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Solvability for second-order nonlocal boundary value problems with $\dim(\ker M) = 2$

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Abstract
The existence of at least one solution to the second-order nonlocal boundary value problems on the real line is investigated by using an extension of Mawhin's continuation theorem.

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1 Introduction

Boundary value problems on an infinite interval arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and in various applications such as an unsteady flow of gas through a semi-infinite porous medium, theory of drain flows and plasma physics. For an extensive collection of results to boundary value problems on unbounded domains, we refer the reader to a monograph by Agarwal and O'Regan [1]. The study of nonlocal elliptic boundary value problems was investigated by Bicadze and Samarskiĭ [2], and later continued by Il'in and Moiseev [3] and Gupta [4]. Since then, the existence of solutions for nonlocal boundary value problems has received a great deal of attention in the literature. For more recent results, we refer the reader to [5–22] and the references therein.

In this paper, we consider the following second-order nonlinear differential equation with integral boundary conditions:

$$\begin{cases} (c\varphi_p(u'))'(t) = f(t,u(t),u'(t)), \quad \text{a.e. } t \in (-\infty,\infty), \\ \lim_{t\to-\infty}(c\varphi_p(u'))(t) = \int_{-\infty}^{\infty} g(s)(c\varphi_p(u'))(s)\,ds, \\ \lim_{t\to\infty}(c\varphi_p(u'))(t) = \int_{-\infty}^{\infty} h(s)(c\varphi_p(u'))(s)\,ds, \end{cases} \tag{1}$$

where $\varphi_p(s) := |s|^{p-2}s$, $p > 1$, $f : (-\infty,\infty)^3 \to (-\infty,\infty)$ is a Carathéodory function, *i.e.*, $f = f(t,u,v)$ is Lebesgue measurable in t for all $(u,v) \in (-\infty,\infty)^2$ and continuous in (u,v) for almost all $t \in (-\infty,\infty)$. Throughout this paper, we assume that the following assumptions hold:

(H1) $g,h \in L^1(-\infty,\infty)$ satisfy $\int_{-\infty}^{\infty} g(s)\,ds = \int_{-\infty}^{\infty} h(s)\,ds = 1$;

(H2) $c : (-\infty,\infty) \to (0,\infty)$ is a continuous function which satisfy
$\varphi_p^{-1}(\frac{1}{c}) \in L^1_{\text{loc}}(-\infty,\infty) \setminus L^1(-\infty,\infty)$;

(H3) let $w(t) := \int_0^t \varphi_p^{-1}(\frac{1}{c(s)})\,ds$, and there exist nonnegative measurable functions α, β and γ such that $(1+|w|)^{p-1}\alpha, \beta/c, \gamma \in L^1(-\infty,\infty)$ and

$$\left|f(t,u,v)\right| \le \alpha(t)|u|^{p-1} + \beta(t)|v|^{p-1} + \gamma(t), \quad \text{a.e. } t \in (-\infty,\infty);$$

(H4) there exists a function $k(t)$ such that $(1+|w(\cdot)|)e^{-k(\cdot)} \in L^1(-\infty,\infty)$ and

$$\triangle := a_{11}a_{22} - a_{12}a_{21} \neq 0,$$

where $a_{11} := Q_2(w(\cdot)e^{-k(\cdot)})$, $a_{12} := -Q_1(w(\cdot)e^{-k(\cdot)})$, $a_{21} := -Q_2(e^{-k(\cdot)})$, $a_{22} := Q_1(e^{-k(\cdot)})$, and $Q_1, Q_2 : L^1(-\infty,\infty) \to (-\infty,\infty)$ will be defined in Section 3.

A boundary value problem is called a resonance one if the corresponding homogeneous boundary value problem has a non-trivial solution. Resonance problems can be expressed as an abstract equation $Lx = Nx$, where L is a noninvertible operator. When L is linear, Mawhin's continuation theorem [23] is an efficient tool in finding solutions for these problems. However, it is not suitable for the case L is nonlinear. Recently, Ge and Ren [24] extended Mawhin's continuation theorem from the case of linear L to the case of quasi-linear L. The purpose of this paper is to establish the sufficient conditions for the existence of solutions to the problem (1) on the real line at resonance with $\dim(\ker L) = 2$ by using an extension of Mawhin's continuation theorem [24].

2 Preliminaries

In this section, we recall some definitions and theorems. Let X and Y be two Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively.

Definition 2.1 A continuous operator $M : X \cap \operatorname{dom} M \to Y$ is said to be quasi-linear if

(i) $\operatorname{Im} M := M(X \cap \operatorname{dom} M)$ is a closed subset of Y;
(ii) $\operatorname{Ker} M := \{x \in X \cap \operatorname{dom} M : Mx = 0\}$ is linearly homeomorphic to $(-\infty,\infty)^n$ for some $n < \infty$.

Definition 2.2 Let $M : X \cap \operatorname{dom} M \to Y$ be a quasi-linear operator. Let $X_1 = \operatorname{Ker} M$ and $\Omega \subset X$ be an open and bounded set with the origin $\theta_X \in \Omega$. Then $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is said to be M-compact in $\overline{\Omega}$ if $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is a continuous operator, and there exist a vector subspace Y_1 of Y satisfying $\dim Y_1 = \dim X_1$ and an operator $R : \overline{\Omega} \times [0,1] \to X_2$ being continuous and compact such that, for $\lambda \in [0,1]$,

(i) $(I-Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I-Q)Y$;
(ii) $QN_\lambda x = \theta_Y$, $\lambda \in (0,1) \Leftrightarrow QN_1 x = \theta_Y$;
(iii) $R(\cdot,0)$ is the zero operator and $R(\cdot,\lambda)|_{\Sigma_\lambda} = (I-P)|_{\Sigma_\lambda}$, where $\Sigma_\lambda = \{x \in \overline{\Omega} : Mx = N_\lambda x\}$;
(iv) $M[P + R(\cdot,\lambda)] = (I-Q)N_\lambda$.

Here, X_2 is a complement space of X_1 in X, θ_Y is the origin of Y and $P : X \to X_1$, $Q : Y \to Y_1$ are projections.

Now, we give an extension of Mawhin's continuation theorem [24].

Theorem 2.3 *Let $\Omega \subset X$ be an open and bounded set with $\theta_X \in \Omega$. Suppose that M : $X \cap \operatorname{dom} M \to Y$ is a quasi-linear operator and $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is M-compact. In addition, if the following conditions hold*:

(A1) $Mx \neq N_\lambda x$ *for every* $(u,\lambda) \in (\operatorname{dom} M \cap \partial\Omega) \times (0,1)$;
(A2) $\deg\{JQN_1, \Omega \cap \operatorname{Ker} M, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to \operatorname{Ker} M$ *is a homeomorphism with* $J(\theta_X) = \theta_Y$,

then the abstract equation $Mx = N_1 x$ *has at least one solution in* $\overline{\Omega}$.

Finally, we give a theorem which is useful to show the compactness of operators defined on an infinite interval.

Theorem 2.4 [1] *Let Z be the space of all bounded continuous functions on $(-\infty,\infty)$ and $S \subset Z$. Then S is relatively compact in Z if the following conditions hold*:
(i) *S is bounded in Z*;
(ii) *S is equicontinuous on any compact interval of $(-\infty,\infty)$*;
(iii) *S is equiconvergent at $\pm\infty$, that is, given $\epsilon > 0$, there exists a constant $T = T(\epsilon) > 0$ such that $|\phi(t) - \phi(\infty)| < \epsilon$ (respectively, $|\phi(t) - \phi(-\infty)| < \epsilon$) for all $t > T$ (respectively, $t < -T$) and all $\phi \in S$.*

3 Main result

Let X be the set of the functions $u \in C^1(-\infty,\infty)$ such that

$$\frac{u}{1+|w|}, \varphi_p^{-1}(c)u' \in L^\infty(-\infty,\infty),$$

where w is the function in the assumption (H3). Then X is a Banach space equipped with a norm $\|u\|_X = \|u\|_1 + \|u\|_2$, where

$$\|u\|_1 = \sup_{t\in(-\infty,\infty)} \frac{|u(t)|}{1+|w(t)|} \quad \text{and} \quad \|u\|_2 = \sup_{t\in(-\infty,\infty)} \bigl|\bigl(\varphi_p^{-1}(c)u'\bigr)(t)\bigr|.$$

Let Y denote the Banach space $L^1(-\infty,\infty)$ equipped with a usual norm

$$\|h\|_Y = \int_{-\infty}^{\infty} \bigl|h(s)\bigr|\,ds.$$

Remark 3.1
(1) It is well known that, for any $u, v \in (-\infty,\infty)$ and $q > 0$,

$$|u+v|^q \le \max\bigl\{1, 2^{q-1}\bigr\}\bigl(|u|^q + |v|^q\bigr).$$

Thus, $\varphi_p^{-1}(u+v) \le \alpha_p(\varphi_p^{-1}(u) + \varphi_p^{-1}(v))$ for all $u, v \ge 0$, where $\alpha_p := \max\{1, 2^{\frac{2-p}{p-1}}\}$.
(2) Since $\varphi_p^{-1}(\frac{1}{c}) \in L^1_{\mathrm{loc}}(-\infty,\infty) \setminus Y$, then w is a continuous function which satisfies $\lim_{t\to\infty} w(t) = \infty$ and $\lim_{t\to-\infty} w(t) = -\infty$.
(3) For any continuous functions $w(t)$, we can choose a function $k(t)$ which satisfies $(1+|w(\cdot)|)e^{-k(\cdot)} \in Y$. For example, put $k(t) = \int_0^t (1+|w(s)|)\,ds$, then $(1+|w(\cdot)|)e^{-k(\cdot)} \in Y$.

Define $M : X \cap \operatorname{dom} M \to Y$ by $Mu = (c\varphi_p(u'))'$, where

$$\operatorname{dom} M = \Big\{ u : \big(c\varphi_p\big(u'\big)\big)' \in Y, \lim_{t\to-\infty} \big(c\varphi_p\big(u'\big)\big)(t) = \int_{-\infty}^{\infty} g(s)\big(c\varphi_p\big(u'\big)\big)(s)\,ds,$$
$$\text{and} \lim_{t\to\infty} \big(c\varphi_p\big(u'\big)\big)(t) = \int_{-\infty}^{\infty} h(s)\big(c\varphi_p\big(u'\big)\big)(s)\,ds \Big\}.$$

Then $M : X \cap \operatorname{dom} M \to Y$ is continuous. Let Ω be an open bounded subset of X such that $\operatorname{dom} M \cap \overline{\Omega} \neq \emptyset$. For $\lambda \in [0,1]$, define $N_\lambda : \Omega \to Y$ by $N_\lambda x = \lambda f(\cdot, x, x')$. By (H3) and the Lebesgue dominated convergence theorem, N_λ is continuous. Denote N_1 by N. Then problem (1) is equivalent to $Mx = Nx$, $x \in \operatorname{dom} M$. Define $Q_1, Q_2 : Y \to (-\infty, \infty)$ by

$$Q_1(y) := \int_{-\infty}^{\infty} g(s) \int_{-\infty}^{s} y(\tau)\,d\tau\,ds, \qquad Q_2(y) := \int_{-\infty}^{\infty} h(s) \int_{s}^{\infty} y(\tau)\,d\tau\,ds.$$

Then $Q_1, Q_2 : Y \to (-\infty, \infty)$ are continuous.

Lemma 3.2 *Assume that* (H1) *and* (H2) *hold. Then the operator* $M : X \cap \operatorname{dom} M \to Y$ *is quasi-linear. Moreover,* $\operatorname{Ker} M = \{a + bw : a, b \in (-\infty, \infty)\}$ *and* $\operatorname{Im} M = \{y \in Y : Q_1(y) = Q_2(y) = 0\}$.

Proof Clearly, $\operatorname{Ker} M = \{a + bw : a, b \in (-\infty, \infty)\}$, and it is linearly homeomorphic to $(-\infty, \infty)^2$. Next, we show that

$$\operatorname{Im} M = \big\{ y \in Y : Q_1(y) = Q_2(y) = 0 \big\}.$$

Let $y \in \operatorname{Im} M$. Then there exists $x \in X \cap \operatorname{dom} M$ such that

$$\big(c\varphi_p\big(x'\big)\big)'(t) = y(t), \quad t \in (-\infty, \infty).$$

For $t \in (-\infty, \infty)$,

$$\big(c\varphi_p\big(x'\big)\big)(t) = \big(c\varphi_p\big(x'\big)\big)(-\infty) + \int_{-\infty}^{t} y(s)\,ds$$

and

$$\int_{-\infty}^{\infty} g(s)\big(c\varphi_p\big(x'\big)\big)(s)\,ds = \big(c\varphi_p\big(x'\big)\big)(-\infty) + \int_{-\infty}^{\infty} g(s) \int_{-\infty}^{s} y(\tau)\,d\tau\,ds.$$

Thus $Q_1(y) = 0$. In a similar manner, $Q_2(y) = 0$.

On the other hand, let $y \in Y$ satisfying $Q_1(y) = Q_2(y) = 0$. Take

$$x(t) = \int_0^t \varphi_p^{-1}\left(\frac{1}{c(s)}\right) \varphi_p^{-1}\left(\int_0^s y(\tau)\,d\tau\right) ds.$$

Then $x \in X \cap \operatorname{dom} M$, and $(c\varphi_p(x'))' = y \in \operatorname{Im} M$. Thus, $\operatorname{Im} M = \{y \in Y : Q_1(y) = Q_2(y) = 0\}$. Since $Q_1, Q_2 : Y \to (-\infty, \infty)$ are continuous, $\operatorname{Im} M$ is closed in Y. Consequently, M is a quasi-linear operator.

Let $T_1, T_2 : Y \to Y$ be linear operators which are defined as follows:

$$T_1 y = \frac{1}{\triangle}\big(a_{11}Q_1(y) + a_{12}Q_2(y)\big)e^{-k(\cdot)}$$

and

$$T_2 y = \frac{1}{\triangle}\big(a_{21}Q_1(y) + a_{22}Q_2(y)\big)e^{-k(\cdot)},$$

where a_{ij} $(i,j = 1,2)$ are the constants in the assumption in (H4). Then, by direct calculations,

$$T_1(T_1 y) = T_1 y, \qquad T_1\big((T_2 y)w\big) = 0, \qquad T_2(T_1 y) = 0 \quad \text{and} \quad T_2\big((T_2 y)w\big) = T_2 y.$$

Define the bounded linear operators $Q : Y \to Y_1$ and $P : X \to X_1$ by

$$Q(y) = T_1 y + (T_2 y)w, \qquad P(x) = x(0) + \big(\varphi_p^{-1}(c)x'\big)(0)w,$$

where $X_1 := \operatorname{Ker} M$ and $Y_1 := \operatorname{Im} Q = \{(a + bw(\cdot))e^{-k(\cdot)} : a, b \in (-\infty, \infty)\}$. Then $Q : Y \to Y_1$, $P : X \to X_1$ are projections, and $\dim Y_1 = \dim X_1 = 2$. By (H4), $\triangle \neq 0$, and it follows from Lemma 3.2 that $\operatorname{Im} M = \operatorname{Ker} Q$. □

Lemma 3.3 *Assume that* (H1)-(H4) *hold. Assume that* Ω *is an open bounded subset of* X *such that* $\operatorname{dom} M \cap \overline{\Omega} \neq \emptyset$. *Then* $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ *is* M*-compact on* $\overline{\Omega}$.

Proof Let $X_2 := \operatorname{Ker} P$. Then X_2 is a complement space of X_1 in X, *i.e.*, $X = X_1 \oplus X_2$. Define $R : \overline{\Omega} \times [0,1] \to X_2$, for $t \in (-\infty, \infty)$, by

$$\begin{aligned} R(x,\lambda)(t) = \int_0^t \varphi_p^{-1}\left(\frac{1}{c(s)}\right)\bigg[&\varphi_p^{-1}\left(\big(c\varphi_p(x')\big)(0) + \lambda \int_0^s (I - Q)Nx(\tau)\,d\tau\right) \\ &- \big(\varphi_p^{-1}(c)x'\big)(0)\bigg]\,ds. \end{aligned}$$

Since Ω is bounded, there exists a constant $r > 0$ such that $\|x\|_X \le r$ for any $x \in \overline{\Omega}$. For $x \in \overline{\Omega}$, and for almost all $t \in (-\infty, \infty)$, by (H3)

$$\begin{aligned} \big|(Nx)(t)\big| &= \big|f\big(t, x(t), x'(t)\big)\big| \\ &\le \big(1 + |w(t)|\big)^{p-1}\alpha(t)\left(\frac{|x(t)|}{1 + w(t)}\right)^{p-1} + \frac{\beta}{c}(t)\big|\big(\varphi_p^{-1}(c)x'\big)(t)\big|^{p-1} + \gamma(t) \\ &\le \left(\big(1 + |w(t)|\big)^{p-1}\alpha(t) + \frac{\beta}{c}(t)\right)\|x\|_X^{p-1} + \gamma(t), \end{aligned} \tag{2}$$

which implies that

$$\|Nx\|_Y \le r^{p-1}\left\|\big(1 + |w|\big)^{p-1}\alpha + \frac{\beta}{c}\right\|_Y + \|\gamma\|_Y =: l_r.$$

Since $|Q_1(Nx)| \le \|Nx\|_Y$ and $|Q_2(Nx)| \le \|Nx\|_Y$, for $x \in \overline{\Omega}$,

$$\begin{aligned}\bigl|Q(Nx)(t)\bigr| &\le \bigl|T_1(Nx)(t)\bigr| + \bigl|\bigl(T_2(Nx)(t)\bigr)w(t)\bigr| \\ &\le \frac{1}{|\triangle|}\bigl(|a_{11}|\bigl|Q_1(Nx)\bigr| + |a_{12}|\bigl|Q_2(Nx)\bigr| \\ &\quad + \bigl(|a_{21}|\bigl|Q_1(Nx)\bigr| + |a_{22}|\bigl|Q_2(Nx)\bigr|\bigr)\bigl|w(t)\bigr|\bigr)e^{-k(t)} \\ &\le \|Nx\|_Y\bigl(M_1 + M_2\bigl|w(t)\bigr|\bigr)e^{-k(t)} \le l_r\bigl(M_1 + M_2\bigl|w(t)\bigr|\bigr)e^{-k(t)}. \end{aligned} \tag{3}$$

Here,

$$M_1 := \frac{1}{|\triangle|}\bigl[|a_{11}| + |a_{12}|\bigr] \quad \text{and} \quad M_2 := \frac{1}{|\triangle|}\bigl[|a_{21}| + |a_{22}|\bigr].$$

Thus

$$\bigl\|Q(Nx)\bigr\|_Y \le D\|Nx\|_Y, \tag{4}$$

where

$$D := \int_{-\infty}^{\infty}\bigl(M_1 + M_2\bigl|w(\tau)\bigr|\bigr)e^{-k(\tau)}\,d\tau. \tag{5}$$

First, we prove that $R : \overline{\Omega} \times [0,1] \to X_2$ is compact by using Theorem 2.4. Let $Z = C(-\infty,\infty) \cap L^{\infty}(-\infty,\infty)$ with the usual sup norm. For $x \in \overline{\Omega}$,

$$\begin{aligned}&\sup_{t\in(-\infty,\infty)}\frac{|R(x,\lambda)(t)|}{1+|w(t)|} \\ &\quad\le \left(\varphi_p^{-1}\left(\bigl|\bigl(c\varphi_p(x')\bigr)(0)\bigr| + \int_{-\infty}^{\infty}\bigl|Nx(\tau)\bigr| + \bigl|Q(Nx)(\tau)\bigr|\,d\tau\right) + \bigl|\bigl(\varphi_p^{-1}(c)x'\bigr)(0)\bigr|\right) \\ &\quad\le (\alpha_p+1)r + \alpha_p\bigl((1+D)\|Nx\|_Y\bigr)^{\frac{1}{p-1}} \le (\alpha_p+1)r + \alpha_p\bigl((1+D)l_r\bigr)^{\frac{1}{p-1}}\end{aligned}$$

and

$$\begin{aligned}&\sup_{t\in(-\infty,\infty)}\bigl|\bigl(\varphi_p^{-1}(c)R(x,\lambda)'\bigr)(t)\bigr| \\ &\quad= \sup_{t\in(-\infty,\infty)}\left|\varphi_p^{-1}\left(\bigl(c\varphi_p(x')\bigr)(0) + \lambda\int_0^t (I-Q)Nx(\tau)\,d\tau\right) - \bigl(\varphi_p^{-1}(c)x'\bigr)(0)\right| \\ &\quad\le (\alpha_p+1)r + \alpha_p\bigl((1+D)\|Nx\|_Y\bigr)^{\frac{1}{p-1}} \le (\alpha_p+1)r + \alpha_p\bigl((1+D)l_r\bigr)^{\frac{1}{p-1}}.\end{aligned}$$

Here α_p is the constant in Remark 3.1(1). Thus $\{\frac{R(x,\lambda)}{1+|w(x)|} : x \in \overline{\Omega}\}$ and $\{\varphi_p^{-1}(c)R(x,\lambda)' : x \in \overline{\Omega}\}$ are bounded in Z.

Let $T > 0$ and let $\epsilon > 0$ be given. First, for any $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, we have

$$\begin{aligned}&\left|\frac{R(x,\lambda)(t_1)}{1+w(t_1)} - \frac{R(x,\lambda)(t_2)}{1+w(t_2)}\right| \\ &\quad\le \frac{w(t_2)-w(t_1)}{(1+w(t_1))(1+w(t_2))}\bigl|w(t_1)\bigr|\bigl((\alpha_p+1)r + \alpha_p\bigl((1+D)\|Nx\|_Y\bigr)^{\frac{1}{p-1}}\bigr)\end{aligned}$$

$$\begin{aligned}&+\frac{1}{1+w(t_2)}\big(w(t_2)-w(t_1)\big)\big((\alpha_p+1)r+\alpha_p\big((1+D)\|Nx\|_Y\big)^{\frac{1}{p-1}}\big)\\&\leq 2\big(w(t_2)-w(t_1)\big)\big((\alpha_p+1)r+\alpha_p\big((1+D)\|Nx\|_Y\big)^{\frac{1}{p-1}}\big)\end{aligned}$$

and

$$\begin{aligned}&\big|\big(\varphi_p^{-1}(c)R(x,\lambda)'\big)(t_1)-\big(\varphi_p^{-1}(c)R(x,\lambda)'\big)(t_2)\big|\\&\quad=\bigg|\varphi_p^{-1}\bigg(\big(c\varphi_p(x')\big)(0)+\lambda\int_0^{t_1}(I-Q)Nx(\tau)\,d\tau\bigg)\\&\qquad-\varphi_p^{-1}\bigg(\big(c\varphi_p(x')\big)(0)+\lambda\int_0^{t_2}(I-Q)Nx(\tau)\,d\tau\bigg)\bigg|.\end{aligned}$$

By (2) and (3), there exists $z\in Y$ such that

$$\big|(I-Q)Nx\big|\leq z\quad\text{for all } x\in\overline{\Omega},\tag{6}$$

and since φ_p^{-1} and w are uniformly continuous on a compact interval in $(-\infty,\infty)$, there exists $\delta_1>0$ such that if $|t_1-t_2|<\delta_1$ with $t_1,t_2\in[0,T]$, then

$$\left|\frac{R(x,\lambda)(t_1)}{1+|w(t_1)|}-\frac{R(x,\lambda)(t_2)}{1+|w(t_2)|}\right|<\frac{\epsilon}{2}$$

and

$$\big|\big(\varphi_p^{-1}(c)R(x,\lambda)'\big)(t_1)-\big(\varphi_p^{-1}(c)R(x,\lambda)'\big)(t_2)\big|<\frac{\epsilon}{2}.$$

In a similar manner, there exists $\delta_2>0$ such that if $|t_1-t_2|<\delta_2$ with $t_1,t_2\in[-T,0]$, then

$$\left|\frac{R(x,\lambda)(t_1)}{1+|w(t_1)|}-\frac{R(x,\lambda)(t_2)}{1+|w(t_2)|}\right|<\frac{\epsilon}{2}$$

and

$$\big|\big(\varphi_p^{-1}(c)R(x,\lambda)'\big)(t_1)-\big(\varphi_p^{-1}(c)R(x,\lambda)'\big)(t_2)\big|<\frac{\epsilon}{2}.$$

Letting $\delta=\min\{\delta_1,\delta_2\}>0$, if $|t_1-t_2|<\delta$ with $t_1,t_2\in[-T,T]$, then

$$\left|\frac{R(x,\lambda)(t_1)}{1+|w(t_1)|}-\frac{R(x,\lambda)(t_2)}{1+|w(t_2)|}\right|<\epsilon$$

and

$$\big|\big(\varphi_p^{-1}(c)R(x,\lambda)'\big)(t_1)-\big(\varphi_p^{-1}(c)R(x,\lambda)'\big)(t_2)\big|<\epsilon.$$

Consequently, $\{\frac{R(x,\lambda)}{1+|w(x)|}:x\in\overline{\Omega}\}$ and $\{\varphi_p^{-1}(c)R(x,\lambda)':x\in\overline{\Omega}\}$ are equicontinuous on any compact intervals in $(-\infty,\infty)$.

For $x \in \overline{\Omega}$, by L'Hôspital's rule,

$$\begin{aligned}
&\lim_{t\to\infty} \frac{R(x,\lambda)(t)}{1+|w(t)|} \\
&\quad= \lim_{t\to\infty} \frac{1}{1+|w(t)|} \int_0^t \varphi_p^{-1}\left(\frac{1}{c(s)}\right)\left[\varphi_p^{-1}\left(\left(c\varphi_p(x')\right)(0)\right.\right. \\
&\qquad \left.\left. + \lambda \int_0^s (I-Q)Nx(\tau)\,d\tau\right) - \left(\varphi_p^{-1}(c)x'\right)(0)\right] ds \\
&\quad= \lim_{t\to\infty} \varphi_p^{-1}\left(\left(c\varphi_p(x')\right)(0) + \lambda \int_0^t (I-Q)Nx(\tau)\,d\tau\right) - \left(\varphi_p^{-1}(c)x'\right)(0) \\
&\quad= \varphi_p^{-1}\left(\left(c\varphi_p(x')\right)(0) + \lambda \int_0^\infty (I-Q)Nx(\tau)\,d\tau\right) - \left(\varphi_p^{-1}(c)x'\right)(0)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\to\infty} \left(\varphi_p^{-1}(c)R(x,\lambda)'\right)(t) \\
&\quad= \varphi_p^{-1}\left(\left(c\varphi_p(x')\right)(0) + \lambda \int_0^\infty (I-Q)Nx(\tau)\,d\tau\right) - \left(\varphi_p^{-1}(c)x'\right)(0).
\end{aligned}$$

In a similar manner,

$$\begin{aligned}
&\lim_{t\to-\infty} \frac{R(x,\lambda)(t)}{1+|w(t)|} \\
&\quad= \varphi_p^{-1}\left(-\left(c\varphi_p(x')\right)(0) + \lambda \int_{-\infty}^0 (I-Q)Nx(\tau)\,d\tau\right) + \left(\varphi_p^{-1}(c)x'\right)(0)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\to-\infty} \left(\varphi_p^{-1}(c)R(x,\lambda)'\right)(t) \\
&\quad= \varphi_p^{-1}\left(\left(c\varphi_p(x')\right)(0) - \lambda \int_{-\infty}^0 (I-Q)Nx(\tau)\,d\tau\right) - \left(\varphi_p^{-1}(c)x'\right)(0).
\end{aligned}$$

By (6), we conclude that $\{\frac{R(x,\lambda)}{1+|w(x)|} : x \in \Omega\}$ and $\{\varphi_p^{-1}(c)R(x,\lambda)' : x \in \Omega\}$ are equiconvergent at $\pm\infty$. Thus, $R : \overline{\Omega} \times [0,1] \to X_2$ is compact in view of Theorem 2.4.

Next, we prove that $R : \overline{\Omega} \times [0,1] \to X_2$ is continuous. Let $\{(x_n,\lambda_n)\}$ be a sequence in $\overline{\Omega} \times [0,1]$ such that $x_n \to x$ in X and $\lambda_n \to \lambda$ in $(-\infty,\infty)$ as $n \to \infty$. Then $\{x_n\}$ is bounded in X and $x_n(t) \to x(t)$ pointwise as $n \to \infty$. Since R is compact, there exists a subsequence $\{(x_{n_k},\lambda_{n_k})\}$ of $\{(x_n,\lambda_n)\}$ such that $R(x_{n_k},\lambda_{n_k})(t) \to L$ in X as $n_k \to \infty$. By the Lebesgue dominated convergence theorem, $R(x_n,\lambda_n)(t) \to R(x,\lambda)(t)$ as $n \to \infty$. Thus, $L \equiv R(x,\lambda)$. By a standard argument, $R : \overline{\Omega} \times [0,1] \to X_2$ is continuous.

Finally, we show that (i)-(iv) hold in Definition 2.2. Since $Q(I-Q)N_\lambda(\overline{\Omega}) = 0$, $(I-Q)N_\lambda(\overline{\Omega}) \in \operatorname{Ker} Q = \operatorname{Im} M$. For $y \in \operatorname{Im} M$, $Qy = 0$, and $y = (I-Q)y \in (I-Q)Y$. Consequently, $(I-Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I-Q)Y$. Since $N_\lambda x = \lambda Nx$ for any $x \in \overline{\Omega}$,

$$QN_\lambda x = 0, \quad \lambda \in (0,1) \quad \Leftrightarrow \quad QNx = 0,$$

and $R(\cdot,0)=\theta_X$. For $x\in\Sigma_\lambda=\{x\in\overline{\Omega}:Mx=N_\lambda x\}$, $N_\lambda x=(c\varphi_p(x'))'\in\operatorname{Im}M=\operatorname{Ker}Q$. Then, for $x\in X$ and $t\in(-\infty,\infty)$,

$$\begin{aligned}
&R(x,\lambda)(t)\\
&\quad=\int_0^t\varphi_p^{-1}\left(\frac{1}{c(s)}\right)\left[\varphi_p^{-1}\left(\bigl(c\varphi_p(x')\bigr)(0)+\int_0^s(I-Q)N_\lambda x(\tau)\,d\tau\right)-\bigl(\varphi_p^{-1}(c)x'\bigr)(0)\right]ds\\
&\quad=\int_0^t\varphi_p^{-1}\left(\frac{1}{c(s)}\right)\left[\varphi_p^{-1}\left(\bigl(c\varphi_p(x')\bigr)(0)+\int_0^s N_\lambda x(\tau)\,d\tau\right)-\bigl(\varphi_p^{-1}(c)x'\bigr)(0)\right]ds\\
&\quad=\int_0^t\varphi_p^{-1}\left(\frac{1}{c(s)}\right)\bigl[\bigl(\varphi_p^{-1}(c)x'\bigr)(s)-\bigl(\varphi_p^{-1}(c)x'\bigr)(0)\bigr]\,ds\\
&\quad=x(t)-\bigl(x(0)+\bigl(\varphi_p^{-1}(c)x'\bigr)(0)w(t)\bigr)=(I-P)x(t).
\end{aligned}$$

On the other hand, for $x\in X$ and $t\in(-\infty,\infty)$,

$$\begin{aligned}
&M\bigl[Px+R(x,\lambda)\bigr](t)\\
&\quad=M\left[x(0)+\bigl(\varphi_p^{-1}(c)x'\bigr)(0)w(t)+\int_0^t\varphi_p^{-1}\left(\frac{1}{c(s)}\right)\left[\varphi_p^{-1}\left(\bigl(c\varphi_p(x')\bigr)(0)\right.\right.\right.\\
&\qquad\left.\left.\left.+\lambda\int_0^s(I-Q)Nx(\tau)\,d\tau\right)-\bigl(\varphi_p^{-1}(c)x'\bigr)(0)\right]ds\right]\\
&\quad=(I-Q)N_\lambda x(t).
\end{aligned}$$

Thus, N_λ is M-compact on $\overline{\Omega}$. □

Now, we give the main result in this paper.

Theorem 3.4 *Assume that* (H1)-(H4) *hold. Assume also that the following hold*:

(H5) *there exist positive constants A and B such that if $|x(t)|>A$ for every $t\in[-B,B]$ or $|(c\varphi_p(x'))(t)|>A$ for every $t\in(-\infty,\infty)$, then $Q(Nx)\neq0$, i.e., either $Q_1(Nx)\neq0$ or $Q_2(Nx)\neq0$*;

(H6) *there exists a positive constant C such that if $|a|>C$ or $|b|>C$, then either*
   (1) *$aQ_1(N(a+bw(\cdot)))+bQ_2(N(a+bw(\cdot)))<0$ or*
   (2) *$aQ_1(N(a+bw(\cdot)))+bQ_2(N(a+bw(\cdot)))>0$.*

Then problem (1) *has at least one solution in X provided that*

$$\bigl\|(1+|w|)^{p-1}\alpha\bigr\|_Y+\left\|\frac{\beta}{c}\right\|_Y<\left(\frac{1}{2\alpha_p^3+(4+|w(B)|+2(1+D)^{\frac{1}{p-1}})\alpha_p^2}\right)^{p-1}. \tag{7}$$

Here, D is the constant defined in (5).

Proof We divide the proof into three steps.

Step 1. Let

$$\Omega_1=\bigl\{x\in\operatorname{dom}M:Mx=N_\lambda x,\ \text{for some }\lambda\in(0,1)\bigr\}.$$

We will prove that Ω_1 is bounded. For $x\in\Omega_1$, $Mx=N_\lambda x\in\operatorname{Im}M=\operatorname{Ker}Q$.

Thus

$$Q_1(Nx) = Q_2(Nx) = 0.$$

By (H5), there exist $t_0 \in [-B, B]$ and $t_1 \in (-\infty, \infty)$ such that

$$\left|x(t_0)\right| \leq A \quad \text{and} \quad \left|\left(c\varphi_p\left(x'\right)\right)(t_1)\right| \leq A,$$

which imply that

$$\begin{aligned}\left|\left(c\varphi_p\left(x'\right)\right)(0)\right| &= \left|\left(c\varphi_p\left(x'\right)\right)(t_1) - \left(c\varphi_p\left(x'\right)\right)(t_1) + \left(c\varphi_p\left(x'\right)\right)(0)\right| \\ &\leq \left|\left(c\varphi_p\left(x'\right)\right)(t_1)\right| + \left|\int_0^{t_1} \left(c\varphi_p\left(x'\right)\right)'(s)\,ds\right| \\ &\leq A + \|Mx\|_Y \leq A + \|Nx\|_Y,\end{aligned}$$

and $|(\varphi_p^{-1}(c)x')(0)| \leq (A + \|Nx\|_Y)^{\frac{1}{p-1}} \leq \alpha_p A^{\frac{1}{p-1}} + \alpha_p \|Nx\|_Y^{\frac{1}{p-1}}$. Then we have

$$\begin{aligned}\left|x(0)\right| &= \left|x(t_0) - \int_0^{t_0} \varphi_p^{-1}\left(\frac{1}{c(s)}\right)\varphi_p^{-1}\left[\left(c\varphi_p\left(x'\right)\right)(t_1) + \int_{t_1}^{s} \left(c\varphi_p\left(x'\right)\right)'(\tau)\,d\tau\right] ds\right| \\ &\leq \left|x(t_0)\right| + \left|w(B)\right|\left[\varphi_p^{-1}\left(\left|\left(c\varphi_p\left(x'\right)\right)(t_1)\right| + \int_0^{\infty} \left|\left(c\varphi_p\left(x'\right)\right)'(\tau)\right| d\tau\right)\right] \\ &\leq A + \left|w(B)\right|\left(\alpha_p A^{\frac{1}{p-1}} + \alpha_p \|Nx\|_Y^{\frac{1}{p-1}}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\|Px\|_X &= \|Px\|_1 + \|Px\|_2 \leq \left|x(0)\right| + 2\left|\left(\varphi_p^{-1}(c)x'\right)(0)\right| \\ &\leq A + \alpha_p\left(2 + \left|w(B)\right|\right)A^{\frac{1}{p-1}} + \alpha_p\left(2 + \left|w(B)\right|\right)\|Nx\|_Y^{\frac{1}{p-1}}.\end{aligned}$$

On the other hand, by (4),

$$\begin{aligned}\frac{|R(x,\lambda)(t)|}{1 + |w(t)|} &= \frac{1}{1 + |w(t)|}\left|\int_0^t \varphi_p^{-1}\left(\frac{1}{c(s)}\right)\left[\varphi_p^{-1}\left(\left(c\varphi_p\left(x'\right)\right)(0) + \lambda\int_0^s (I-Q)Nx(\tau)\,d\tau\right)\right.\right. \\ &\quad \left.\left. - \left(\varphi_p^{-1}(c)x'\right)(0)\right] ds\right| \\ &\leq (\alpha_p + 1)\varphi_p^{-1}\left(\left|\left(c\varphi_p\left(x'\right)\right)(0)\right|\right) + \alpha_p\varphi_p^{-1}\left(\|Nx\|_Y + \|QNx\|_Y\right) \\ &\leq (\alpha_p + 1)\left(A + \|Nx\|_Y\right)^{\frac{1}{p-1}} + \alpha_p(1 + D)^{\frac{1}{p-1}}\|Nx\|_Y^{\frac{1}{p-1}} \\ &= \alpha_p(\alpha_p + 1)A^{\frac{1}{p-1}} + \alpha_p\left(\alpha_p + 1 + (1 + D)^{\frac{1}{p-1}}\right)\|Nx\|_Y^{\frac{1}{p-1}}\end{aligned}$$

and

$$\begin{aligned}&\left|\left(\varphi_p^{-1}(c)\left(R(x,\lambda)\right)'\right)(t)\right| \\ &\quad \leq \alpha_p(\alpha_p + 1)A^{\frac{1}{p-1}} + \alpha_p\left(\alpha_p + 1 + (1 + D)^{\frac{1}{p-1}}\right)\|Nx\|_Y^{\frac{1}{p-1}}.\end{aligned}$$

Thus,

$$\left\|R(x,\lambda)\right\|_X \le 2\alpha_p(\alpha_p+1)A^{\frac{1}{p-1}} + 2\alpha_p\left(\alpha_p+1+(1+D)^{\frac{1}{p-1}}\right)\|Nx\|_Y^{\frac{1}{p-1}}.$$

It follows that

$$\begin{aligned}
\|x\|_X &= \left\|Px+(I-P)x\right\|_X \\
&\le \|Px\|_X + \left\|(I-P)x\right\|_X \\
&= \|Px\|_X + \left\|R(x,\lambda)\right\|_X \\
&\le A + \alpha_p\left(2+\left|w(B)\right|\right)A^{\frac{1}{p-1}} + \alpha_p\left(2+\left|w(B)\right|\right)\|Nx\|_Y{}^{\frac{1}{p-1}} + 2\alpha_p(\alpha_p+1)A^{\frac{1}{p-1}} \\
&\quad + 2\alpha_p\left(\alpha_p+1+(1+D)^{\frac{1}{p-1}}\right)\|Nx\|_Y^{\frac{1}{p-1}} \\
&\le A + \alpha_p\left(2\alpha_p+4+\left|w(B)\right|\right)A^{\frac{1}{p-1}} + \alpha_p^2\left(2\alpha_p+4+\left|w(B)\right|+2(1+D)^{\frac{1}{p-1}}\right)\|\gamma\|_Y^{\frac{1}{p-1}} \\
&\quad + \alpha_p^2\left(2\alpha_p+4+\left|w(B)\right|+2(1+D)^{\frac{1}{p-1}}\right)\left(\left\|\left(1+|w|\right)^{p-1}\alpha\right\|_Y + \left\|\frac{\beta}{c}\right\|_Y\right)^{\frac{1}{p-1}}\|x\|_X.
\end{aligned}$$

By (7), Ω_1 is bounded.

Step 2. Define a homeomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} M$ by

$$J\left(\left(a+bw(\cdot)\right)e^{k(\cdot)}\right) = a_{22}a - a_{12}b + (-a_{21}a + a_{11}b)w(\cdot).$$

Assume (H6)(1) holds, *i.e.*, there exists a positive constant C such that if $|a| > C$ or $|b| > C$, then $aQ_1(N(a+bw(\cdot))) + bQ_2(N(a+bw(\cdot))) < 0$. Let

$$\Omega_2 = \left\{x \in \ker M : -\lambda x + (1-\lambda)JQNx = 0,\ \text{for some } \lambda \in [0,1]\right\}.$$

Let $x \in \Omega_2$. Then $x = a + bw$ for some $a, b \in (-\infty, \infty)$. If $\lambda = 0$, $JQN(a+bw(\cdot)) = 0$. Since J is homeomorphism, $QN(a+bw(\cdot)) = 0$. By (H6), we obtain $|a| \le C$ and $|b| \le C$. If $\lambda = 1$, then $a = b = 0$.

For $\lambda \in (0,1)$, by $\lambda x = (1-\lambda)JQNx$, we obtain

$$\lambda a = (1-\lambda)Q_1\left(N\left(a+bw(\cdot)\right)\right), \qquad \lambda b = (1-\lambda)Q_2\left(N\left(a+bw(\cdot)\right)\right).$$

If $|a| > C$ or $|b| > C$, then, by (H6)(1), we obtain

$$\lambda\left(a^2+b^2\right) = (1-\lambda)\left(aQ_1N\left(a+bw(\cdot)\right) + bQ_2N\left(a+bw(\cdot)\right)\right) < 0,$$

which is a contradiction. Thus, Ω_2 is bounded.

In the case that (H6)(2) holds, we take

$$\Omega_2 = \left\{x \in \ker M : \lambda x + (1-\lambda)JQNx = 0 \text{ for some } \lambda \in [0,1]\right\},$$

and it follows that Ω_2 is bounded in a similar manner.

Step 3. Take an open bounded set $\Omega \supset \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \{0\}$ in X. By Step 1, (A1) $Mu \neq N_\lambda u$ for every $(u,\lambda) \in (\operatorname{dom} M \cap \partial\Omega) \times (0,1)$.

Now we will show that

(A2) $\deg(JQN, \Omega \cap \ker M, 0) \neq 0$.

Let $H(x,\lambda) = \pm\lambda x + (1-\lambda)JQNx$. By Step 2, we know that $H(x,\lambda) \neq 0$, for every $(x,\lambda) \in (\ker M \cap \partial\Omega) \times [0,1]$. Thus, by the homotopy property of the degree, we obtain

$$\begin{aligned}\deg(JQN, \Omega \cap \ker M, 0) &= \deg\bigl(H(\cdot,0), \Omega \cap \ker M, 0\bigr)\\ &= \deg\bigl(H(\cdot,1), \Omega \cap \ker M, 0\bigr)\\ &= \deg(\pm I, \Omega \cap \ker M, 0) = \pm 1 \neq 0.\end{aligned}$$

By Theorem 2.3, $Mx = Nx$ has at least one solution in $\operatorname{dom} M \cap \Omega$, and consequently problem (1) has at least one solution in X. □

4 Example

Consider the following second-order nonlinear differential equation:

$$\begin{cases}(|u'|u')'(t) = f(t,u(t),u'(t)), & \text{a.e. } t \in (-\infty,\infty),\\ \lim_{t\to-\infty}(|u'|u')(t) = \int_{-\infty}^{\infty} g(s)(|u'|u')(s)\,ds, & \\ \lim_{t\to\infty}(|u'|u')(t) = \int_{-\infty}^{\infty} h(s)(|u'|u')(s)\,ds, & \end{cases} \tag{8}$$

where

$$g(t) = \begin{cases}-2t, & t \in [-1,0],\\ 0, & \text{otherwise},\end{cases} \qquad h(t) = \begin{cases}2t, & t \in [0,1],\\ 0, & \text{otherwise}.\end{cases}$$

Define $f : (-\infty,\infty)^3 \to (-\infty,\infty)$ by $f(t,u,v) = \alpha(t)u + \beta(t)v + \overline{\gamma}(t)$, where

$$\alpha(t) = \begin{cases}10^{-3}e^t, & t \in [-1,0],\\ 0, & \text{otherwise},\end{cases} \qquad \beta(t) = \begin{cases}10^{-4}e^{-t}, & t \geq 1,\\ 0, & \text{otherwise},\end{cases}$$

$$\overline{\gamma}(t) = \begin{cases}e^t + 15e^{-1} - 6, & t \in [-1,0],\\ 0, & \text{otherwise}.\end{cases}$$

Then

$$\begin{aligned}\bigl|f(t,u,v)\bigr| &\leq \alpha(t)|u| + \beta(t)|v| + \bigl|\overline{\gamma}(t)\bigr| \leq \alpha(t)\bigl(|u|^2+1\bigr) + \beta(t)\bigl(|v|^2+1\bigr) + \bigl|\overline{\gamma}(t)\bigr|\\ &= \alpha(t)|u|^2 + \beta(t)|v|^2 + \gamma(t),\end{aligned}$$

where $\gamma(t) = \alpha(t) + \beta(t) + |\overline{\gamma}(t)|$. Since $c(t) = 1$ for $t \in (-\infty,\infty)$ and $p = 3$, $w(t) = t$, and thus (H1), (H2), and (H3) hold.

For $y \in Y$,

$$Q_1(y) = \int_{-\infty}^{-1} y(\tau)\,d\tau + \int_{-1}^{0} y(\tau)\tau^2\,d\tau$$

and

$$Q_2(y) = \int_1^\infty y(\tau)\,d\tau + \int_0^1 y(\tau)\tau^2\,d\tau.$$

Take $k(t) = |t|$, then

$$a_{11} = -14e^{-1} + 6, \qquad a_{12} = -14e^{-1} + 6, \qquad a_{21} = 4e^{-1} - 2, \qquad a_{22} = -4e^{-1} + 2,$$

and

$$\triangle = a_{11}a_{22} - a_{12}a_{21} = \left(-14e^{-1} + 6\right)\left(-8e^{-1} + 4\right) > 0.$$

Thus, (H4) holds.

Take $B = 1$ and $A = 1$. If $|x(t)| > 1$, for $t \in [-1,1]$, then $|Q_1(Nx)| = |10^{-3}\int_{-1}^0 t^2 e^{-|t|}x(t)\,dt| > |10^{-3}\int_{-1}^0 t^2 e^{-|t|}\,dt| = 10^{-3}(-5e^{-1} + 2) > 0$. If $|(|x'|x')(t)| > 1$ for $t \in (-\infty,\infty)$, then $|x'(t)| > 1$ for $t \in (-\infty,\infty)$, and

$$\left|Q_2(Nx)\right| = \left|10^{-4}\int_1^\infty e^{-|t|}x'(t)\,dt\right| > \left|10^{-4}\int_1^\infty e^{-|t|}\,dt\right| = 10^{-4}e^{-1} > 0.$$

Thus, (H5) holds.

For any $C > 0$, if $|a| > C$ or $|b| > C$, then

$$\begin{aligned} &aQ_1\bigl(N(a + bt)\bigr) + bQ_2\bigl(N(a + bt)\bigr) \\ &\quad= 10^{-3}\left(-5e^{-1} + 2\right)a^2 + 10^{-3}\left(16e^{-1} - 6\right)ab + 10^{-4}e^{-1}b^2 \\ &\quad= 10^{-3}\left(-5e^{-1} + 2\right)\left(a + \frac{8e^{-1} - 3}{-5e^{-1} + 2}b\right)^2 + 10^{-4}\left(e^{-1} - 10\frac{(8e^{-1} - 3)^2}{-5e^{-1} + 2}\right)b^2 > 0. \end{aligned}$$

Thus, (H6)(2) is satisfied.

Since $p = 3$, $\alpha_p = 1$, $B = 1$, and $D = \frac{1}{-2e^{-1}+1} + \frac{1}{-7e^{-1}+3}$, we have

$$\begin{aligned} \left\|\left(1 + |w|\right)^2\alpha\right\|_Y + \left\|\frac{\beta}{c}\right\|_Y &< 4 \times 10^{-4} \\ &< \left(\frac{1}{2 + (5 + 2(1 + \frac{1}{-2e^{-1}+1} + \frac{1}{-7e^{-1}+3})^{\frac{1}{2}})}\right)^2 \end{aligned}$$

and (7) holds. Consequently, there exists at least one solution to problem (8) in view of Theorem 3.4.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final draft.

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