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New results for perturbed second-order impulsive differential equation on the half-line

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Abstract

By using a variational method and some critical points theorems, we establish some results on the multiplicity of solutions for second-order impulsive differential equation depending on two real parameters on the half-line. In addition, two examples to illustrate our results are given.

Keywords: impulsive differential equation; variational methods; critical points; half-line

1 Introduction

In this paper we consider the following boundary value problem with impulsive effects:

$$\begin{cases} -u''(t) + m^2u(t) = \lambda f(t, u(t)), & t \in [0, +\infty), t \neq t_j, \\ \Delta(u'(t_j)) = \mu I_j(u(t_j)), & j = 1, 2, \dots, p, \\ u'(0^+) = g(u(0)), & u'(+\infty) = 0, \end{cases} \quad (1.1)$$

where m is a non-zero constant; λ and μ are referred to as two control parameters, $I_j \in C(\mathbf{R}, \mathbf{R})$ for $1 \leq j \leq p$, $0 = t_0 < t_1 < t_2 < \dots < t_p < +\infty$, $\Delta(u'(t_j)) = u'(t_j^+) - u'(t_j^-) = \lim_{t \rightarrow t_j^+} u'(t) - \lim_{t \rightarrow t_j^-} u'(t)$; $f : [0, +\infty) \times \mathbf{R} \rightarrow \mathbf{R}$ is an L^2 -Carathéodory function, and $g : \mathbf{R} \rightarrow \mathbf{R}$ be a Lipschitz continuous function with the Lipschitz constant $L > 0$; *i.e.*

$$|g(s_1) - g(s_2)| \leq L|s_1 - s_2| \quad (1.2)$$

for all $s_1, s_2 \in \mathbf{R}$, satisfying $g(0) = 0$.

Boundary value problems on the half-line, arising naturally in the study of radially symmetric solutions of nonlinear elliptic equations and various physical phenomena [1], have been studied extensively and a variety of new results can be found in the papers [2–6] and the references cited therein. Criteria for the existence of solutions or multiplicities of positive solutions are established for the boundary value problem on the half-line. The main tools used in the literature for such a problem are the coincidence degree theory of Mawhin, fixed point arguments together with the lower and upper solutions method. For example the readers are referred to [1–8] and the references therein.

Recently, many researchers pay more attention to the impulsive boundary value problems, such as Dirichlet boundary value problem, periodic boundary value problem, two

point boundary value problem and so on (see for example, [9, 10] and references therein). The existence or multiplicity of solutions for impulsive boundary value problems (IBVP) on the half-line has been studied by many authors [11–14]. Kaufmann *et al.* [13] investigated the following impulsive boundary value problem:

$$\begin{cases} -(qu')'(t) = f(t, u(t)), & t \in [0, +\infty), t \neq t_j, \\ \Delta(u'(t_j)) = I_j(u(t_j)), & j = 1, 2, \dots, p, \\ \alpha u'(0) - \beta \lim_{t \rightarrow 0^+} q(t)u'(t) = 0, \\ \gamma \lim_{t \rightarrow \infty} u(t) + \delta \lim_{t \rightarrow \infty} q(t)u'(t) = 0, \end{cases} \quad (1.3)$$

where $q \in C[0, \infty) \cap C^1[0, \infty)$ and $q(t) > 0$ for all $t > 0$; and $f : [0, +\infty) \times \mathbf{R} \rightarrow \mathbf{R}$ is continuous. By using the fixed point theorem, the existence of at least one solution for IBVP (1.3) is obtained.

In [14], Li and Nieto considered the existence of multiple positive solutions of the following IBVP on the half-line:

$$\begin{cases} u''(t) + q(t)f(t, u(t)) = 0, & t \in [0, +\infty), t \neq t_j, \\ \Delta(u(t_j)) = I_j(u(t_j)), & j = 1, 2, \dots, p, \\ \alpha u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), & \lim_{t \rightarrow \infty} u'(t) = 0, \end{cases}$$

where $q \in C([0, \infty), [0, \infty))$ and $f : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous. By using a fixed point theorem due to Avery and Peterson, the existence of at least three positive solutions is obtained.

On the other hand, critical point theory and variational methods are proved to be a powerful tool in studying the existence of solutions for the impulsive differential equations [15–28]. For some recent works on the theory of critical point theory and variational methods we refer the readers to [29].

In the case $m = \mu = 1$, Chen and Sun [17] studied and presented some results on the existence and multiplicity of solutions for IBVP (1.1) by using a variational method and a three critical points theorem due to Bonanno and Marano (see Theorem 2.1 of [30]). The result is as follows.

Theorem 1.1 ([17, Theorem 3.1]) *Suppose that the following conditions hold.*

- (H1) $g(u), I_j(u)$ are nondecreasing, and $g(u)u \geq 0, I_j(u)u \geq 0$ for any $u \in \mathbf{R}$.
- (H2) There exist positive constants a, l with $l < 2, b \in L^1[0, +\infty)$ and $c \in L^2[0, +\infty)$ such that

$$F(t, u) \leq b(t)(a + |u|^l), \quad f(t, u) \leq c(t)|u|^{l-1}, \quad \text{for a.e. } t \in [0, +\infty) \text{ and all } u \in \mathbf{R},$$

where $F(t, u) = \int_0^u f(t, s) ds$.

- (H3) There exist two constants $d, k > 0$ such that

$$\frac{d^2}{M^2} < k^2 + 2 \sum_{j=1}^p \int_0^{ke^{-t_j}} I_j(s) ds + 2 \int_0^k g(s) ds.$$

- (H4) $\frac{M^2 \int_0^{+\infty} \max_{|\xi| \leq d} F(t, \xi) dt}{d^2} < \frac{\int_0^{+\infty} F(t, ke^{-t}) dt}{k^2 + 2 \sum_{j=1}^p \int_0^{ke^{-t_j}} I_j(s) ds + 2 \int_0^k g(s) ds}$.

Then, for each $\lambda \in \left[\frac{k^2 + \sum_{j=1}^p \int_0^k I_j(s) ds + \int_0^k g(s) ds}{\int_0^{+\infty} F(t, ke^{-t}) dt}, \frac{d^2}{2M^2 \int_0^{+\infty} \max_{|\xi| \leq d} F(t, \xi) dt} \right]$, IBVP (1.1) has at least three classical solutions.

Soon after, in the case $m = \mu = 1$, by using the variant fountain theorems (see Theorem 2.2 of [31]), Dai and Zhang [25] obtained some existence theorems of solutions for IBVP (1.1) when the function g and the impulsive functions I_j ($j = 1, 2, \dots, p$) satisfies the following superlinear growth conditions:

(H1') $I_j(u)$ ($j = 1, 2, \dots, p$), $g(u)$ satisfy $\int_0^u I_j(s) ds \geq 0$, $\int_0^u g(s) ds \geq 0$ for any $u \in \mathbf{R}$; and there exist positive constants $\gamma_j > \alpha - 1$, $\theta > \alpha - 1$ ($1 < \alpha < 2$) and q_j, q ($j = 1, 2, \dots, p$) such that $I_j(u) \leq q_j |u|^{\gamma_j}$, $g(u) \leq q |u|^\theta$, $\forall u \in \mathbf{R}$.

However, there is no work for IBVP (1.1) when the parameter $\mu \neq 1$ and f is an L^2 -Carathéodory function. As a result, the goal of this paper is to fill the gap in this area. Our aim is to establish a precise open interval $\Lambda \subseteq (0, +\infty)$, for each $\lambda \in \Lambda$, there exists a $\delta > 0$ such that for each $\mu \in [0, \delta)$, IBVP (1.1) admits at least three classical solutions.

The remainder of the paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we will state and prove the main results of the paper, and also two examples are presented to illustrate our main results.

2 Preliminaries

In this section, we first introduce some notations and some necessary definitions.

Set

$$W := \{u : [0, +\infty) \rightarrow \mathbf{R} \text{ is absolutely continuous, } u' \in L^2[0, +\infty)\}.$$

Denote the Sobolev space X by

$$X := \left\{ u \in W : \int_0^{+\infty} (|u'(t)|^2 + m^2 |u(t)|^2) dt < \infty \right\},$$

with the inner product

$$(u, v) = \int_0^{+\infty} (u'(t)v'(t) + m^2 u(t)v(t)) dt,$$

which induces the norm

$$\|u\|_X := \left(\int_0^{+\infty} (|u'(t)|^2 + m^2 |u(t)|^2) dt \right)^{1/2}.$$

Obviously, X is a reflexive Banach space. We define the norm in $L^2([0, +\infty))$ as $\|u\|_2 = (\int_0^{+\infty} |u(t)|^2 dt)^{1/2}$, and let $\mathcal{C} := \{u \in C([0, +\infty)) : \sup_{t \in [0, +\infty)} |u(t)| < +\infty\}$, with the norm $\|u\|_\infty = \sup_{t \in [0, +\infty)} |u(t)|$. Then \mathcal{C} is a Banach space. In addition, X is continuously embedded into \mathcal{C} , and therefore, there exist two constants M_1, M_2 such that

$$\|u\|_\infty \leq M_1 \|u\|_X, \quad \text{for any } u \in X, \tag{2.1}$$

and

$$\|u\|_2 \leq M_2 \|u\|_X, \quad \text{for any } u \in X. \tag{2.2}$$

Suppose that $u \in C[0, +\infty)$. By a classical solution of IBVP (1.1), we mean a function

$$u \in \{z \in C([0, +\infty)) : z|_{[t_j, t_{j+1}]} \in C^2([t_j, t_{j+1}]), j = 0, 1, 2, \dots, p-1; \\ z|_{[t_p, +\infty)} \in C^2([t_p, +\infty))\}$$

that satisfies the equation in IBVP (1.1) a.e. on $[0, +\infty)$, the limits $u'(t_j^+), u'(t_j^-), j = 1, 2, \dots, p$ exist and the impulsive conditions in IBVP (1.1) hold; $u'(0^+), u'(+\infty)$ exist, and the boundary conditions in IBVP (1.1) also are met.

For each $u \in X$, put

$$\Phi(u) = \frac{1}{2} \|u\|_X^2 + G(u(0)), \tag{2.3}$$

$$\Psi(u) = \int_0^{+\infty} F(t, u(t)) dt - \frac{\mu}{\lambda} \sum_{j=1}^p \int_0^{u(t_j)} I_j(s) ds, \tag{2.4}$$

where $G(u(0)) = \int_0^{u(0)} g(s) ds, F(t, u) = \int_0^u f(t, s) ds$.

It is clear that Ψ is differentiable at any $u \in X$ and

$$\Psi'(u)(v) = \int_0^{+\infty} f(t, u(t))v(t) dt - \frac{\mu}{\lambda} \sum_{j=1}^p I_j(u(t_j))v(t_j), \tag{2.5}$$

for any $v \in X$.

Recall that a function $f : [0, +\infty) \times \mathbf{R} \rightarrow \mathbf{R}$ is said to be an L^2 -Carathéodory function, if

- (S1) $t \rightarrow f(t, u)$ is measurable for every $u \in \mathbf{R}$;
- (S2) $u \rightarrow f(t, u)$ is continuous for almost every $t \in [0, +\infty)$;
- (S3) for every $\rho > 0$ there exists a function $l_\rho \in L^2([0, +\infty))$ such that

$$\sup_{|u| \leq \rho} |f(t, u)| \leq l_\rho(t), \quad \text{for almost every } t \in [0, +\infty).$$

If we assume that the function f satisfies the further condition

- (S3') there exists a function $l \in L^2([0, +\infty))$ such that

$$\sup_{u \in \mathbf{R}} |f(t, u)| \leq l(t), \quad \text{for almost every } t \in [0, +\infty),$$

then one has the following result.

Lemma 2.1 *Suppose that condition (S3') holds. Then $\Psi' : X \rightarrow X^*$ is a compact operator. In particular, $\Psi : X \rightarrow \mathbf{R}$ is a weakly sequentially continuous functional.*

Proof Let Ω be a bounded set in X and let $\{a_n\}$ be a sequence in $\overline{\Psi'(\Omega)}$. Then there is a sequence $\{u_n\}$ in Ω such that $b_n = \Psi'(u_n)$ and $\|a_n - b_n\|_{X^*} < \frac{1}{n}$ for all $n \in \mathbf{N}$. Due to X being reflexive, there exists a subsequence u_{n_k} converging weakly to $u \in X$. We can prove that

$\{u_n\}$ has a subsequence which converges almost everywhere in $[0, +\infty)$ to the function u . In fact, given a number $r > 0$, let $\Omega_r := \Omega([0, r])$. Then we easily infer that $\{u_n\}$ is bounded in Ω_r . Pick $r = 1$, the Rellich-Kondrachov theorem [32, Theorem IX.16] yields a subsequence $\{u_n^{(1)}\}$ of $\{u_n\}$ such that $\lim_{n \rightarrow \infty} u_n^{(1)}(t) = u(t)$ at most all points $t \in [0, 1)$. Applying this argument again, with 1 replaced by 2, we also obtain a sequence $\{u_n^{(2)}\}$ of $\{u_n\}$ such that $\lim_{n \rightarrow \infty} u_n^{(2)}(t) = u(t)$ at almost all points $t \in [0, 2)$. Thus, the sequence $\{u_n^{(n)}\}$ clearly complies with the conclusion. Without loss of generality we write $\{u_n^{(n)}\}$ as $\{u_{n_k}\}$. Therefore, $\{f(t, u_{n_k})\}$ converges to $\{f(t, u(t))\}$ a.e. on $[0, \infty)$. From (2.1), (2.2), and (2.5), we have

$$\begin{aligned} & |\Psi'(u_{n_k})(v) - \Psi'(u)(v)| \\ & \leq \int_0^\infty |f(t, u_{n_k}(t)) - f(t, u(t))| \cdot |v(t)| \, dt + \frac{\mu}{\lambda} \sum_{j=1}^p |I_j(u_{n_k}(t_j)) - I_j(u(t_j))| \cdot |v(t_j)| \\ & \leq \left(\int_0^\infty |f(t, u_{n_k}(t)) - f(t, u(t))|^2 \, dt \right)^{1/2} \cdot \left(\int_0^\infty |v(t)|^2 \, dt \right)^{1/2} \\ & \quad + \frac{\mu}{\lambda} \sum_{j=1}^p |I_j(u_{n_k}(t_j)) - I_j(u(t_j))| \cdot \|v\|_\infty \\ & \leq M_2 \left(\int_0^\infty |f(t, u_{n_k}(t)) - f(t, u(t))|^2 \, dt \right)^{1/2} + \frac{\mu}{\lambda} M_1 \sum_{j=1}^p |I_j(u_{n_k}(t_j)) - I_j(u(t_j))| \end{aligned}$$

for all $v \in X$ with $\|v\|_X \leq 1$. Hence, from (S3'), the Lebesgue dominated convergence theorem and continuity of I_j show that the sequence $\{\Psi'(u_{n_k})\}$ converges to $\Psi'(u)$ in X^* . Therefore, taking into account that

$$\|a_{n_k} - \Psi'(u)\|_{X^*} \leq \|a_{n_k} - b_{n_k}\|_{X^*} + \|b_{n_k} - \Psi'(u)\|_{X^*},$$

the sequence $\{a_{n_k}\}$ converges in $\overline{\Psi'(\Omega)}$ and the compactness is proved.

Finally, it follows from Corollary 41.9 of [33, p.236] that Ψ is a weakly sequentially continuous functional. This completes the proof. \square

By standard arguments, we find that Φ is a Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_0^{+\infty} (u'(t)v'(t) + m^2 u(t)v(t)) \, dt + g(u(0))v(0), \tag{2.6}$$

for any $v \in X$.

Definition 2.2 A function $u \in X$ is said to be a weak solution IBVP (1.1) if u satisfies

$$\begin{aligned} & \int_0^{+\infty} u'(t)v'(t) \, dt + m^2 \int_0^{+\infty} u(t)v(t) \, dt - \lambda \int_0^{+\infty} f(t, u(t))v(t) \, dt \\ & \quad + \mu \sum_{j=1}^p I_j(u(t_j))v(t_j) + g(u(0))v(0) = 0, \end{aligned}$$

for any $v \in X$.

It is easy to verify that $u \in X$ is a weak solution to IBVP (1.1) if and only if u is a classical solution of IBVP (1.1).

Arguing in a standard way, it is easy to prove that the critical points of the functional $E_\lambda := \Phi(u) - \lambda\Psi(u)$ are the weak solution of IBVP (1.1) and so they are classical solutions.

The main tools to prove our results in Section 3 are the following critical points theorems.

Theorem 2.3 ([18]) *Let X be a reflexive real Banach space; $\Phi : X \rightarrow \mathbf{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* , $\Psi : X \rightarrow \mathbf{R}$ be a sequentially weakly upper semicontinuous, continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that $\Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{v} \in X$, with $r < \Phi(\bar{v})$ such that*

- (i) $\sup_{\Phi(u) \leq r} \Psi(u) < r\Psi(\bar{v})/\Phi(\bar{v})$,
- (ii) for each λ in

$$\Lambda_r := \left(\frac{\Phi(\bar{v})}{\Psi(\bar{v})}, \frac{r}{\sup_{\Phi(u) \leq r} \Psi(u)} \right),$$

the functional $\Phi - \lambda\Psi$ is coercive. Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X .

Theorem 2.4 ([34]) *Let X be a reflexive real Banach space; $\Phi : X \rightarrow \mathbf{R}$ be a convex, coercive and continuously Gâteaux differentiable functionals whose derivative admits a continuous inverse on X^* , $\Psi : X \rightarrow \mathbf{R}$ be a continuously Gâteaux differentiable functionals whose derivative is compact, such that*

- (1) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
- (2) for each $\lambda > 0$ and for every u_1, u_2 which are local minimum for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has

$$\inf_{s \in [0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are two positive constants r_1, r_2 , and $\bar{v} \in X$, with $2r_1 < \Phi(\bar{v}) < \frac{r_2}{2}$, such that

- (b1) $\frac{\sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)}{r_1} < \frac{2\Psi(\bar{v})}{3\Phi(\bar{v})}$;
- (b2) $\frac{\sup_{u \in \Phi^{-1}(-\infty, r_2)} \Psi(u)}{r_2} < \frac{\Psi(\bar{v})}{3\Phi(\bar{v})}$.

Then, for each $\lambda \in (\frac{3\Phi(\bar{v})}{2\Psi(\bar{v})}, \min\{\frac{r_1}{\sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)}, \frac{r_2}{2 \sup_{u \in \Phi^{-1}(-\infty, r_2)} \Psi(u)}\})$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(-\infty, r_2)$.

3 Main results

Lemma 3.1 *Suppose that*

- (C0) *there exist constants $\alpha_j, \beta_j > 0$, and $\sigma_j \in [0, 1)$ ($j = 1, 2, \dots, p$) such that*

$$|I_j(u)| \leq \alpha_j + \beta_j |u(t_j)|^{\sigma_j}, \quad \text{for any } u \in \mathbf{R}, j = 1, 2, \dots, p.$$

Then, for each $u \in X$, we have

$$\left| \sum_{j=1}^p \int_0^{u(t_j)} I_j(s) ds \right| \leq \sum_{j=1}^p \left(\alpha_j \|u\|_\infty + \frac{\beta_j}{\sigma_j + 1} \|u\|_\infty^{\sigma_j + 1} \right). \tag{3.1}$$

Proof By the condition (C0), we have

$$\left| \int_0^{u(t_j)} I_j(s) ds \right| \leq \alpha_j |u(t_j)| + \frac{\beta_j}{\sigma_j + 1} |u(t_j)|^{\sigma_j + 1}.$$

Thus, (3.1) is proved. □

Now we can state and prove our main results.

Theorem 3.2 *Assume that (C0) and $M_1^2 L < 1$ hold. Let $f : [0, +\infty) \times \mathbf{R} \rightarrow \mathbf{R}$ be an L^2 -Carathéodory function such that (S3') satisfies. Furthermore, suppose that there exist two positive constants a and b such that*

$$\begin{aligned} \text{(C1)} \quad & a^2 < \frac{M_1^2 b^2 (m^2 - 2M_1^2 L + 1)}{2}; \\ \text{(C2)} \quad & \frac{\int_0^{+\infty} F(t, be^{-t}) dt}{b^2 (m^2 + 2M_1^2 L + 1)} > \frac{M_1^2 \cdot (\int_0^{+\infty} \sup_{|\xi| \leq a} |f(t, \xi)|^2 dt)^{1/2}}{2a(1 - M_1^2 L)}, \text{ and } \int_0^{+\infty} F(t, be^{-t}) dt > 0. \end{aligned}$$

Then, for each λ in

$$\Lambda_1 := \left(\frac{b^2 (m^2 + 2M_1^2 L + 1)}{4 \int_0^{+\infty} F(t, be^{-t}) dt}, \frac{(1 - M_1^2 L)a}{2M_1^2 \cdot (\int_0^{+\infty} \sup_{|\xi| \leq a} |f(t, \xi)|^2 dt)^{1/2}} \right), \tag{3.2}$$

there exists

$$\delta_1 := \min \left\{ \frac{(1 - M_1^2 L)a - 2M_1^2 \lambda \cdot (\int_0^{+\infty} \sup_{|\xi| \leq a} |f(t, \xi)|^2 dt)^{1/2}}{2M_1^2 (\sum_{j=1}^p (\alpha_j a + \frac{\beta_j}{\sigma_j + 1} a^{\sigma_j + 1}))}, \frac{4\lambda \int_0^{+\infty} F(t, be^{-t}) dt - b^2 (m^2 + 2M_1^2 L + 1)}{4 \sum_{j=1}^p (\alpha_j b + \frac{\beta_j}{\sigma_j + 1} b^{\sigma_j + 1})} \right\} \tag{3.3}$$

such that for each $\mu \in [0, \delta_1)$, IBVP (1.1) has at least three distinct classical solutions.

Proof Obviously, under the condition (S3'), $\Psi : X \rightarrow \mathbf{R}$ is weakly sequentially lower semi-continuous and Gâteaux differentiable functional.

Note that as (1.2) holds for every $s_1, s_2 \in \mathbf{R}$ and $g(0) = 0$, one has

$$|g(s)| = |g(s) - g(0)| \leq L|s|, \quad \text{for every } s \in \mathbf{R}.$$

Furthermore, for any $u \in X$, one has

$$\Phi(u) = \frac{1}{2} \|u\|_X^2 + \int_0^{u(0)} g(s) ds \leq \frac{1}{2} \|u\|_X^2 + \int_0^{u(0)} L|s| ds \leq \frac{1 + M_1^2 L}{2} \|u\|_X^2 \tag{3.4}$$

and

$$\Phi(u) \geq \frac{1}{2} \|u\|_X^2 - \int_0^{u(0)} |g(s)| ds \geq \frac{1}{2} \|u\|_X^2 - \int_0^{u(0)} L|s| ds \geq \frac{1 - M_1^2 L}{2} \|u\|_X^2. \tag{3.5}$$

So Φ is coercive.

Next, we show that $\Phi' : X \rightarrow X^*$ admits a Lipschitz continuous inverse. For any $u \in X \setminus \{0\}$, it follows from (2.6) and (1.2) that

$$\langle \Phi'(u), u \rangle = \int_0^{+\infty} (|u'(t)|^2 + m^2|u(t)|^2) dt + g(u(0))u(0) \geq (1 - M_1^2L) \|u\|_X^2.$$

By the assumption $M_1^2L < 1$, it turns out that

$$\lim_{\|u\|_X \rightarrow +\infty} \frac{\langle \Phi'(u), u \rangle}{\|u\|_X} = +\infty,$$

that is, Φ' is coercive.

For any $u, v \in X$,

$$\begin{aligned} \langle \Phi'(u) - \Phi'(v), u - v \rangle &= \langle u - v, u - v \rangle + (g(u(0)) - g(v(0)))(u(0) - v(0)) \\ &\geq (1 - M_1^2L) \|u - v\|_X^2, \end{aligned}$$

so Φ' is a strongly monotone operator. By [35, Theorem 26.A], one finds that $(\Phi')^{-1}$ exists and is Lipschitz continuous on X^* . Hence the functionals Φ and Ψ satisfy the regularity assumptions of Theorem 2.3.

Furthermore for any fixed λ , and μ as in (3.2), (3.3). Put $r = \frac{(1-M_1^2L)a^2}{2M_1^2}$. Taking (2.1) into account, for every $u \in X$ such that $\Phi(u) \leq r$, we have $\sup_{t \in [0, +\infty)} |u(t)| \leq a$. Therefore, it follows from (2.4), (3.1), and the Hölder inequality that

$$\begin{aligned} \sup_{\Phi(u) \leq r} \Psi(u) &= \sup_{\Phi(u) \leq r} \left\{ \int_0^{+\infty} F(t, u(t)) dt - \frac{\mu}{\lambda} \sum_{j=1}^p \int_0^{u(t_j)} I_j(s) ds \right\} \\ &\leq \sup_{\Phi(u) \leq r} \left\{ \int_0^{+\infty} \sup_{|\xi| \leq a} |f(t, \xi)| |u(t)| dt + \frac{\mu}{\lambda} \sum_{j=1}^p \left(\alpha_j \|u\|_\infty + \frac{\beta_j}{\sigma_j + 1} \|u\|_\infty^{\sigma_j + 1} \right) \right\} \\ &\leq \sup_{\Phi(u) \leq r} \left(\int_0^{+\infty} \sup_{|\xi| \leq a} |f(t, \xi)|^2 dt \right)^{1/2} \cdot \|u\|_2 + \frac{\mu}{\lambda} \sum_{j=1}^p \left(\alpha_j a + \frac{\beta_j}{\sigma_j + 1} a^{\sigma_j + 1} \right) \\ &\leq a \cdot \left(\int_0^{+\infty} \sup_{|\xi| \leq a} |f(t, \xi)|^2 dt \right)^{1/2} + \frac{\mu}{\lambda} \sum_{j=1}^p \left(\alpha_j a + \frac{\beta_j}{\sigma_j + 1} a^{\sigma_j + 1} \right), \end{aligned}$$

which implies

$$\sup_{\Phi(u) \leq r} \frac{\Psi(u)}{r} \leq \frac{2M_1^2}{(1 - M_1^2L)a} \left\{ \left(\int_0^{+\infty} \sup_{|\xi| \leq a} |f(t, \xi)|^2 dt \right)^{1/2} + \frac{\mu}{\lambda} \sum_{j=1}^p \left(\alpha_j + \frac{\beta_j}{\sigma_j + 1} a^{\sigma_j} \right) \right\}.$$

Since $\mu < \delta_1$, one has

$$\sup_{\Phi(u) \leq r} \frac{\Psi(u)}{r} < \frac{1}{\lambda}. \tag{3.6}$$

Put $v_0(t) = be^{-t}$. Obviously, $v_0 \in X$, $\|v_0\|^2 = \frac{(m^2+1)b^2}{2}$. By a similar reasoning to (3.4) and (3.5), we get

$$\frac{m^2 - 2M_1^2L + 1}{4}b^2 \leq \Phi(v_0) \leq \frac{2M_1^2L + m^2 + 1}{4}b^2.$$

From (C1) we get $r < \Phi(v_0)$. It follows from (2.4) that

$$\begin{aligned} \Psi(v_0) &= \int_0^{+\infty} F(t, v_0(t)) dt - \frac{\mu}{\lambda} \sum_{j=1}^p \int_0^{v_0(t_j)} I_j(s) ds \\ &\geq \int_0^{+\infty} F(t, v_0(t)) dt - \frac{\mu}{\lambda} \sum_{j=1}^p \left(\alpha_j \|v_0(t_j)\|_\infty + \frac{\beta_j}{\sigma_j + 1} \|v_0(t_j)\|_\infty^{\sigma_j+1} \right) \\ &\geq \int_0^{+\infty} F(t, be^{-t}) dt - \frac{\mu}{\lambda} \sum_{j=1}^p \left(\alpha_j b + \frac{\beta_j}{\sigma_j + 1} b^{\sigma_j+1} \right), \end{aligned}$$

then

$$\frac{\Psi(v_0)}{\Phi(v_0)} \geq \frac{\int_0^{+\infty} F(t, be^{-t}) dt - \frac{\mu}{\lambda} \sum_{j=1}^p \left(\alpha_j b + \frac{\beta_j}{\sigma_j+1} b^{\sigma_j+1} \right)}{\frac{(m^2+2M_1^2L+1)b^2}{4}}.$$

Since $\mu < \delta_1$, one has

$$\frac{\Psi(v_0)}{\Phi(v_0)} > \frac{1}{\lambda}. \tag{3.7}$$

Combining with (3.6) and (3.7), condition (i) of Theorem 2.3 is fulfilled.

Next we will prove the coercivity of the functional E_λ .

Taking into account (S3') and the Hölder inequality, one has

$$\begin{aligned} \int_0^{+\infty} F(t, u(t)) dt &\leq \int_0^{+\infty} \left(\int_0^u \sup_{\xi \in \mathbb{R}} |f(t, \xi)| d\xi \right) dt \\ &\leq \int_0^{+\infty} \left(\int_0^u l(t) d\xi \right) dt \\ &\leq \int_0^{+\infty} |l(t)| |u(t)| dt \\ &\leq \left(\int_0^{+\infty} l^2(t) dt \right)^{1/2} \cdot \left(\int_0^{+\infty} |u(t)|^2 dt \right)^{1/2} \\ &= \|l\|_2 \cdot \|u\|_2 < +\infty, \end{aligned} \tag{3.8}$$

combining with (2.2) and Lemma 3.1, it follows that

$$\begin{aligned} E_\lambda(u) &= \Phi(u) - \lambda \Psi(u) \\ &= \frac{1}{2} \|u\|_X^2 + G(u(0)) - \lambda \int_0^{+\infty} F(t, u(t)) dt + \mu \sum_{j=1}^p \int_0^{u(t_j)} I_j(s) ds \end{aligned}$$

$$\begin{aligned}
 &\geq \frac{1 - M_1^2 L}{2} \|u\|_X^2 - \lambda \int_0^{+\infty} \left(\int_0^u \sup_{\xi \in \mathbf{R}} |f(t, \xi)| d\xi \right) dt \\
 &\quad - \mu \sum_{j=1}^p \left(\alpha_j \|u\|_\infty + \frac{\beta_j}{\sigma_j + 1} \|u\|_\infty^{\sigma_j + 1} \right) \\
 &\geq \frac{1 - M_1^2 L}{2} \|u\|_X^2 - \lambda \|l\|_2 \cdot \|u\|_2 - \mu \sum_{j=1}^p \left(\alpha_j M_1 \|u\|_X + \frac{\beta_j M_1^{\sigma_j + 1}}{\sigma_j + 1} \|u\|_X^{\sigma_j + 1} \right) \\
 &\geq \frac{1 - M_1^2 L}{2} \|u\|_X^2 - \left(\lambda M_2 \|l\|_2 + \mu M_1 \sum_{j=1}^p \alpha_j \right) \|u\|_X - \mu \sum_{j=1}^p \frac{\beta_j M_1^{\sigma_j + 1}}{\sigma_j + 1} \|u\|_X^{\sigma_j + 1}.
 \end{aligned}$$

Since $M_1^2 L < 1$, $1 \leq \sigma_j + 1 < 2$, the above inequality implies that $\lim_{\|u\|_X \rightarrow +\infty} E_\lambda(u) = +\infty$, so $E_\lambda(u)$ is coercive. According to Theorem 2.3, it follows that, for each

$$\lambda \in \Lambda_1,$$

the functional $E_\lambda(u) = \Phi(u) - \lambda \Psi(u)$ has at least three distinct critical points, i.e. IBVP (1.1) has at least three distinct weak solutions. This completes the proof. \square

Let

$$I_r^* := \sum_{j=1}^p \min_{|\xi| \leq r} \int_0^\xi I_j(s) ds, \quad \text{for any } r > 0,$$

and

$$\theta = \frac{2M_1^2}{1 - M_1^2 L}.$$

Theorem 3.3 Assume that $M_1^2 L < 1$ holds, and $f : [0, +\infty) \times \mathbf{R} \rightarrow \mathbf{R}$ be an L^2 -Carathéodory function such that (S3') satisfies, and $f(t, u) \geq 0$ for all $(t, u) \in [0, +\infty) \times \mathbf{R}$. Furthermore, suppose that there exist a function $\omega \in X$ and two positive constants c_1, c_2 with $\frac{4c_1}{1 - M_1^2 L} < \|\omega\|_X^2 < \frac{c_2}{1 + M_1^2 L}$ such that

(D1)

$$\frac{\int_0^{+\infty} \sup_{|u| \leq \sqrt{\theta c_1}} F(t, u) dt}{c_1} < \frac{4 \int_0^{+\infty} F(t, \omega(t)) dt}{3(1 + M_1^2 L) \|\omega\|_X^2};$$

(D2)

$$\frac{\int_0^{+\infty} \sup_{|u| \leq \sqrt{\theta c_2}} F(t, u) dt}{c_2} < \frac{2 \int_0^{+\infty} F(t, \omega(t)) dt}{3(1 + M_1^2 L) \|\omega\|_X^2}.$$

Then, for each λ in

$$\Lambda_2 := \left(\frac{3(1 + M_1^2 L) \|\omega\|_X^2}{4 \int_0^{+\infty} F(t, \omega(t)) dt}, \min \left\{ \frac{c_1}{\int_0^{+\infty} \sup_{|u| \leq \sqrt{\theta c_1}} F(t, u) dt}, \frac{c_2}{2 \int_0^{+\infty} \sup_{|u| \leq \sqrt{\theta c_2}} F(t, u) dt} \right\} \right), \tag{3.9}$$

and for every negative continuous function $I_j, j = 1, 2, \dots, p$, and $g(u) \leq 0$ for all $u \in \mathbf{R}$, there exists

$$\delta_2 := \min \left\{ \frac{\lambda \int_0^{+\infty} \sup_{|u| \leq \sqrt{\theta c_1}} F(t, u) dt - c_1}{I^*_{\sqrt{\theta c_1}}}, \frac{\lambda \int_0^{+\infty} \sup_{|u| \leq \sqrt{\theta c_2}} F(t, u) dt - \frac{c_2}{2}}{I^*_{\sqrt{\theta c_2}}} \right\}$$

such that, for each $\mu \in [0, \delta_2)$, IBVP (1.1) has three distinct classical solutions $u_i, i = 1, 2, 3$, with $0 < \|u_i\|_X \leq \sqrt{\frac{2c_2}{1-M_1^2 L}}$.

Proof In order to apply Theorem 2.4 to IBVP (1.1), we take the functionals $\Phi, \Psi : X \rightarrow \mathbf{R}$ as given in (2.3) and (2.4). Obviously, Φ and Ψ satisfy the conditions (1) of Theorem 2.4. Now we will prove that the functional $E_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ satisfies the assumption (2) of Theorem 2.4. Let u^* and u^{**} be two local minima for $E_\lambda(u)$. Then u^* and u^{**} are critical points for $E_\lambda(u)$, which implies that u^*, u^{**} are weak solutions of IBVP (1.1). In particular u^* and u^{**} are nonnegative. Indeed, with no loss of generality we may assume that u_0 be a weak solution of IBVP (1.1), and the set $\Omega = \{t \in (0, +\infty) : u_0(t) < 0\}$ is nonempty and of positive measure. Furthermore, taking into account that u_0 is a weak solution, one has

$$\int_0^{+\infty} u_0''(t)v(t) dt - m^2 \int_0^{+\infty} u_0(t)v(t) dt + \lambda \int_0^{+\infty} f(t, u_0(t))v(t) dt = 0,$$

for all $v \in X$.

Put $\bar{v}(t) = \max\{0, -u_0(t)\}$ for all $t \in [0, +\infty)$. Clearly $\bar{v} \in X$ and we deduce that

$$\begin{aligned} 0 &= \sum_{j=0}^p \int_{t_j}^{t_{j+1}} u_0''(t)\bar{v}(t) dt - m^2 \int_0^{+\infty} u_0(t)\bar{v}(t) dt + \lambda \int_0^{+\infty} f(t, u_0(t))\bar{v}(t) dt \\ &= \sum_{j=0}^p u_0'(t)\bar{v}(t)|_{t_j}^{t_{j+1}} - \int_0^{+\infty} u_0'(t)\bar{v}'(t) dt - m^2 \int_0^{+\infty} u_0(t)\bar{v}(t) dt \\ &\quad + \lambda \int_0^{+\infty} f(t, u_0(t))\bar{v}(t) dt \\ &= -\sum_{j=1}^p \Delta u_0'(t_j)\bar{v}(t_j) - \int_0^{+\infty} u_0'(t)\bar{v}'(t) dt - m^2 \int_0^{+\infty} u_0(t)\bar{v}(t) dt \\ &\quad + \lambda \int_0^{+\infty} f(t, u_0(t))\bar{v}(t) dt - g(u_0(0))\bar{v}(0) \\ &= -\mu \sum_{j=1}^p \Delta I_j(u_0(t_j))\bar{v}(t_j) + \int_0^{+\infty} (\bar{v}'(t))^2 dt + m^2 \int_0^{+\infty} (\bar{v}(t))^2 dt \\ &\quad + \lambda \int_0^{+\infty} f(t, u_0(t))\bar{v}(t) dt - g(u_0(0))\bar{v}(0) \\ &\geq \|\bar{v}\|^2, \end{aligned}$$

which implies $\bar{v}(t) = 0$ for $t \in [0, +\infty)$. Hence, $u_0 \equiv 0$ on Ω , which is absurd. Then we obtain $u^*(t) \geq 0, u^{**}(t) \geq 0$ for all $t \in [0, +\infty)$. So, one has $su^* + (1-s)u^{**} \geq 0$ for every $s \in [0, 1]$, which implies that

$$f(t, su^*(t) + (1-s)u^{**}(t)) \geq 0,$$

and

$$I_j(su^*(t_j) + (1-s)u^{**}(t_j)) \leq 0, \quad j = 1, 2, \dots, p.$$

Consequently, $\Psi(su^* + (1-s)u^{**}) \geq 0$ for every $s \in [0, 1]$.

From (3.4) and (3.5), we have

$$\frac{1 - M_1^2 L}{2} \|u\|_X^2 \leq \Phi(u) \leq \frac{1 + M_1^2 L}{2} \|u\|_X^2, \tag{3.10}$$

for all $u \in X$.

Note that by the condition $\frac{4c_1}{1 - M_1^2 L} < \|\omega\|^2 < \frac{c_2}{1 + M_1^2 L}$, we get $2c_1 < \Phi(\omega) < \frac{c_2}{2}$. It follows from the definition of Φ , (2.1), and (3.10) that

$$\begin{aligned} \Phi^{-1}(-\infty, r) &= \{u \in X : \Phi(u) < r\} \\ &\subseteq \left\{ u \in X : \|u\|_X < \sqrt{\frac{2r}{1 - M_1^2 L}} \right\} \\ &\subseteq \left\{ u \in X : \|u\|_\infty < \sqrt{\frac{2M_1^2 r}{1 - M_1^2 L}} \right\} \\ &= \{u \in X : |u(t)| < \sqrt{\theta r} \text{ for all } t \in [0, +\infty)\}. \end{aligned}$$

So, we have

$$\begin{aligned} \sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u) &= \sup_{u \in \Phi^{-1}(-\infty, r)} \left\{ \int_0^{+\infty} F(t, u(t)) dt - \frac{\mu}{\lambda} \sum_{j=1}^p \int_0^{u(t_j)} I_j(s) ds \right\} \\ &\leq \int_0^{+\infty} \sup_{|u| \leq \sqrt{\theta r}} F(t, u(t)) dt - \frac{\mu}{\lambda} I_{\sqrt{\theta r}}^*. \end{aligned}$$

Therefore, due to the assumption (D1), one gets

$$\begin{aligned} \frac{1}{c_1} \cdot \sup_{u \in \Phi^{-1}(-\infty, c_1)} \Psi(u) &\leq \frac{\int_0^{+\infty} \sup_{|u| \leq \sqrt{\theta c_1}} F(t, u(t)) dt - \frac{\mu}{\lambda} I_{\sqrt{\theta c_1}}^*}{c_1} \\ &< \frac{1}{\lambda} \leq \frac{4}{3(1 + M_1^2 L)} \cdot \frac{\int_0^{+\infty} F(t, \omega(t)) dt}{\|\omega\|_X^2} \\ &\leq \frac{2}{3} \frac{\Psi(\omega)}{\Phi(\omega)}. \end{aligned}$$

From the assumption (D2), one infers that

$$\begin{aligned} \frac{1}{c_2} \cdot \sup_{u \in \Phi^{-1}(-\infty, c_2)} \Psi(u) &\leq \frac{\int_0^{+\infty} \sup_{|u| \leq \sqrt{\theta c_2}} F(t, u(t)) dt - \frac{\mu}{\lambda} I_{\sqrt{\theta c_2}}^*}{c_2} \\ &< \frac{1}{2\lambda} \leq \frac{2}{3(1 + M_1^2 L)} \cdot \frac{\int_0^{+\infty} F(t, \omega(t)) dt}{\|\omega\|_X^2} \\ &\leq \frac{1}{3} \frac{\Psi(\omega)}{\Phi(\omega)}. \end{aligned}$$

So, the conditions (b1) and (b2) of Theorem 2.4 are satisfied. Then by means of Theorem 2.4, IBVP (1.1) admits at least three distinct weak solutions u_i ($i = 1, 2, 3$) in X , such that $0 < \|u_i\|_X \leq \sqrt{\frac{2c_2}{1-M_1^2L}}$. This completes the proof. \square

Example 3.4 Consider the following boundary value problem with impulsive effect:

$$\begin{cases} -u''(t) + \frac{1}{2}u(t) = \lambda f(t, u(t)), & t \in [0, +\infty), t \neq t_1, \\ \Delta(u'(t_1)) = \mu I_1(u(t_1)), \\ u'(0^+) = g(u(0)), \quad u'(+\infty) = 0, \end{cases} \quad (3.11)$$

where $m = \frac{\sqrt{2}}{2}$, $I_1(u) = 2 - \sqrt{|u(t_1)|}$. Choose $\alpha_1 = 2$, $\beta_1 = 1$, $\sigma_1 = \frac{1}{2}$, then the condition (C0) is satisfied. M_1 is a positive constant defined in (2.1). When M_1 lies in different intervals, we can choose different f and g satisfying the conditions. Hence we only consider one case. If $M_1 \leq \frac{\sqrt{2}}{4}$, we may choose $g(x) = \frac{1}{4M_1^2} \sin x$ and

$$f(t, u) = \begin{cases} M_1 e^{-t}, & u \leq \frac{M_1}{2}, \\ \frac{4e^{-t}u^2}{M_1}, & \frac{M_1}{2} < u \leq M_1, \\ \frac{4e^{-t}M_1^2}{u}, & u > M_1, \end{cases}$$

where M_1 is given in (2.1). Then

$$F(t, u) = \begin{cases} M_1 e^{-t}u, & u \leq \frac{M_1}{2}, \\ \frac{4e^{-t}u^3}{3M_1} + \frac{M_1^2 e^{-t}}{3}, & \frac{M_1}{2} < u \leq M_1, \\ 4e^{-t}M_1^2 \ln u + (\frac{5}{3} - \ln M_1)M_1^2 e^{-t}, & u > M_1. \end{cases}$$

Obviously, f satisfies (S3') with $l(t) = 4M_1 e^{-t}$, and g satisfies (1.2) with $L = \frac{1}{4M_1^2}$, and $M_1^2 L < 1$. Take $a = \frac{M_1^2}{2}$, $b = M_1$. By simple calculations (C1) and (C2) are satisfied. Applying Theorem 3.2, IBVP (3.11) admits at least three distinct classical solutions for each $\lambda \in (\frac{24}{35}, \frac{3\sqrt{2}}{16M_1})$, and for each

$$0 < \mu < \min \left\{ \frac{9 - 24\sqrt{2}\lambda M_1}{(48 + 8\sqrt{2}M_1)M_1^2}, \frac{(35\lambda - 24)M_1}{96 + 32\sqrt{M_1}} \right\}.$$

Example 3.5 Consider the following problem:

$$\begin{cases} -u''(t) + u(t) = \lambda f(t, u(t)), & t \in [0, +\infty), t \neq t_j, \\ \Delta(u'(t_1)) = \mu I_1(u(t_1)), \\ u'(0^+) = g(u(0)), \quad u'(+\infty) = 0, \end{cases} \quad (3.12)$$

where $m = 1$, $I_1(u) = -1 - |u(t_1)|$. Then $I_1(u) \leq 0$ for any $u \in \mathbf{R}$. M_1 is a positive constant defined in (2.1). When M_1 lies in different intervals, we can choose different f and g satisfying the conditions. For example, if $\frac{\sqrt{3}}{48} \leq M_1 < 1$, we may choose $g(x) = -\frac{1-M_1^2}{M_1^2}(1 - \cos x)$ and

$$f(t, u) = \begin{cases} u^2 e^{-t}, & u \leq 4M_1, \\ \frac{256M_1^4 e^{-t}}{u^2}, & u > 4M_1, \end{cases}$$

where M_1 is given in (2.1). Then

$$F(t, u) = \begin{cases} \frac{1}{3}u^3e^{-t}, & u \leq 4M_1, \\ -\frac{256M_1^4e^{-t}}{u} + \frac{256M_1^3e^{-t}}{3}, & u > 4M_1. \end{cases}$$

Obviously, $M_1^2L < 1$, $f(t, u) \geq 0$ for all $(t, u) \in [0, +\infty) \times \mathbf{R}$ and satisfies (S3') with $l(t) = 16M_1^2e^{-t}$. g satisfies (1.2) with $L = \frac{1-M_1^2}{M_1^2} > 0$, and $g(u) \leq 0$ for any $u \in \mathbf{R}$. Take $\omega(t) = 4M_1e^{-t}$, $c_1 = \frac{M_1^2}{288}$, and $c_2 = 800M_1^2$. Then $\theta = 2$, $\frac{4c_1}{1-M_1^2L} < \|\omega\|_X^2 = 16M_1^2 < \frac{c_2}{1+M_1^2L}$. By simple calculations (D1) and (D2) are satisfied. Applying Theorem 3.2, for each $\lambda \in (\frac{18-9M_1^2}{4M_1}, \frac{18}{M_1})$ and for each

$$0 < \mu < \min \left\{ \frac{M_1(1 - \frac{\lambda M_1}{18})}{M_1 + 24}, \frac{10M_1(1 - \frac{18\lambda M_1}{125})}{20M_1 + 1} \right\},$$

IBVP (3.12) admits at least three distinct classical solutions u_i ($i = 1, 2, 3$) with $0 < \|u_i\|_X \leq 40$.

We observe that in Example 3.4 and Example 3.5 the functions f , g , and the impulsive term do not satisfy the conditions (H1), (H2) of Theorem 3.1 in [17] or the conditions of Theorem 3.2 in [25]. Hence, the problem (3.11) and (3.12) cannot be dealt with by the results of [17, 25].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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References

1. Agarwal, RP, O'Regan, D: Infinite Interval Problems for Differential, Difference and Integral Equations. Kluwer Academic, Dordrecht (2001)
2. Struwe, M: Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Springer, Berlin (2008)
3. Eloe, PW, Kaufmann, ER, Tisdell, CC: Multiple solutions of a boundary value problem on an unbounded domain. *Dyn. Syst. Appl.* **15**(1), 53-63 (2006)
4. Lian, H, Wang, P, Ge, W: Unbounded upper and lower solutions method for Sturm-Liouville boundary value problem on infinite intervals. *Nonlinear Anal. TMA* **70**, 2627-2633 (2009)
5. Yan, B, O'Regan, D, Agarwal, RP: Unbounded solutions for singular boundary value problems on the semi-infinite interval: upper and lower solutions and multiplicity. *J. Comput. Appl. Math.* **197**, 365-386 (2006)
6. Zhao, Y, Chen, H, Xu, C: Existence of multiple solutions for three-point boundary-value problems on infinite intervals in Banach spaces. *Electron. J. Differ. Equ.* **2012**, 44 (2012)
7. Chen, S, Zhang, Y: Singular boundary value problems on a half-line. *J. Math. Anal. Appl.* **195**, 449-468 (1995)
8. Liu, B, Li, J, Liu, L: Existence and uniqueness for an m -point boundary value problem at resonance on infinite intervals. *Comput. Math. Appl.* **64**, 1677-1690 (2012)
9. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
10. Benchohra, M, Henderson, J, Ntouyas, S: Theory of Impulsive Differential Equations. Contemporary Mathematics and Its Applications. Hindawi Publishing Corporation, New York (2006)
11. Liu, L, Liu, Z, Wu, Y: Infinite boundary value problems for n th-order nonlinear impulsive integro-differential equations in Banach spaces. *Nonlinear Anal. TMA* **67**, 2670-2679 (2007)
12. Yan, B: Boundary value problems on the half-line with impulsive and infinite delay. *J. Math. Anal. Appl.* **259**, 94-114 (2001)

13. Kaufmann, ER, Kosmatov, N, Raffoul, YN: A second-order boundary value problem with impulsive effects on an unbounded domain. *Nonlinear Anal. TMA* **69**, 2924-2929 (2008)
14. Li, J, Nieto, JJ: Existence of positive solutions for multiple boundary value problems on the half-line with impulsive. *Bound. Value Probl.* **2009**, Article ID 834158 (2009)
15. Gomes, M, Sanchez, L: A variational approach to some boundary value problems in the half-line. *Z. Angew. Math. Phys.* **56**, 192-209 (2005)
16. Nieto, JJ, O'Regan, D: Variational approach to impulsive differential equations. *Nonlinear Anal., Real World Appl.* **10**, 680-690 (2009)
17. Chen, H, Sun, J: An application of variational method to second-order impulsive differential equation on the half-line. *Appl. Math. Comput.* **217**, 1863-1869 (2010)
18. Bonanno, G, Marano, SA: On the structure of the critical set of non-differentiable functions with a weak compactness condition. *Appl. Anal.* **89**, 1-10 (2010)
19. Sun, J, Chen, H, Nieto, JJ, Otero-Novoa, M: Multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects. *Nonlinear Anal. TMA* **72**, 4575-4586 (2010)
20. Sun, J, Chen, H: Multiple positive solutions for multi-point boundary value problems with a p -Laplacian on the half-line. *J. Appl. Math. Comput.* **33**, 173-191 (2010)
21. Zhang, H, Li, Z: Periodic and homoclinic solutions generated by impulses. *Nonlinear Anal., Real World Appl.* **12**, 39-51 (2011)
22. Chen, P, Tang, X: Existence and multiplicity of solutions for second-order impulsive differential equations with Dirichlet problems. *Appl. Math. Comput.* **218**, 11775-11789 (2012)
23. Tian, Y, Ge, W: Variational methods to Sturm-Liouville boundary value problem for impulsive differential equations. *Nonlinear Anal. TMA* **72**, 277-287 (2010)
24. Tian, Y, Ge, W: Multiple solutions of impulsive Sturm-Liouville boundary value problem via lower and upper solutions and variational methods. *J. Math. Anal. Appl.* **387**, 475-489 (2012)
25. Dai, B, Zhang, D: The existence and multiplicity of solutions for second-order impulsive differential equations on the half-line. *Results Math.* **63**, 135-149 (2013)
26. Ricceri, B: A general variational principle and some of its applications. *J. Comput. Appl. Math.* **113**, 401-410 (2000)
27. Xiao, J, Nieto, JJ, Luo, Z: Multiplicity of solutions for nonlinear second order impulsive differential equations with linear derivative dependence via variational methods. *Commun. Nonlinear Sci. Numer. Simul.* **17**, 426-432 (2012)
28. Xu, J, Wei, Z, Ding, Y: Existence of weak solutions for p -Laplacian problem with impulsive effects. *Taiwan. J. Math.* **17**(2), 501-515 (2013)
29. Mawhin, J, Willem, M: *Critical Point Theory and Hamiltonian Systems*. Springer, Berlin (1989)
30. Bonanno, G, Riccobono, G: Multiplicity results for Sturm-Liouville boundary value problems. *Appl. Math. Comput.* **210**, 294-297 (2009)
31. Zou, W: *Variant fountain theorems and their applications*. *Manuscr. Math.* **104**, 343-358 (2001)
32. Brézis, H: *Analyse Fonctionnelle-Théorie et Applications*. Masson, Paris (1983)
33. Zeidler, E: *Nonlinear Functional Analysis and Its Applications. III*. Springer, Berlin (1985)
34. Bonanno, G, Candito, P: Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities. *J. Differ. Equ.* **244**, 3031-3059 (2008)
35. Zeidler, E: *Nonlinear Functional Analysis and Its Applications. II/B*. Springer, Berlin (1990)

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