T_1(\mu_0)$ is a compact and connected subset of $\mathbb{R}$. Hence without loss of generality, we may assume that

$$T_1(\mu_0) = \left[y_1^{(n-1)}(x_2, \mu_0), y_2^{(n-1)}(x_2, \mu_0)\right].$$

Take any open set U with $T_1(\mu_0) \subset U$. Then there exists $\varepsilon > 0$ such that

$$\left(y_1^{(n-1)}(x_2, \mu_0) - \varepsilon, y_2^{(n-1)}(x_2, \mu_0) + \varepsilon\right) \subset U.$$

Thus from Lemma 3.6, there exists $\rho > 0$ such that if $|\mu - \mu_0| < \rho$, then, for any $y^{(n-1)}(x_2, \mu) \in S_\mu$, there exists $y^{(n-1)}(x_2, \mu_0) \in S_{\mu_0} = T_1(\mu_0)$ for which

$$\left|y^{(n-1)}(x_2, \mu) - y^{(n-1)}(x_2, \mu_0)\right| < \varepsilon,$$

and so $S_\mu \subset U$. Hence T_1 is upper semi-continuous at $\mu = \mu_0$.

The upper semi-continuity of T_2 on $\mathbb{R}$ can be shown similarly. □

Theorem 3.1 *Suppose that* $(\bar{\mathrm{H}}_1)$, $(\bar{\mathrm{H}}_2)$, $(\bar{\mathrm{H}}_3)$, $(\bar{\mathrm{H}}_4)$, $(\bar{\mathrm{H}}_5)$, *and* $(\bar{\mathrm{H}}_7)$ *hold. Then BVP* (1.1), (1.2) *has at least one solution.*

Proof We consider two cases as follows.

Case 1. Suppose there exists $\mu_0 \in \mathbb{R}$ such that $S_{\mu_0} \cap J_{\mu_0} \neq \emptyset$. Then BVP (1.1), (3.1) with $\mu = \mu_0$ and BVP (1.1), (3.2) with $\mu = \mu_0$ have solutions $y(x, \mu_0)$ and $z(x, \mu_0)$, respectively, such that $y^{(n-1)}(x_2,\mu_0) = z^{(n-1)}(x_2,\mu_0)$. Since $y^{(n-2)}(x_2,\mu_0) = \mu_0 = z^{(n-2)}(x_2,\mu_0)$, by $(\bar{\mathrm{H}}_4)$ it is easy to see that $y^{(i)}(x_2,\mu_0) = z^{(i)}(x_2,\mu_0)$, $i = 0,1,\ldots,n-3$. Hence, if we let

$$u(x) := \begin{cases} y(x,\mu_0), & x \in [x_2,x_3], \\ z(x,\mu_0), & x \in [x_1,x_2], \end{cases}$$

then $u(x)$ is a solution of BVP (1.1), (1.2).

Case 2. Suppose for any $\mu \in \mathbb{R}$, $S_\mu \cap J_\mu = \emptyset$. Then by Lemma 3.3 and 3.5, there exist μ_1 and μ_2 with $\mu_1 < \mu_2$, such that

$$S_{\mu_1} > J_{\mu_1}, \qquad S_{\mu_2} < J_{\mu_2}, \qquad S_{\mu_1} > S_{\mu_2}.$$

In fact, let us take any $\mu_0 \in \mathbb{R}$ and $z^{(n-1)}(x_2,\mu_0) \in J_{\mu_0}$. Then by Lemma 3.5, there exists some $y_1^{(n-1)}(x_2,\nu_1) \in S_{\nu_1}$ such that $y_1^{(n-1)}(x_2,\nu_1) > z^{(n-1)}(x_2,\mu_0)$. Take μ_1 with $\mu_1 < \min\{\nu_1,\mu_0\}$. Then by Lemma 3.3, we have

$$S_{\mu_1} \geq \left\{y_1^{(n-1)}(x_2,\nu_1)\right\} > \left\{z^{(n-1)}(x_2,\mu_0)\right\} \geq J_{\mu_1}.$$

Also by Lemma 3.5, there exists some $y_2^{(n-1)}(x_2,\nu_2) \in S_{\nu_2}$ such that

$$y_2^{(n-1)}(x_2,\nu_2) < \min\left\{z^{(n-1)}(x_2,\mu_1), y_1^{(n-1)}(x_2,\nu_1)\right\}.$$

Again take $\mu_2 > \max\{\mu_1,\nu_2\}$. Then by Lemma 3.3, we have

$$S_{\mu_2} \leq \left\{y_2^{(n-1)}(x_2,\nu_2)\right\} < \left\{z^{(n-1)}(x_2,\mu_1)\right\} \leq J_{\mu_2}$$

and

$$S_{\mu_1} \geq \left\{y_1^{(n-1)}(x_2,\nu_1)\right\} > \left\{y_2^{(n-1)}(x_2,\nu_2)\right\} \geq S_{\mu_2}.$$

Now we apply a bisection argument as follows. Set $a_0 = \mu_1$, $b_0 = \mu_2$. Then we have two cases, *i.e.*,

$$S_{\frac{a_0+b_0}{2}} > J_{\frac{a_0+b_0}{2}} \quad \text{or} \quad S_{\frac{a_0+b_0}{2}} < J_{\frac{a_0+b_0}{2}}.$$

If $S_{\frac{a_0+b_0}{2}} > J_{\frac{a_0+b_0}{2}}$, set $a_1 = \frac{a_0+b_0}{2}$ and $b_1 = b_0$. If $S_{\frac{a_0+b_0}{2}} < J_{\frac{a_0+b_0}{2}}$, set $a_1 = a_0$ and $b_1 = \frac{a_0+b_0}{2}$. In summary, there exist $a_1, b_1 \in [a_0,b_0]$ such that

$$a_1 < b_1, \qquad b_1 - a_1 = \frac{1}{2}(b_0 - a_0), \qquad S_{a_1} > J_{a_1}, \qquad S_{b_1} < J_{b_1}.$$

By continuing this bisection process, we can get sequences $\{a_m\}_{m=1}^\infty$ and $\{b_m\}_{m=1}^\infty$ with $a_m, b_m \in [a_{m-1}, b_{m-1}] \subset [a_0,b_0]$, $m = 1,2,\ldots,$ such that

$$a_m < b_m, \qquad b_m - a_m = \frac{1}{2^m}(b_0 - a_0), \qquad S_{a_m} > J_{a_m}, \qquad S_{b_m} < J_{b_m}.$$

Hence by the nested interval theorem, there uniquely exists $\xi \in \mathbb{R}$ such that $\xi \in \bigcap_{m=1}^\infty [a_m,b_m]$, actually a_m, b_m squeeze to the common limit ξ.

Suppose $S_\xi > J_\xi$. Then since both S_ξ and J_ξ are compact and connected subsets of $\mathbb{R}$ and $S_\xi \cap J_\xi = \emptyset$, there exist two open interval U_S and U_J such that $U_S \supset S_\xi$, $U_J \supset J_\xi$ and $U_S \cap U_J = \emptyset$. Consequently $U_S > U_J$. Since both T_1 and T_2 are upper semi-continuous on $\mathbb{R}$ by Lemma 3.7, there exists $\rho > 0$ such that if $|\mu - \xi| < \rho$ then $T_1(\mu) = S_\mu \subset U_S$ and $T_2(\mu) = J_\mu \subset U_J$, and thus $S_\mu > J_\mu$. On the other hand since $b_m \to \xi$ as $m \to \infty$, there exists $m_0 \in \mathbb{N}$ such that $|b_{m_0} - \xi| < \rho$, consequently $S_{b_{m_0}} > J_{b_{m_0}}$, which is a contradiction.

If $S_\xi < J_\xi$, then we can similarly obtain a contradiction. Hence the case 2 cannot occur. This completes the proof of the theorem. □

Theorem 3.2 *Suppose that* $(\bar{\mathrm{H}}_1)$, $(\bar{\mathrm{H}}_2)$, $(\bar{\mathrm{H}}_3)$, $(\bar{\mathrm{H}}_4)$, $(\bar{\mathrm{H}}_5)$, *and* $(\bar{\mathrm{H}}_6)$ *hold. Then BVP* (1.1), (1.2) *has exactly one solution.*

Proof Since $(\bar{\mathrm{H}}_5)$ and $(\bar{\mathrm{H}}_6)$ imply $(\bar{\mathrm{H}}_7)$, by Theorem 3.1, BVP (1.1), (1.2) has at least one solution.

Now we need to show the uniqueness. By Theorem 3.1, BVP (1.1), (1.2) has a solution $u(x)$, for which we denote

$$u(x) := \begin{cases} y(x, \mu_0), & x \in [x_2, x_3], \\ z(x, \mu_0), & x \in [x_1, x_2]. \end{cases}$$

Let $v(x)$ be any solution of BVP (1.1), (1.2), and let $z_1(x) = v(x)$ for $x_1 \le x \le x_2$, $y_1(x) = v(x)$ for $x_2 \le x \le x_3$ and $v^{(n-2)}(x_2) = \mu^*$. Then $y_1(x)$ and $z_1(x)$ are the solutions of BVP (1.1), (3.1) with $\mu = \mu^*$ and BVP (1.1), (3.2) with $\mu = \mu^*$, respectively.

If $\mu^* > \mu_0$, then by Lemma 3.3 we have

$$y^{(n-1)}(x_2, \mu_0) > y_1^{(n-1)}(x_2) = z_1^{(n-1)}(x_2) \ge z^{(n-1)}(x_2, \mu_0),$$

which is a contradiction.

If $\mu^* < \mu_0$, then by Lemma 3.3 we have

$$z^{(n-1)}(x_2, \mu_0) \ge z_1^{(n-1)}(x_2) = y_1^{(n-1)}(x_2) > y^{(n-1)}(x_2, \mu_0),$$

which is also a contradiction. Hence $\mu^* = \mu_0$. Consequently by Lemma 3.2, we get $z_1(x) = z(x, \mu_0)$ for $x_1 \le x \le x_2$ and $y_1(x) = y(x, \mu_0)$ for $x_2 \le x \le x_3$. Thus $u(x) \equiv v(x)$ on $[x_1, x_3]$. This completes the proof of the theorem. □

Remark 3.1 Theorem 3.2 includes the results of [1, 2, 4, 12–14] as particular cases.

It is easy to see that the linear boundary conditions in the next corollary satisfy $(\bar{\mathrm{H}}_4)$, $(\bar{\mathrm{H}}_5)$, and $(\bar{\mathrm{H}}_6)$.

Corollary 3.1 *Suppose that* $(\bar{\mathrm{H}}_1)$, $(\bar{\mathrm{H}}_2)$, *and* $(\bar{\mathrm{H}}_3)$ *hold. Suppose further that* $a_i a_{i+1} \le 0$, $i = 0, 1, \ldots, n-2$, $\sum_{i=0}^{n-1} |a_i| > 0$; $b_{ii} b_{ij} \le 0$, $i = 0, 1, \ldots, n-3$, $j = i+1, i+2, \ldots, n-1$, $|b_{ii}| > 0$, $i = 0, 1, \ldots, n-3$; $c_i c_{i+1} \ge 0$, $i = 0, 1, \ldots, n-2$, $\sum_{i=0}^{n-2} |c_i| > 0$. *Then, for any* $\lambda_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, *the three-point boundary value problem of* (1.1) *with linear boundary conditions*

$$\begin{cases} \sum_{i=0}^{n-1} a_i y^{(i)}(x_1) = \lambda_0, \\ \sum_{j=i}^{n-1} b_{ij} y^{(j)}(x_2) = \lambda_{i+1}, \quad i = 0,1,\ldots,n-3, \\ \sum_{i=0}^{n-1} c_i y^{(i)}(x_3) = \lambda_{n-1} \end{cases}$$

has exactly one solution.

By using the transformations $x = -t$ and $y(x) = (-1)^n z(t)$, from Theorem 3.1 we can easily obtain the following.

Theorem 3.3 *Suppose that* $(\bar{\mathrm{H}}_1)$, $(\bar{\mathrm{H}}_2)$, $(\bar{\mathrm{H}}_3)$, $(\bar{\mathrm{H}}_4)$, $(\bar{\mathrm{H}}_5)$, *and* $(\bar{\mathrm{H}}_7')$ *hold. Then BVP* (1.1), (1.2) *has at least one solution.*

Similarly to the proof of Theorem 3.2, from Theorem 3.3 we can get the following.

Theorem 3.4 *Suppose that* $(\bar{\mathrm{H}}_1)$, $(\bar{\mathrm{H}}_2)$, $(\bar{\mathrm{H}}_3)$, $(\bar{\mathrm{H}}_4)$, $(\bar{\mathrm{H}}_5)$, *and* $(\bar{\mathrm{H}}_6')$ *hold. Then BVP* (1.1), (1.2) *has exactly one solution.*

It is easy to see that the linear boundary conditions in the next corollary satisfy $(\bar{\mathrm{H}}_4)$, $(\bar{\mathrm{H}}_5)$, and $(\bar{\mathrm{H}}_6')$.

Corollary 3.2 *Suppose that* $(\bar{\mathrm{H}}_1)$, $(\bar{\mathrm{H}}_2)$, *and* $(\bar{\mathrm{H}}_3)$ *hold. Suppose further that* $a_i a_{i+1} \le 0$, $i = 0,1,\ldots,n-2$, $\sum_{i=0}^{n-2} |a_i| > 0$; $b_{ii} b_{ij} \le 0$, $i = 0,1,\ldots,n-3$, $j = i+1, i+2,\ldots,n-1$, $|b_{ii}| > 0$, $i = 0,1,\ldots,n-3$; $c_i c_{i+1} \ge 0$, $i = 0,1,\ldots,n-2$, $\sum_{i=0}^{n-1} |c_i| > 0$. *Then, for any* $\lambda_i \in \mathbb{R}$, $i = 0,1,\ldots,n-1$, *the three-point boundary value problems of* (1.1) *with linear boundary conditions*

$$\begin{cases} \sum_{i=0}^{n-1} a_i y^{(i)}(x_1) = \lambda_0, \\ \sum_{j=i}^{n-1} b_{ij} y^{(j)}(x_2) = \lambda_{i+1}, \quad i = 0,1,\ldots,n-3, \\ \sum_{i=0}^{n-1} c_i y^{(i)}(x_3) = \lambda_{n-1} \end{cases}$$

has exactly one solution.

Finally, as an application, we give an example to demonstrate our results.

Example 3.1 Consider a third-order three-point boundary value problem

$$y''' = (\sin x)\frac{y^3}{1+y^2} + (\cos x)\arctan y' + |y''| + 1, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \tag{3.4}$$

$$\begin{cases} a_0 y(-\frac{\pi}{2}) + a_1 y'(-\frac{\pi}{2}) + a_2 y''(-\frac{\pi}{2}) = \lambda_0, \\ b_0 y(0) + b_1 y'(0) + b_2 y''(0) = \lambda_1, \\ c_0 y(\frac{\pi}{2}) + c_1 y'(\frac{\pi}{2}) + c_2 y''(\frac{\pi}{2}) = \lambda_2, \end{cases} \tag{3.5}$$

where $a_i, b_i, c_i, \lambda_i \in \mathbb{R}$, $i = 0,1,2$, are constants.

Let

$$f(x,y,z,w) = (\sin x)\frac{y^3}{1+y^2} + (\cos x)\arctan z + |w| + 1 \quad \text{on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \mathbb{R}^3.$$

Then it is easy to check that the assumptions ($\bar{H}_1$), ($\bar{H}_2$), and ($\bar{H}_3$) are satisfied. Hence from either Corollary 3.1 or Corollary 3.2, BVP (3.4), (3.5) has exactly one solution under either of the following conditions:

(i) $a_0a_1 \le 0$, $a_1a_2 \le 0$, $|a_0| + |a_1| + |a_1| > 0$;
(ii) $b_0b_1 \le 0$, $b_0b_2 \le 0$, $b_0 \ne 0$;
(iii) $c_0c_1 \ge 0$, $c_1c_2 \ge 0$, $|c_0| + |c_1| > 0$,

or the following conditions:

(i′) $a_0a_1 \le 0$, $a_1a_2 \le 0$, $|a_0| + |a_1| > 0$;
(ii′) $b_0b_1 \le 0$, $b_0b_2 \le 0$, $b_0 \ne 0$;
(iii′) $c_0c_1 \ge 0$, $c_1c_2 \ge 0$, $|c_0| + |c_1| + |c_2| > 0$.

We note that the results of [1, 2, 4, 12–14] cannot guarantee that the above third-order three-point boundary value problem has a unique solution, unless $b_1 = 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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