Wang et al. *Boundary Value Problems* (2015) 2015:13
DOI 10.1186/s13661-014-0273-5

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

The mixed boundary value problem for the inhomogeneous Cimmino system

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Abstract
In this article, we first propose a kind of mixed boundary value problem for the inhomogeneous Cimmino system, which consists of first order linear partial differential equations in $\mathbb{R}^4$. Then, by using the one-to-one correspondence between the theory of quaternion valued hyperholomorphic functions and that of Cimmino system's solutions, we transform the problem as stated above into a problem related to the ψ-hyperholomorphic functions in quaternionic analysis. Moreover, we show the boundedness, Hölder continuity, and generalized derivatives of a kind of singular integral operator ${}^{\psi}T_{\mathbb{C}^2}[g]$ related to ψ-hyperholomorphic functions in quaternionic analysis. Lastly, the solution of the mixed boundary value problem for the inhomogeneous Cimmino system is explicitly described.

Keywords: Cimmino system; quaternionic analysis; ψ-hyperholomorphic functions; Cimmino singular integral operator; mixed boundary value problem

1 Introduction

The skew field of quaternions $\mathbb{H}$ gives an example of a noncommutative Clifford algebra with minimal dimension. It serves as a very convenient model of general Clifford constructions. Today, quaternionic analysis is regarded as a broadly accepted branch of classical analysis offering a successful generalization of complex analysis. It studies functions defined on domains in $\mathbb{R}^3$ or $\mathbb{R}^4$ with values in the skew field of real quaternions $\mathbb{H}$. This theory is centered around the concept of ψ-hyperholomorphic functions related to a so-called structural set ψ of $\mathbb{H}^3$ or $\mathbb{H}^4$, respectively.

Quaternionic analysis initiated new solution methods for boundary value problems in several research areas of mathematical physics, in particular in planar fluids, quantum field theory, electromagnetic wave equations *etc.* Many scholars and experts have studied some boundary and initial value problems in higher dimensions by using them, such as Gürlebeck, Sprössig, Adler, Alesker, Yang, and so on [1–5].

The Cimmino system (1.1) offers a natural and elegant generalization to the four-dimensional case of that of Cauchy-Riemann. Cimmino, Dragomir and Lanconelli have done a lot of research on it [6, 7]. Recently, Abreu Blaya *et al.* [8] studied the Dirichlet boundary value problem for the inhomogeneous Cimmino system (1.2). We have

$$\begin{cases} \frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} = 0, \\ \frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} - \frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2} = 0, \\ \frac{\partial f_0}{\partial x_2} - \frac{\partial f_1}{\partial x_3} - \frac{\partial f_2}{\partial x_0} + \frac{\partial f_3}{\partial x_1} = 0, \\ \frac{\partial f_0}{\partial x_3} + \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0} = 0, \end{cases} \tag{1.1}$$

where f_m ($m = 0,1,2,3$) are continuously differentiable $\mathbb{R}$-valued functions in $\Omega \subset \mathbb{R}^4$. The corresponding inhomogeneous Cimmino system is as follows:

$$\begin{cases} \frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} = g_0, \\ \frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} - \frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2} = g_1, \\ \frac{\partial f_0}{\partial x_2} - \frac{\partial f_1}{\partial x_3} - \frac{\partial f_2}{\partial x_0} + \frac{\partial f_3}{\partial x_1} = g_2, \\ \frac{\partial f_0}{\partial x_3} + \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0} = g_3, \end{cases} \tag{1.2}$$

where f_m are as stated above, $g_m \in L_p(\Omega, \mathbb{R})$ ($m = 0,1,2,3$).

In this article, we will study a kind of mixed boundary value problem for the inhomogeneous Cimmino system (1.2) by using the quaternionic analysis approach. This article is organized as follows. In Section 2, we recall some basic knowledge of quaternionic analysis. In Section 3, we construct a singular integral operator and study some of its properties. In Section 4, we first propose a kind of mixed boundary value problem for the inhomogeneous Cimmino system (1.2); then we obtain an integral representation of the solution of the mixed boundary value problem by using the one-to-one correspondence between the theory of quaternion valued hyperholomorphic functions and that of a Cimmino system's solutions.

2 Preliminaries

Quaternionic analysis studies functions defined on $\mathbb{R}^4$ with their values in quaternion algebra space $\mathbb{H}$, which is a four-dimensional vector space with basis e, i, j, k. The basis element e is a unit element, henceforth we shall abbreviate e to 1. Also, i, j, k satisfy the following multiplication rule:

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j.$$

An arbitrary element of the quaternion algebra space $\mathbb{H}$ can be written as $x = x_0 + ix_1 + jx_2 + kx_3$, $x_m \in \mathbb{R}$ ($m = 0,1,2,3$), and $\bar{x} = x_0 - ix_1 - jx_2 - kx_3$. The norm for an element $x \in \mathbb{H}$ is taken to be $|x| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$ and satisfies $|\bar{x}| = |x|$, $|x + y| \leq |x| + |y|$, $|xy| = |x||y|$. Obviously, $\overline{xy} = \bar{y}\bar{x}$ and $x\bar{x} = \bar{x}x = |x|^2$. In addition, suppose the imaginary unit of $\mathbb{C}$ is identified with the basis element i in quaternion algebra space $\mathbb{H}$, then for arbitrary $z \in \mathbb{C}$, we have $z = x_0 + ix_1$ and its complex conjugate $\bar{z} = x_0 - ix_1$. In this way it is easily seen that $zj = j\bar{z}$.

By means of the mapping $x_0 + ix_1 + jx_2 + kx_3 \to (x_0 + ix_1) + (x_2 + ix_3)j$ ($\to (x_0, x_1, x_2, x_3)$), one can see $\mathbb{H}$ as $\mathbb{C}^2$ (or $\mathbb{R}^4$). From now on, an arbitrary element $\xi \in \mathbb{H}$ can be written as $\xi = z_1 + z_2 j$, $z_1, z_2 \in \mathbb{C}$. From the multiplication rule as stated above, for arbitrary $\xi = z_1 + z_2 j$, $\eta = \varsigma_1 + \varsigma_2 j \in \mathbb{H}$, $z_1, z_2, \varsigma_1, \varsigma_2 \in \mathbb{C}$. We have $\xi\eta = (z_1\varsigma_1 - z_2\bar{\varsigma}_2) + (z_1\varsigma_2 + z_2\bar{\varsigma}_1)j$, $\bar{\xi} = \overline{z_1 + z_2 j} = \bar{z}_1 + \overline{z_2 j} = \bar{z}_1 - z_2 j$, and $\xi\bar{\xi} = \bar{\xi}\xi = |z_1|^2 + |z_2|^2 = |\xi|^2$.

Let $\Omega \subset \mathbb{R}^4$ be a nonempty open bounded connected set and the boundary $\Gamma = \partial\Omega$ be a differentiable, oriented, and compact Liapunov surface. The functions f which are defined in Ω with values in $\mathbb{H}$ can be expressed as $f(x) = f_0 + f_1 i + f_2 j + f_3 k$, where f_m ($m = 0,1,2,3$) are continuously differentiable $\mathbb{R}$-valued functions in $\Omega \subset \mathbb{R}^4$. On $C^{(1)}(\Omega, \mathbb{H})$, we define the differential operators ${}^{\psi}D$ and ${}^{\bar{\psi}}D$ as follows:

$${}^{\psi}D = 2\left(\frac{\partial}{\partial \bar{z}_1} - j\frac{\partial}{\partial \bar{z}_2}\right), \qquad {}^{\bar{\psi}}D = 2\left(\frac{\partial}{\partial z_1} + j\frac{\partial}{\partial \bar{z}_2}\right),$$

where

$$\frac{\partial}{\partial \bar{z}_1} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1}\right), \qquad \frac{\partial}{\partial \bar{z}_2} = \frac{1}{2}\left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_3}\right),$$
$$\frac{\partial}{\partial z_1} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1}\right), \qquad \frac{\partial}{\partial z_2} = \frac{1}{2}\left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial x_3}\right).$$

Obviously, the differential operators ${}^{\psi}D$ and ${}^{\bar{\psi}}D$ can be written as

$${}^{\psi}D = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}, \qquad {}^{\bar{\psi}}D = \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3},$$

which are associated to the structural set $\psi = \{1, i, -j, k\}$ and $\bar{\psi} = \{1, -i, j, -k\}$, respectively.

Let $\Delta_{\mathbb{R}^4} = \sum_{m=0}^{3} \partial_{x_m}^2$, then the following equalities hold on $C^{(2)}(\Omega, \mathbb{H})$:

$${}^{\psi}D^{\bar{\psi}}D = {}^{\bar{\psi}}D^{\psi}D = \Delta_{\mathbb{R}^4} \cong \Delta_{\mathbb{C}^2} \cong \Delta_{\mathbb{H}}.$$

Taking into account that the multiplication in $\mathbb{H}$ is noncommutative, the functions f are called left ψ-hyperholomorphic in Ω if ${}^{\psi}D[f](\xi) = 0$ $(\xi \in \Omega)$. The functions g are called right ψ-hyperholomorphic in Ω if $[g]^{\psi}D(\xi) = 0$ $(\xi \in \Omega)$.

Denote by Θ_4 the fundamental solution of the Laplace operator

$$\Theta_4(\xi) = -\frac{1}{4\pi^2}\frac{1}{|\xi|^2},$$

and by $\mathcal{K}_{\psi}$ the fundamental solution of the operator ${}^{\psi}D$:

$$\mathcal{K}_{\psi}(\xi) = {}^{\bar{\psi}}D[\Theta_4] = [\Theta_4]^{\bar{\psi}}D = \frac{1}{2\pi^2}\frac{\xi_{\bar{\psi}}}{|\xi|^4} = \frac{1}{2\pi^2}\frac{\bar{z}_1 + \bar{z}_2 j}{(|z_1|^2 + |z_2|^2)^2}.$$

Then the corresponding Cauchy type integral operator is

$${}^{\psi}K_{\Gamma}[f](\xi) = \int_{\Gamma} \mathcal{K}_{\psi}(\eta - \xi) n_{\psi}(\eta) f(\eta)\, d\Gamma_{\eta} = \int_{\Gamma} \mathcal{K}_{\psi}(\eta - \xi)\, d\sigma_{\eta} f(\eta),$$

and the Teodorescu type integral operator is

$${}^{\psi}T_{\Omega}[f](\xi) = -\int_{\Omega} \mathcal{K}_{\psi}(\eta - \xi) f(\eta)\, d\Omega_{\eta}.$$

In this article, $g(x) \in L_p(\mathbb{C}^2, \mathbb{H})$ means that $g(x) \in L_p(E, \mathbb{H})$, $g_{\sigma}(x) = |x|^{-\sigma} g(\frac{\bar{x}}{|x|^2}) \in L_p(E, \mathbb{H})$, in which $E = \{\xi \mid |\xi| \leq 1\}$, σ is a real number, $\|g\|_{L_p} = \|g\|_{L_p(E)} + \|g_{\sigma}\|_{L_p(E)}$, $p \geq 1$. The following fundamental statements are widely known to hold and can be found in [1, 9, 10], respectively.

Definition 2.1 Suppose that the functions f, g, φ are defined in Ω with values in $\mathbb{H}$ and $f, g \in L_1(\Omega, \mathbb{H})$. If for arbitrary $\varphi \in C_0^{\infty}(\Omega, \mathbb{H})$, f, g satisfy

$$\int_{\Omega} [\varphi]^{\psi}D(\xi) g(\xi)\, d\Omega_{\xi} + \int_{\Omega} \varphi(\xi) f(\xi)\, d\Omega_{\xi} = 0,$$

then f is called a generalized derivative of the function g, denoted by $f = {}^{\psi}D[g]$.

Lemma 2.1 ([9]) *If* $\sigma_1, \sigma_2 > 0$, $0 \le \gamma \le 1$, *then we have*

$$\left|\sigma_1^\gamma - \sigma_2^\gamma\right| \le |\sigma_1 - \sigma_2|^\gamma.$$

Lemma 2.2 (Integral form of the quaternionic Stokes formula [1]) *Let* $\Omega, \Gamma = \partial\Omega$ *be as stated above and* $f, g \in C^{(1)}(\Omega, \mathbb{H})$, *then*

$$\int_\Gamma g(\xi) n_\psi(\xi) f(\xi)\, d\Gamma_\xi = \int_\Omega \bigl([g]^\psi D(\xi) \cdot f(\xi) + g(\xi) \cdot {}^\psi D[f](\xi)\bigr)\, d\Omega_\xi.$$

Lemma 2.3 (Borel-Pompeiu quaternionic formula [1]) *Let* $\Omega, \Gamma = \partial\Omega$ *be as stated above and* $f \in C^{(1)}(\Omega, \mathbb{H})$, *then for arbitrary* $\xi \in \Omega$, *we have*

$$\int_\Gamma \mathcal{K}_\psi(\eta - \xi)\, d\sigma_\eta f(\eta) - \int_\Omega \mathcal{K}_\psi(\eta - \xi)^\psi D[f](\eta)\, d_{\Omega_\eta} = f(\xi)$$

and

$$\int_\Gamma f(\eta)\, d\sigma_\eta \mathcal{K}_\psi(\eta - \xi) - \int_\Omega [f]^\psi D(\eta) \mathcal{K}_\psi(\eta - \xi)\, d_{\Omega_\eta} = f(\xi).$$

Lemma 2.4 (Hadamard lemma [10]) *Suppose* Ω *be as stated above. If* α', β' *satisfy* $0 < \alpha', \beta' < 4$, $\alpha' + \beta' > 4$, *then for all* $x_1, x_2 \in \mathbb{R}^4$ *and* $x_1 \neq x_2$, *we have*

$$\int_\Omega |t - x_1|^{-\alpha'} |t - x_2|^{-\beta'}\, dt \le M_0\bigl(\alpha', \beta'\bigr) |x_1 - x_2|^{4-\alpha'-\beta'}.$$

3 Some useful properties of the Cimmino singular integral operator

By means of the idea as stated above, we suppose

$$\begin{aligned}
&\xi = z_1 + z_2 j \in \mathbb{H}, \quad z_1 = x_0 + ix_1, z_2 = x_2 + ix_3 \in \mathbb{C}, \\
&\eta = \varsigma_1 + \varsigma_2 j \in \mathbb{H}, \quad \varsigma_1 = y_0 + iy_1, \varsigma_2 = y_2 + iy_3 \in \mathbb{C}, \\
&f(\xi) = f(z_1, z_2) = u_1(z_1, z_2) + u_2(z_1, z_2) j, \quad u_1 = f_0 + if_1, u_2 = f_2 + if_3 \in \mathbb{C}, \\
&g(\xi) = g(z_1, z_2) = v_1(z_1, z_2) + v_2(z_1, z_2) j, \quad v_1 = g_0 + ig_1, v_2 = g_2 + ig_3 \in \mathbb{C}.
\end{aligned}$$

Then system (1.1) can be written as

$$(1.1) \quad \Longleftrightarrow \quad \begin{cases} \partial_{\bar z_1} u_1 + \partial_{z_2} \bar u_2 = 0, \\ \partial_{\bar z_2} u_1 - \partial_{z_1} \bar u_2 = 0 \end{cases} \quad \Longleftrightarrow \quad {}^\psi D[f] = 0. \tag{3.1}$$

Moreover, if the pair (u_1, u_2) of continuously differentiable (up to the second order) complex-valued functions give a solution of system (1.1) then

$$\Delta_{\mathbb{R}^4} u_1 \cong \Delta_{\mathbb{C}^2} u_1 = 4\bigl(\partial^2_{z_1 \bar z_1} + \partial^2_{z_2 \bar z_2}\bigr) u_1 = 0.$$

A similar observation is valid for $\bar u_2$, so u_1, $\bar u_2$ are complex-valued harmonic functions, *i.e.* the set of solutions of system (1.1) contains all holomorphic functions of two complex variables.

Similarly, system (1.2) can be written as

$$(1.2) \quad \Longleftrightarrow \quad \begin{cases} 2\partial_{\bar{z}_1} u_1 + 2\partial_{z_2} \bar{u}_2 = v_1, \\ 2\partial_{\bar{z}_2} u_1 - 2\partial_{z_1} \bar{u}_2 = v_2 \end{cases} \quad \Longleftrightarrow \quad {}^{\psi}D[f] = g. \tag{3.2}$$

The generalized Teodorescu type integral operator ${}^{\psi}T_{\mathbb{C}^2}[g]$ can be written as

$$\begin{aligned} {}^{\psi}T_{\mathbb{C}^2}[g](z_1, z_2) &= {}^{\psi}T_{\mathbb{C}^2}[g](\xi) \\ &= -\int_{\mathbb{C}^2} \mathcal{K}_{\psi}(\eta - \xi) g(\eta)\, d_{\mathbb{C}^2_{\eta}} \\ &= \frac{1}{2\pi^2} \int_{\mathbb{C}^2} \frac{(\bar{z}_1 - \bar{\varsigma}_1) + (\bar{z}_2 - \bar{\varsigma}_2)j}{(|z_1 - \varsigma_1|^2 + |z_2 - \varsigma_2|^2)^2} g(\varsigma_1, \varsigma_2)\, d_{\mathbb{C}^2_{\varsigma_1, \varsigma_2}} \\ &= {}^{\psi}T^{(1)}_{\mathbb{C}^2}[g](z_1, z_2) + {}^{\psi}T^{(2)}_{\mathbb{C}^2}[g](z_1, z_2) j, \end{aligned}$$

where the Cimmino singular integral operators ${}^{\psi}T^{(1)}_{\mathbb{C}^2}[g]$, ${}^{\psi}T^{(2)}_{\mathbb{C}^2}[g]$ are as follows:

$$\begin{aligned} {}^{\psi}T^{(1)}_{\mathbb{C}^2}[g](z_1, z_2) &= \frac{1}{2\pi^2} \int_{\mathbb{C}^2} \frac{(\bar{z}_1 - \bar{\varsigma}_1)}{(|z_1 - \varsigma_1|^2 + |z_2 - \varsigma_2|^2)^2} g(\varsigma_1, \varsigma_2)\, d_{\mathbb{C}^2_{\varsigma_1, \varsigma_2}}, \\ {}^{\psi}T^{(2)}_{\mathbb{C}^2}[g](z_1, z_2) &= \frac{1}{2\pi^2} \int_{\mathbb{C}^2} \frac{(\bar{z}_2 - \bar{\varsigma}_2)}{(|z_1 - \varsigma_1|^2 + |z_2 - \varsigma_2|^2)^2} g(\varsigma_1, \varsigma_2)\, d_{\mathbb{C}^2_{\varsigma_1, \varsigma_2}}. \end{aligned}$$

Theorem 3.1 *Let E be as stated above. If $g \in L_p(\mathbb{C}^2, \mathbb{H})$, $4 < p < +\infty$, then we have*

(1) $|{}^{\psi}T_{\mathbb{C}^2}[g](\xi)| \le M_1(p)\|g\|_{L_p}$, $\xi \in \mathbb{C}^2 \cong \mathbb{R}^4$,

(2) ${}^{\psi}T_{\mathbb{C}^2}[g] \in C_{\beta}(\mathbb{C}^2, \mathbb{H}) \cong C_{\beta}(\mathbb{R}^4, \mathbb{H})$ $(0 < \beta = 1 - 4/p < 1)$,

(3) ${}^{\psi}D({}^{\psi}T_{\mathbb{C}^2}[g])(\xi) = g(\xi)$, $\xi \in \mathbb{C}^2 \cong \mathbb{R}^4$.

Proof (1) First, we have

$$\begin{aligned} \left|{}^{\psi}T_{\mathbb{C}^2}[g](\xi)\right| &\le \frac{1}{2\pi^2} \left| \int_E \frac{(\bar{z}_1 - \bar{\varsigma}_1) + (\bar{z}_2 - \bar{\varsigma}_2)j}{(|z_1 - \varsigma_1|^2 + |z_2 - \varsigma_2|^2)^2} g(\eta)\, dE_{\eta} \right| \\ &\quad + \frac{1}{2\pi^2} \left| \int_{\mathbb{C}^2 - E} \frac{(\bar{z}_1 - \bar{\varsigma}_1) + (\bar{z}_2 - \bar{\varsigma}_2)j}{(|z_1 - \varsigma_1|^2 + |z_2 - \varsigma_2|^2)^2} g(\eta)\, d_{(\mathbb{C}^2 - E)_{\eta}} \right| \\ &= O_1 + O_2. \end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned} O_1 &= \frac{1}{2\pi^2} \left| \int_E \frac{(\bar{z}_1 - \bar{\varsigma}_1) + (\bar{z}_2 - \bar{\varsigma}_2)j}{(|z_1 - \varsigma_1|^2 + |z_2 - \varsigma_2|^2)^2} g(\eta)\, dE_{\eta} \right| \\ &\le \frac{1}{2\pi^2} \int_E \frac{|z_1 - \varsigma_1| + |z_2 - \varsigma_2|}{|\xi - \eta|^4} \left|g(\eta)\right| dE_{\eta} \\ &\le \frac{1}{\pi^2} \int_E \frac{1}{|\xi - \eta|^3} \left|g(\eta)\right| dE_{\eta} \le \frac{1}{\pi^2} \|g\|_{L_p} \left(\int_E \frac{1}{|\xi - \eta|^{3q}}\, dE_{\eta} \right)^{\frac{1}{q}}, \end{aligned}$$

where $1/p + 1/q = 1$.

When $\xi \in \bar{E}$, because of $4 < p < +\infty$, $1/p + 1/q = 1$, we have $1 < q < 4/3$. Thus $\int_E \frac{1}{|\xi-\eta|^{3q}}\,dE_\eta$ is bounded. Hence we have

$$\left(\int_E \frac{1}{|\xi-\eta|^{3q}}\,dE_\eta\right)^{\frac{1}{q}} \le J_1.$$

When $\xi \in \mathbb{C}^2 - \bar{E}$, by Lemma 2.1 and the generalized spherical coordinate, we have

$$\left(\int_E \frac{1}{|\xi-\eta|^{3q}}\,dE_\eta\right)^{\frac{1}{q}} \le J_2\left(\int_{d_0}^{d_0+2} \rho^{3-3q}\,d\rho\right)^{\frac{1}{q}} \le J_3,$$

where $\rho = |\xi - \eta|$, $d_0 = d(\xi, \bar{E})$.

Therefore, for $\forall \xi \in \mathbb{C}^2 \cong \mathbb{R}^4$, we can obtain

$$O_1 \le M_1'(p)\|g\|_{L_p}, \quad \xi \in \mathbb{C}^2 \cong \mathbb{R}^4,$$

where $M_1'(p) = \max\{J_1/\pi^2, J_3/\pi^2\}$.

For $\eta \in \mathbb{C}^2 - E$, we suppose that $\eta = \frac{\bar{\eta}'}{|\eta'|^2}$, then we have $|\eta'| \le 1$. Thus by $g \in L_p(\mathbb{C}^2, \mathbb{H})$, similar to the proof as stated above, we have

$$O_2 \le M_1''(p)\|g\|_{L_p}.$$

Therefore, we obtain

$$\left|{}^{\psi}T_{\mathbb{C}^2}[g](\xi)\right| \le M_1(p)\|g\|_{L_p}, \quad \xi \in \mathbb{C}^2 \cong \mathbb{R}^4,$$

where $M_1(p) = M_1'(p) + M_1''(p)$.

(2) For arbitrary $\xi', \xi'' \in \mathbb{C}^2 \cong \mathbb{R}^4$, $\xi' \ne \xi''$, we have

$$\begin{aligned}
&\left|{}^{\psi}T_{\mathbb{C}^2}[g](\xi') - {}^{\psi}T_{\mathbb{C}^2}[g](\xi'')\right| \\
&\quad = \frac{1}{2\pi^2}\left|\int_{\mathbb{C}^2}\left[\frac{(\bar{z}_1' - \bar{\varsigma}_1) + (\bar{z}_2' - \bar{\varsigma}_2)j}{|\xi'-\eta|^4} - \frac{(\bar{z}_1'' - \bar{\varsigma}_1) + (\bar{z}_2'' - \bar{\varsigma}_2)j}{|\xi''-\eta|^4}\right]g(\eta)\,d_{\mathbb{C}^2{}_\eta}\right| \\
&\quad \le \frac{1}{2\pi^2}\left|\int_E\left[\frac{(\bar{z}_1' - \bar{\varsigma}_1) + (\bar{z}_2' - \bar{\varsigma}_2)j}{|\xi'-\eta|^4} - \frac{(\bar{z}_1'' - \bar{\varsigma}_1) + (\bar{z}_2'' - \bar{\varsigma}_2)j}{|\xi''-\eta|^4}\right]g(\eta)\,dE_\eta\right| \\
&\qquad + \frac{1}{2\pi^2}\left|\int_{\mathbb{C}^2-E}\left[\frac{(\bar{z}_1' - \bar{\varsigma}_1) + (\bar{z}_2' - \bar{\varsigma}_2)j}{|\xi'-\eta|^4} - \frac{(\bar{z}_1'' - \bar{\varsigma}_1) + (\bar{z}_2'' - \bar{\varsigma}_2)j}{|\xi''-\eta|^4}\right]g(\eta)\,d_{(\mathbb{C}^2-E)_\eta}\right| \\
&\quad = O_3 + O_4
\end{aligned}$$

and

$$\begin{aligned}
O_3 &\le \frac{1}{2\pi^2}\left|\int_E\left[\frac{(\bar{z}_1' - \bar{\varsigma}_1)}{|\xi'-\eta|^4} - \frac{(\bar{z}_1'' - \bar{\varsigma}_1)}{|\xi''-\eta|^4}\right]g(\eta)\,dE_\eta\right| \\
&\quad + \frac{1}{2\pi^2}\left|\int_E\left[\frac{(\bar{z}_2' - \bar{\varsigma}_2)j}{|\xi'-\eta|^4} - \frac{(\bar{z}_2'' - \bar{\varsigma}_2)j}{|\xi''-\eta|^4}\right]g(\eta)\,dE_\eta\right| \\
&= \left|{}^{\psi}T_E^{(1)}[g](z_1', z_2') - {}^{\psi}T_E^{(1)}[g](z_1'', z_2'')\right| + \left|{}^{\psi}T_E^{(2)}[g](z_1', z_2') - {}^{\psi}T_E^{(2)}[g](z_1'', z_2'')\right| \\
&= I_1 + I_2.
\end{aligned} \tag{3.3}$$

Since

$$\begin{aligned}
&\left|\frac{\bar{z}_1'-\bar{\varsigma}_1}{|\xi'-\eta|^4}-\frac{\bar{z}_1''-\bar{\varsigma}_1}{|\xi''-\eta|^4}\right| \\
&\quad=\left|\frac{(\bar{z}_1'-\bar{\varsigma}_1)|\xi''-\eta|^2(|z_1''-\varsigma_1|^2+|z_2''-\varsigma_2|^2)}{|\xi'-\eta|^4|\xi''-\eta|^4}\right. \\
&\qquad\left.-\frac{(|z_1'-\varsigma_1|^2+|z_2'-\varsigma_2|^2)|\xi'-\eta|^2(\bar{z}_1''-\bar{\varsigma}_1)}{|\xi'-\eta|^4|\xi''-\eta|^4}\right| \\
&\quad\le\frac{|(\bar{z}_1'-\bar{\varsigma}_1)|\xi''-\eta|^2|z_1''-\varsigma_1|^2-|z_1'-\varsigma_1|^2|\xi'-\eta|^2(\bar{z}_1''-\bar{\varsigma}_1)|}{|\xi'-\eta|^4|\xi''-\eta|^4} \\
&\qquad+\frac{|(\bar{z}_1'-\bar{\varsigma}_1)|\xi''-\eta|^2|z_2''-\varsigma_2|^2-|z_2'-\varsigma_2|^2|\xi'-\eta|^2(\bar{z}_1''-\bar{\varsigma}_1)|}{|\xi'-\eta|^4|\xi''-\eta|^4} \\
&\quad=\mathcal{K}_1(\xi',\xi'',\eta)+\mathcal{K}_2(\xi',\xi'',\eta).
\end{aligned}\tag{3.4}$$

Thus

$$I_1\le\frac{1}{2\pi^2}\int_E\mathcal{K}_1(\xi',\xi'',\eta)\left|g(\eta)\right|dE_\eta+\frac{1}{2\pi^2}\int_E\mathcal{K}_2(\xi',\xi'',\eta)\left|g(\eta)\right|dE_\eta=I_{1_1}+I_{1_2}.\tag{3.5}$$

Again, because of

$$\begin{aligned}
&\mathcal{K}_1(\xi',\xi'',\eta) \\
&\quad=\frac{|(\bar{z}_1'-\bar{\varsigma}_1)|\xi''-\eta|^2|z_1''-\varsigma_1|^2-|z_1'-\varsigma_1|^2|\xi'-\eta|^2(\bar{z}_1''-\bar{\varsigma}_1)|}{|\xi'-\eta|^4|\xi''-\eta|^4} \\
&\quad=\frac{|(\bar{z}_1'-\bar{\varsigma}_1)|\xi''-\eta|^2(z_1''-\varsigma_1)(\bar{z}_1''-\bar{\varsigma}_1)-(\bar{z}_1'-\bar{\varsigma}_1)(z_1'-\varsigma_1)|\xi'-\eta|^2(\bar{z}_1''-\bar{\varsigma}_1)|}{|\xi'-\eta|^4|\xi''-\eta|^4} \\
&\quad=\frac{|\bar{z}_1'-\bar{\varsigma}_1||\,|\xi''-\eta|^2(z_1''-\varsigma_1)-(z_1'-\varsigma_1)|\xi'-\eta|^2\,||\bar{z}_1''-\bar{\varsigma}_1|}{|\xi'-\eta|^4|\xi''-\eta|^4} \\
&\quad\le\frac{|\bar{z}_1'-\bar{\varsigma}_1|[|\xi''-\eta|^2|z_1''-z_1'|+||\xi''-\eta|^2-|\xi'-\eta|^2||z_1'-\varsigma_1|]|\bar{z}_1''-\bar{\varsigma}_1|}{|\xi'-\eta|^4|\xi''-\eta|^4} \\
&\quad\le\frac{|\bar{z}_1'-\bar{\varsigma}_1|[|\xi''-\eta|^2|z_1''-z_1'|+|\xi''-\xi'|(|\xi''-\eta|+|\xi'-\eta|)|z_1'-\varsigma_1|]|\bar{z}_1''-\bar{\varsigma}_1|}{|\xi'-\eta|^4|\xi''-\eta|^4} \\
&\quad\le\frac{|\xi'-\eta|[|\xi''-\eta|^2|\xi''-\xi'|+|\xi''-\xi'|(|\xi''-\eta|+|\xi'-\eta|)|\xi'-\eta|]|\xi''-\eta|}{|\xi'-\eta|^4|\xi''-\eta|^4} \\
&\quad=\left|\xi'-\xi''\right|\sum_{l=1}^{3}\frac{1}{|\xi'-\eta|^{4-l}|\xi''-\eta|^l}.
\end{aligned}\tag{3.6}$$

Thus by (3.5), (3.6), and the Hölder inequality, we have

$$\begin{aligned}
I_{1_1}&\le\frac{1}{2\pi^2}\int_E\sum_{l=1}^{3}\frac{1}{|\xi'-\eta|^{4-l}|\xi''-\eta|^l}\left|g(\eta)\right|dE_\eta\left|\xi'-\xi''\right| \\
&\le\frac{1}{2\pi^2}\|g\|_{L_p}\left|\xi'-\xi''\right|\sum_{l=1}^{3}\left(\int_E\frac{1}{|\xi'-\eta|^{(4-l)q}|\xi''-\eta|^{lq}}\,dE_\eta\right)^{\frac{1}{q}} \\
&=\frac{1}{2\pi^2}\|g\|_{L_p}\left|\xi'-\xi''\right|\sum_{l=1}^{3}\left(I_{1_1}^{(l)}\right)^{\frac{1}{q}}.
\end{aligned}\tag{3.7}$$

Suppose $\alpha_l = (4-l)q$, $\beta_l = lq$ $(l = 1,2,3)$. By $1 < q < 4/3$, we know

$$0 < \alpha_l, \beta_l < 4, \qquad \alpha_l + \beta_l = 4q > 4.$$

Thus, by Lemma 2.4, for $l = 1,2,3$, we have

$$\begin{aligned} I_{1_1}^{(l)} &= \int_E \frac{1}{|\xi' - \eta|^{(4-l)q}|\xi'' - \eta|^{lq}}\, dE_\eta \\ &\le M_0(\alpha_l, \beta_l)\left|\xi' - \xi''\right|^{4-\alpha_l-\beta_l} \\ &= M_0(\alpha_l, \beta_l)\left|\xi' - \xi''\right|^{4-4q}. \end{aligned} \tag{3.8}$$

Thus, by inequalities (3.7) and (3.8), we obtain

$$\begin{aligned} I_{1_1} &\le \frac{1}{2\pi^2}\|g\|_{L_p}\left|\xi' - \xi''\right| \sum_{l=1}^{3}\left(M_0(\alpha_l, \beta_l)\left|\xi' - \xi''\right|^{4-4q}\right)^{\frac{1}{q}} \\ &\le J_4\|g\|_{L_p}\left|\xi' - \xi''\right|^{\beta}, \end{aligned} \tag{3.9}$$

where $0 < \beta = 1 + (4-4q)/q = 1 - 4/p < 1$.

Next, we discuss I_{1_2}.

For arbitrary $\xi', \xi'' \in \mathbb{C}^2 \cong \mathbb{R}^4$, $\xi' \neq \xi''$, we suppose $|\xi' - \xi''| = \delta$ and construct a sphere $B(\xi', 3\delta)$ with the center at ξ' and radius 3δ. Next we discuss I_{1_2} in two cases.

(i) If $B(\xi', 3\delta) \cap \bar{E} \neq \emptyset$, then we may suppose $B(\xi', 3\delta) \cap \bar{E} = \Omega_1$, $\bar{E} - \Omega_1 = \Omega_2$. Thus we have

$$\begin{aligned} I_{1_2} &= \frac{1}{2\pi^2}\int_E \mathcal{K}_2\left(\xi', \xi'', \eta\right)\left|g(\eta)\right| dE_\eta \\ &= \frac{1}{2\pi^2}\int_{\Omega_1} \mathcal{K}_2\left(\xi', \xi'', \eta\right)\left|g(\eta)\right| d\Omega_{1\eta} \\ &\quad + \frac{1}{2\pi^2}\int_{\Omega_2} \mathcal{K}_2\left(\xi', \xi'', \eta\right)\left|g(\eta)\right| d\Omega_{2\eta} \\ &= I_{1_2}^{(1)} + I_{1_2}^{(2)}. \end{aligned} \tag{3.10}$$

Again, by inequality (3.4), the Hölder inequality, and the use of a local generalized spherical coordinate, we have

$$\begin{aligned} I_{1_2}^{(1)} &\le \frac{1}{2\pi^2}\int_{\Omega_1} \frac{|\bar{z}_1' - \bar{\varsigma}_1||\xi'' - \eta|^2|z_2'' - \varsigma_2|^2}{|\xi' - \eta|^4|\xi'' - \eta|^4}\left|g(\eta)\right| d\Omega_{1\eta} \\ &\quad + \frac{1}{2\pi^2}\int_{\Omega_1} \frac{|z_2' - \varsigma_2|^2|\xi' - \eta|^2|\bar{z}_1'' - \bar{\varsigma}_1|}{|\xi' - \eta|^4|\xi'' - \eta|^4}\left|g(\eta)\right| d\Omega_{1\eta} \\ &\le \frac{1}{2\pi^2}\int_{\Omega_1} \frac{1}{|\xi' - \eta|^3}\left|g(\eta)\right| d\Omega_{1\eta} + \frac{1}{2\pi^2}\int_{\Omega_1} \frac{1}{|\xi'' - \eta|^3}\left|g(\eta)\right| d\Omega_{1\eta} \\ &\le J_5\|g\|_{L_p}\left[\left(\int_{\Omega_1} \frac{1}{|\xi' - \eta|^{3q}}\, d\Omega_{1\eta}\right)^{\frac{1}{q}} + \left(\int_{\Omega_1} \frac{1}{|\xi'' - \eta|^{3q}}\, d\Omega_{1\eta}\right)^{\frac{1}{q}}\right] \\ &\le J_6\|g\|_{L_p}\left[\left(\int_0^{3\delta} \frac{1}{\rho^{3q-3}}\, d\rho\right)^{\frac{1}{q}} + \left(\int_0^{4\delta} \frac{1}{\rho^{3q-3}}\, d\rho\right)^{\frac{1}{q}}\right] \end{aligned}$$

$$\begin{aligned}&\le J_7\|g\|_{L_p}\delta^{\frac{4-3q}{q}}=J_7\|g\|_{L_p}\left|\xi'-\xi''\right|^{1-\frac{4}{p}}\\&=J_7\|g\|_{L_p}\left|\xi'-\xi''\right|^{\beta}.\end{aligned}\tag{3.11}$$

In addition,

$$\begin{aligned}&\mathcal{K}_2(\xi',\xi'',\eta)\\&\quad=\frac{|(\bar{z}_1'-\bar{\varsigma}_1)|\xi''-\eta|^2|z_2''-\varsigma_2|^2-|z_2'-\varsigma_2|^2|\xi'-\eta|^2(\bar{z}_1''-\bar{\varsigma}_1)|}{|\xi'-\eta|^4|\xi''-\eta|^4}\\&\quad=\frac{|(\bar{z}_1''-\bar{\varsigma}_1+\bar{z}_1'-\bar{z}_1'')|\xi''-\eta|^2|z_2''-\varsigma_2|^2-|z_2'-\varsigma_2|^2|\xi'-\eta|^2(\bar{z}_1''-\bar{\varsigma}_1)|}{|\xi'-\eta|^4|\xi''-\eta|^4}\\&\quad=\left|\frac{(\bar{z}_1''-\bar{\varsigma}_1)|\xi''-\eta|^2|z_2''-\varsigma_2|^2+(\bar{z}_1'-\bar{z}_1'')|\xi''-\eta|^2|z_2''-\varsigma_2|^2}{|\xi'-\eta|^4|\xi''-\eta|^4}\right.\\&\qquad\left.-\frac{(\bar{z}_1''-\bar{\varsigma}_1)|z_2'-\varsigma_2|^2|\xi'-\eta|^2}{|\xi'-\eta|^4|\xi''-\eta|^4}\right|\\&\quad\le\frac{|z_1''-\varsigma_1||\,|\xi''-\eta|^2|z_2''-\varsigma_2|^2-|z_2'-\varsigma_2|^2|\xi'-\eta|^2|}{|\xi'-\eta|^4|\xi''-\eta|^4}\\&\qquad+\frac{|z_1'-z_1''||\xi''-\eta|^2|z_2''-\varsigma_2|^2}{|\xi'-\eta|^4|\xi''-\eta|^4}.\end{aligned}$$

Again, because of

$$\begin{aligned}&\left|\left|\xi''-\eta\right|^2\left|z_2''-\varsigma_2\right|^2-\left|z_2'-\varsigma_2\right|^2\left|\xi'-\eta\right|^2\right|\\&\quad=\left|\left|\xi''-\eta\right|^2\left|z_2''-\varsigma_2\right|^2-\left|\xi''-\eta\right|^2\left|z_2'-\varsigma_2\right|^2\right.\\&\qquad\left.+\left|\xi''-\eta\right|^2\left|z_2'-\varsigma_2\right|^2-\left|z_2'-\varsigma_2\right|^2\left|\xi'-\eta\right|^2\right|\\&\quad\le\left|\xi''-\eta\right|^2\left|\left|z_2''-\varsigma_2\right|^2-\left|z_2'-\varsigma_2\right|^2\right|+\left|z_2'-\varsigma_2\right|^2\left|\left|\xi''-\eta\right|^2-\left|\xi'-\eta\right|^2\right|\\&\quad\le\left|\xi''-\eta\right|^2\left|z_2''-z_2'\right|\left(\left|z_2''-\varsigma_2\right|+\left|z_2'-\varsigma_2\right|\right)+\left|z_2'-\varsigma_2\right|^2\left|\xi''-\xi'\right|\left(\left|\xi''-\eta\right|+\left|\xi'-\eta\right|\right)\\&\quad\le\left|\xi''-\xi'\right|\sum_{m=0}^{3}\left|\xi''-\eta\right|^{3-m}\left|\xi'-\eta\right|^{m}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\mathcal{K}_2(\xi',\xi'',\eta)\\&\quad\le\frac{|\xi''-\eta||\xi''-\xi'|\sum_{m=0}^{3}|\xi''-\eta|^{3-m}|\xi'-\eta|^m+|\xi'-\xi''||\xi''-\eta|^2|\xi''-\eta|^2}{|\xi'-\eta|^4|\xi''-\eta|^4}\\&\quad=\left|\xi'-\xi''\right|\left(\frac{2}{|\xi'-\eta|^4}+\sum_{l=1}^{3}\frac{1}{|\xi'-\eta|^{4-l}|\xi''-\eta|^l}\right).\end{aligned}\tag{3.12}$$

So by (3.10) and (3.12), we obtain

$$\begin{aligned}I_{12}^{(2)}&=\frac{1}{2\pi^2}\int_{\Omega_2}\mathcal{K}_2(\xi',\xi'',\eta)\left|g(\eta)\right|d\Omega_{2\eta}\\&\le\left|\xi'-\xi''\right|\frac{1}{\pi^2}\int_{\Omega_2}\frac{|g(\eta)|}{|\xi'-\eta|^4}\,d\Omega_{2\eta}\end{aligned}$$

$$
\begin{aligned}
&+ \left|\xi' - \xi''\right| \frac{1}{2\pi^2} \int_{\Omega_2} \sum_{l=1}^{3} \frac{|g(\eta)|}{|\xi' - \eta|^{4-l}|\xi'' - \eta|^{l}} \, d\Omega_{2\eta} \\
&= I_{1_2}^{(2)'} + I_{1_2}^{(2)''}.
\end{aligned} \tag{3.13}
$$

First, similar to the method estimating I_{1_1}, we have

$$
I_{1_2}^{(2)''} \le J_8 \|g\|_{L_p} \left|\xi' - \xi''\right|^{\beta}. \tag{3.14}
$$

Second, when $\eta \in \Omega_2$, $|\xi' - \eta| > 3\delta$, $|\xi'' - \eta| > 2\delta$. Thus we have

$$
\begin{aligned}
&2\delta \le \left|\xi' - \eta\right| - \left|\xi' - \xi''\right| \le \left|\xi'' - \eta\right| \le \left|\xi' - \xi''\right| + \left|\xi' - \eta\right| = \delta + \left|\xi' - \eta\right|, \\
&\delta \le \left|\xi'' - \eta\right| - \left|\xi'' - \xi'\right| \le \left|\xi' - \eta\right| \le \left|\xi' - \xi''\right| + \left|\xi'' - \eta\right| = \delta + \left|\xi'' - \eta\right|.
\end{aligned} \tag{3.15}
$$

So we know

$$
\frac{1}{2} \le \frac{|\xi' - \eta|}{|\xi'' - \eta|} \le \frac{3}{2}. \tag{3.16}
$$

Thus, by (3.16), the Hölder inequality, and Lemma 2.4, we can obtain

$$
\begin{aligned}
I_{1_2}^{(2)'} &= \left|\xi' - \xi''\right| \frac{1}{\pi^2} \int_{\Omega_2} \frac{|g(\eta)|}{|\xi' - \eta|^4} \, d\Omega_{2\eta} \\
&\le \left|\xi' - \xi''\right| \frac{1}{\pi^2} \int_{\Omega_2} \frac{2|g(\eta)|}{|\xi' - \eta|^3|\xi'' - \eta|} \, d\Omega_{2\eta} \\
&\le J_9 \|g\|_{L_p} \left|\xi' - \xi''\right|^{1-\frac{4}{p}} \\
&= J_9 \|g\|_{L_p} \left|\xi' - \xi''\right|^{\beta}.
\end{aligned} \tag{3.17}
$$

So, by (3.13), (3.14), and (3.17), we have

$$
I_{1_2}^{(2)} \le J_{10} \|g\|_{L_p} \left|\xi' - \xi''\right|^{\beta}. \tag{3.18}
$$

Therefore, by (3.10), (3.11), and (3.18), we can obtain

$$
I_{1_2} \le J_{11} \|g\|_{L_p} \left|\xi' - \xi''\right|^{\beta}. \tag{3.19}
$$

(ii) If $B(\xi', 3\delta) \cap \bar{E} = \emptyset$, then for arbitrary $\eta \in E$, we have $|\xi' - \eta| > 3\delta$, $|\xi'' - \eta| > 2\delta$. Thus similar to the method estimating $I_{1_2}^{(2)}$, we have

$$
I_{1_2} \le J_{12} \|g\|_{L_p} \left|\xi' - \xi''\right|^{\beta}. \tag{3.20}
$$

So, by (3.19) and (3.20), we have

$$
I_{1_2} \le J_{13} \|g\|_{L_p} \left|\xi' - \xi''\right|^{\beta}, \quad \xi', \xi'' \in \mathbb{R}^4 \cong \mathbb{C}^2, \tag{3.21}
$$

where $J_{13} = \max\{J_{11}, J_{12}\}$.

Thus, to sum up, by (3.5), (3.9), and (3.21), we obtain

$$I_1 \leq J_{14}\|g\|_{L_p}\left|\xi' - \xi''\right|^{\beta}, \quad \xi', \xi'' \in \mathbb{R}^4 \cong \mathbb{C}^2. \tag{3.22}$$

Similarly, we have

$$I_2 \leq J_{15}\|g\|_{L_p}\left|\xi' - \xi''\right|^{\beta}, \quad \xi', \xi'' \in \mathbb{R}^4 \cong \mathbb{C}^2. \tag{3.23}$$

So, by (3.3), (3.22), and (3.23), we obtain

$$O_3 \leq M_2'(p)\|g\|_{L_p}\left|\xi' - \xi''\right|^{\beta}, \quad \xi', \xi'' \in \mathbb{R}^4 \cong \mathbb{C}^2, \tag{3.24}$$

where $M_2'(p) = J_{14} + J_{15}$.

For $\eta \in \mathbb{C}^2 - E$, we suppose that $\eta = \frac{\bar{\eta}'}{|\eta'|^2}$, then we have $|\eta'| \leq 1$. Thus by $g \in L_p(\mathbb{C}^2, \mathbb{H})$, similar to the proof as stated above, we have

$$O_4 \leq M_2''(p)\|g\|_{L_p}\left|\xi' - \xi''\right|^{\beta}, \quad \xi', \xi'' \in \mathbb{R}^4 \cong \mathbb{C}^2.$$

Therefore, for arbitrary $\xi', \xi'' \in \mathbb{C}^2 \cong \mathbb{R}^4$, $\xi' \neq \xi''$, we obtain

$$\left|{}^{\psi}T_{\mathbb{C}^2}[g](\xi') - {}^{\psi}T_{\mathbb{C}^2}[g](\xi'')\right| \leq M_2(p)\|g\|_{L_p}\left|\xi' - \xi''\right|^{\beta}, \quad \xi', \xi'' \in \mathbb{R}^4 \cong \mathbb{C}^2,$$

where $M_2(p) = M_2'(p) + M_2''(p)$, *i.e.* ${}^{\psi}T_{\mathbb{C}^2}[g] \in C_\beta(\mathbb{C}^2, \mathbb{H}) \cong C_\beta(\mathbb{R}^4, \mathbb{H})$ $(0 < \beta < 1)$.

(3) For arbitrary $\varphi \in C_0^\infty(\mathbb{C}^2, \mathbb{H})$, there exists a bounded closed set $Q \subset \mathbb{C}^2$, such that $\overline{\operatorname{supp}\varphi} \subset\subset Q$. Thus, by $T_{\mathbb{C}^2}[g](\infty) = 0$, Definition 2.1, Lemma 2.3, and the Fubini theorem, we have

$$\begin{aligned}
&\int_{\mathbb{C}^2} [\varphi]^{\psi}D(\xi)^{\psi}T_{\mathbb{C}^2}[g](\xi)\, d_{\mathbb{C}^2_\xi} \\
&\quad = \lim_{d\to\infty} \int_Q [\varphi]^{\psi}D(\xi)^{\psi}T_{\mathbb{C}^2}[g](\xi)\, dQ_\xi \\
&\quad = -\lim_{d\to\infty} \int_Q [\varphi]^{\psi}D(\xi) \int_{\mathbb{C}^2} \mathcal{K}_\psi(\eta - \xi)g(\eta)\, d_{\mathbb{C}^2_\eta}\, dQ_\xi \\
&\quad = \lim_{d\to\infty} \int_{\mathbb{C}^2} \int_Q [\varphi]^{\psi}D(\xi)\mathcal{K}_\psi(\xi - \eta)\, dQ_\xi g(\eta)\, d_{\mathbb{C}^2_\eta} \\
&\quad = \lim_{d\to\infty} \int_{\mathbb{C}^2} \left[\int_{\partial Q} \varphi(\xi)\, d\sigma_\xi \mathcal{K}_\psi(\xi - \eta) - \varphi(\eta)\right] g(\eta)\, d_{\mathbb{C}^2_\eta} \\
&\quad = -\int_{\mathbb{C}^2} \varphi(\eta)g(\eta)\, d_{\mathbb{C}^2_\eta} = -\int_{\mathbb{C}^2} \varphi(\xi)g(\xi)\, d_{\mathbb{C}^2_\xi},
\end{aligned}$$

where $d = \sup_{\xi',\xi''\in Q} |\xi' - \xi''|$. Hence, in the sense of generalized derivatives, we have ${}^{\psi}D({}^{\psi}T_{\mathbb{C}^2}[g])(\xi) = g(\xi)$. □

Remark 3.1 By the process of proof in Theorem 3.1, it is easy to show that ${}^{\psi}T^{(1)}_{\mathbb{C}^2}[g]$, ${}^{\psi}T^{(2)}_{\mathbb{C}^2}[g] \in C_\beta(\mathbb{C}^2, \mathbb{C}) \cong C_\beta(\mathbb{R}^4, \mathbb{C})$ $(0 < \beta < 1)$.

4 Integral representation of solution of the mixed boundary value problem for the inhomogeneous Cimmino system

In this section, let $E = E_1 \times E_2$ be a bounded domain, ∂E_m $(m = 1, 2)$ be simply closed curves in the z_m-plane, and $\partial E_m \in C_\mu^{(1)}$, $0 < \mu < 1$. Without loss of generality, we may consider $\partial E_m = \{z_m \| z_m| = 1\}$ and $E_m = \{z_m \| z_m| < 1\}$ $(m = 1, 2)$. Denote by E_m^+, E_m^- the inner domain and outer domain of ∂E_m, respectively, and $E^{++} = E_1^+ \times E_2^+$, $E^{+-} = E_1^+ \times E_2^-$, $E^{-+} = E_1^- \times E_2^+$, $E^{--} = E_1^- \times E_2^-$, $\Gamma = \partial E_1 \times \partial E_2$.

Problem P The mixed boundary value problem for the inhomogeneous Cimmino system (1.2) is to find a function $f(z_1, z_2) = u_1(z_1, z_2) + u_2(z_1, z_2)j$ satisfying the Cimmino system (1.2) and the following boundary condition:

$$\begin{aligned} u_1^{++}(t_1,t_2) = {} & G_1(t_1,t_2)u_1^{+-}(t_1,t_2) + G_2(t_1,t_2)u_1^{-+}(t_1,t_2) \\ & + G_3(t_1,t_2)u_1^{--}(t_1,t_2) + H(t_1,t_2), \quad t = (t_1,t_2) \in \Gamma, \end{aligned} \tag{4.1}$$

$$u_2(t_1,t_2) = h(t_1,t_2), \quad t = (t_1,t_2) \in \partial E, \tag{4.2}$$

where $u_1 = f_0 + if_1$, $u_2 = f_2 + if_3$, $z_1 = x_0 + ix_1$, $z_2 = x_2 + ix_3$. $G_1(z_1, z_2)$, $G_2(z_1, z_2)$, $G_3(z_1, z_2)$ are analytic in E^{+-}, E^{-+}, E^{--} and are continuous in $\bar{E}^{+-}$, $\bar{E}^{-+}$, $\bar{E}^{--}$, respectively, which have no zero. We have $G_m(t_1, t_2)$ $(m = 1, 2, 3)$, $H(t_1, t_2) \in C_\alpha(\Gamma, \mathbb{C})$, $h(t_1, t_2) \in C_\alpha(\partial E, \mathbb{C})$ $(0 < \alpha < 1)$.

Lemma 4.1 *If $\Psi \in C^{(2)}(E, \mathbb{H})$, $h \in C_\alpha(\partial E, \mathbb{C})$ $(0 < \alpha < 1)$, $g \in L_p(\mathbb{C}^2, \mathbb{H})$ $(4 < p < +\infty)$, then the equation ${}^{\psi}D[\Psi] = 0$ with the boundary condition $\bar{w}_2|_{\partial E} = \bar{h}(t_1, t_2) - \overline{{}^{\psi}T_{\mathbb{C}^2}^{(2)}[g]}(t_1, t_2)$ has the solution $\Psi = w_1 + w_2 j = w_1 + j\bar{w}_2$ and*

$$\begin{aligned} \bar{w}_2(\xi) &= \int_{\partial E} \bigl[\bar{h}(t) - \overline{{}^{\psi}T_{\mathbb{C}^2}^{(2)}[g](t)}\bigr] \frac{\partial}{\partial \nu} G(\xi, t)\, d\partial E_t, \\ w_1(\xi) &= \Phi(\xi) + w_0(\xi) \end{aligned}$$

or

$$\begin{aligned} \bar{w}_2(z_1,z_2) &= \int_{\partial E} \bigl[\bar{h}(t_1,t_2) - \overline{{}^{\psi}T_{\mathbb{C}^2}^{(2)}[g]}(t_1,t_2)\bigr] \frac{\partial}{\partial \nu} G(z_1,z_2,t_1,t_2)\, d\partial E_{t_1,t_2}, \\ w_1(z_1,z_2) &= \Phi(z_1,z_2) + w_0(z_1,z_2), \end{aligned}$$

where ν is the unit outward normal on ∂E, $G(\xi, \eta)$ is the Green's function in $E = E_1 \times E_2$, $\Phi(z_1, z_2)$ is an arbitrary analytic function in $E = E_1 \times E_2$, and

$$\begin{aligned} {}^{\psi}T_{\mathbb{C}^2}^{(2)}[g](t_1,t_2) &= \frac{1}{2\pi^2} \int_{\mathbb{C}^2} \frac{(\bar{t}_2 - \bar{\varsigma}_2)}{(|t_1 - \varsigma_1|^2 + |t_2 - \varsigma_2|^2)^2} g(\varsigma_1, \varsigma_2)\, d_{\mathbb{C}^2_{\varsigma_1,\varsigma_2}}, \\ w_0(z_1,z_2) &= \widetilde{T}_{E_1}[-\partial_{z_2}\bar{w}_2] + \widetilde{T}_{E_2}[\Phi_0], \\ \widetilde{T}_{E_1}[-\partial_{z_2}\bar{w}_2] &= -\frac{1}{\pi} \int_{E_1} \frac{-\partial_{z_2}\bar{w}_2(\varsigma_1, z_2)}{\varsigma_1 - z_1}\, dE_{1\varsigma_1}, \\ \widetilde{T}_{E_2}[\Phi_0] &= -\frac{1}{\pi} \int_{E_2} \frac{\Phi_0(z_1, \varsigma_2)}{\varsigma_2 - z_2}\, dE_{2\varsigma_2}, \\ \Phi_0(z_1,z_2) &= \frac{1}{2\pi i} \int_{\partial E_1} \frac{\partial_{z_1}\bar{w}_2(\varsigma_1, z_2)}{\varsigma_1 - z_1}\, d\partial E_1 \varsigma_1. \end{aligned}$$

Proof From Remark 3.1, we know ${}^{\psi}T^{(2)}_{\mathbb{C}^2}[g] \in C_\beta(\mathbb{C}^2, \mathbb{C}) \cong C_\beta(\mathbb{R}^4, \mathbb{C})$ $(0 < \beta < 1)$. Thus by [9], we have $\bar{h} - \overline{{}^{\psi}T^{(2)}_{\mathbb{C}^2}[g]} \in C_\mu(\partial E, \mathbb{C})$ $(0 < \mu = \min\{\alpha, \beta\} < 1)$. So we may construct

$$\bar{w}_2(\xi) = \int_{\partial E} \bigl[\bar{h}(t) - \overline{{}^{\psi}T^{(2)}_{\mathbb{C}^2}[g]}(t)\bigr] \frac{\partial}{\partial \nu} G(\xi, t)\, d\partial E_t,$$

where ν is the unit outward normal on ∂E, $G(\xi, \eta)$ is the Green's function in $E = E_1 \times E_2$, and

$${}^{\psi}T^{(2)}_{\mathbb{C}^2}[g](t_1, t_2) = \frac{1}{2\pi^2} \int_{\mathbb{C}^2} \frac{(\bar{t}_2 - \bar{\varsigma}_2)}{(|t_1 - \varsigma_1|^2 + |t_2 - \varsigma_2|^2)^2} g(\varsigma_1, \varsigma_2)\, d_{\mathbb{C}^2_{\varsigma_1,\varsigma_2}}.$$

Then $\bar{w}_2(\xi)$ is a complex-value harmonic function in E, *i.e.* $\Delta_{\mathbb{C}^2} \bar{w}_2 = 4(\partial^2_{z_1\bar{z}_1} + \partial^2_{z_2\bar{z}_2})\bar{w}_2 = 0$. Hence

$$\partial_{\bar{z}_1}(\partial_{z_1}\bar{w}_2) = -\partial_{\bar{z}_2}(\partial_{z_2}\bar{w}_2). \tag{4.3}$$

Again, by (3.1), we have

$${}^{\psi}D[\Psi] = 0 \quad \Longleftrightarrow \quad \begin{cases} \partial_{\bar{z}_1} w_1 + \partial_{z_2}\bar{w}_2 = 0, \\ \partial_{\bar{z}_2} w_1 - \partial_{z_1}\bar{w}_2 = 0 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \partial_{\bar{z}_1} w_1 = -\partial_{z_2}\bar{w}_2, \\ \partial_{\bar{z}_2} w_1 = \partial_{z_1}\bar{w}_2. \end{cases} \tag{4.4}$$

By (4.3), we know $-\partial_{z_2}\bar{w}_2$, $\partial_{z_1}\bar{w}_2$ satisfy the compatibility condition

$$\partial_{\bar{z}_2}(-\partial_{z_2}\bar{w}_2) = \partial_{\bar{z}_1}(\partial_{z_1}\bar{w}_2).$$

Thus by Theorem 7.2.1 of Chapter 7 in [11], the general solution $w_1(z_1, z_2)$ of system (4.4) possesses the form

$$w_1(z_1, z_2) = \Phi(z_1, z_2) + w_0(z_1, z_2),$$

where $\Phi(z_1, z_2)$ is an arbitrary analytic function in $E = E_1 \times E_2$ and

$$\begin{aligned} w_0(z_1, z_2) &= \widetilde{T}_{E_1}[-\partial_{z_2}\bar{w}_2] + \widetilde{T}_{E_2}[\Phi_0], \\ \widetilde{T}_{E_1}[-\partial_{z_2}\bar{w}_2] &= -\frac{1}{\pi} \int_{E_1} \frac{-\partial_{z_2}\bar{w}_2(\varsigma_1, z_2)}{\varsigma_1 - z_1}\, dE_{1\varsigma_1}, \\ \widetilde{T}_{E_2}[\Phi_0] &= -\frac{1}{\pi} \int_{E_2} \frac{\Phi_0(z_1, \varsigma_2)}{\varsigma_2 - z_2}\, dE_{2\varsigma_2}, \\ \Phi_0(z_1, z_2) &= \frac{1}{2\pi i} \int_{\partial E_1} \frac{\partial_{z_1}\bar{w}_2(\varsigma_1, z_2)}{\varsigma_1 - z_1}\, d\partial E_1\varsigma_1. \end{aligned}$$

□

Lemma 4.2 *Let G_m $(m = 1, 2, 3)$, H, w_0, E etc. be as stated above. Find a sectionally analytic function $\Phi(z_1, z_2)$ in E^{++}, E^{+-}, E^{-+}, E^{--}, such that $\Phi(z_1, z_2)$ is continuous in E^{++}, E^{+-}, E^{-+}, E^{--} and satisfies the boundary condition*

$$\begin{aligned} \Phi^{++}(t_1, t_2) = {} & G_1(t_1, t_2)\Phi^{+-}(t_1, t_2) + G_2(t_1, t_2)\Phi^{-+}(t_1, t_2) \\ & + G_3(t_1, t_2)\Phi^{--}(t_1, t_2) + (G_1 + G_2 + G_3 - 1)\bigl(w_0(t_1, t_2) \\ & + {}^{\psi}T^{(1)}_{\mathbb{C}^2}[g](t_1, t_2)\bigr) + H(t_1, t_2), \quad t = (t_1, t_2) \in \Gamma, \end{aligned} \tag{4.5}$$

where

$$ {}^{\psi}T^{(1)}_{\mathbb{C}^2}[g](t_1,t_2) = \frac{1}{2\pi^2}\int_{\mathbb{C}^2}\frac{(\bar{t}_1-\bar{\varsigma}_1)}{(|t_1-\varsigma_1|^2+|t_2-\varsigma_2|^2)^2}g(\varsigma_1,\varsigma_2)\,d_{\mathbb{C}^2_{\varsigma_1,\varsigma_2}}. $$

Then the solution has the form

$$ \Phi(z_1,z_2) = \begin{cases} F(z_1,z_2), & z=(z_1,z_2)\in E^{++}, \\ F(z_1,z_2)/G_1(z_1,z_2), & z=(z_1,z_2)\in E^{+-}, \\ F(z_1,z_2)/G_2(z_1,z_2), & z=(z_1,z_2)\in E^{-+}, \\ -F(z_1,z_2)/G_3(z_1,z_2), & z=(z_1,z_2)\in E^{--}, \end{cases} \tag{4.6} $$

where

$$ F(z_1,z_2) = \frac{1}{(2\pi i)^2}\int_{\partial E_1\times\partial E_2}\frac{\widetilde{H}(\varsigma_1,\varsigma_2)}{(\varsigma_1-z_1)(\varsigma_2-z_2)}\,d\partial E_{1\varsigma_1}\,d\partial E_{2\varsigma_2}, $$

and $\widetilde{H} = (G_1+G_2+G_3-1)(w_0+{}^{\psi}T^{(1)}_{\mathbb{C}^2}[g]) + H$.

Proof From Remark 3.1, we know ${}^{\psi}T^{(1)}_{\mathbb{C}^2}[g]\in C_\beta(\mathbb{C}^2,\mathbb{C})\cong C_\beta(\mathbb{R}^4,\mathbb{C})$ $(0<\beta<1)$. Thus by [9], we have $\widetilde{H} = (G_1+G_2+G_3-1)(w_0+{}^{\psi}T^{(1)}_{\mathbb{C}^2}[g]) + H \in C_\mu(\Gamma,\mathbb{C})$ $(0<\mu=\min\{\alpha,\beta\}<1)$. Hence by Theorem 7.1.2 of Chapter 7 in [11], it is not difficult to verify this lemma. □

Theorem 4.1 *Let E, ∂E etc. be as stated above. If $g\in L_p(\mathbb{C}^2,\mathbb{H})$ $(4<p<+\infty)$, then the solution of Problem* P *can be expressed as*

$$ f(\xi) = \Psi(\xi) + {}^{\psi}T_{\mathbb{C}^2}[g](\xi), $$

where ${}^{\psi}D[\Psi]=0$ *and*

$$ \begin{cases} \Psi(\xi) = w_1(\xi)+w_2(\xi)j = w_1(\xi)+j\bar{w}_2(\xi), \\ w_1(\xi) = \Phi(\xi)+w_0(\xi), \\ \bar{w}_2(\xi) = \int_{\partial E}[\bar{h}(t)-\overline{{}^{\psi}T^{(2)}_{\mathbb{C}^2}[g](t)}]\frac{\partial}{\partial\nu}G(\xi,t)\,d\partial E_t, \end{cases} $$

$$ \begin{aligned} {}^{\psi}T_{\mathbb{C}^2}[g](\xi) = {}^{\psi}T_{\mathbb{C}^2}[g](z_1,z_2) &= \frac{1}{2\pi^2}\int_{\mathbb{C}^2}\frac{(\bar{z}_1-\bar{\varsigma}_1)+(\bar{z}_2-\bar{\varsigma}_2)j}{(|z_1-\varsigma_1|^2+|z_2-\varsigma_2|^2)^2}g(\varsigma_1,\varsigma_2)\,d_{\mathbb{C}^2_{\varsigma_1,\varsigma_2}} \\ &= {}^{\psi}T^{(1)}_{\mathbb{C}^2}[g](z_1,z_2) + {}^{\psi}T^{(2)}_{\mathbb{C}^2}[g](z_1,z_2)j, \end{aligned} $$

herein w_0, ${}^{\psi}T^{(2)}_{\mathbb{C}^2}[g]$ are as stated in Lemma 4.1, *Φ, ${}^{\psi}T^{(1)}_{\mathbb{C}^2}[g]$ are as stated in Lemma* 4.2.

Proof By Theorem 3.1, we know ${}^{\psi}D[{}^{\psi}T_{\mathbb{C}^2}[g]](\xi)=g(\xi)$, thus ${}^{\psi}D[\Psi(\xi)+{}^{\psi}T_{\mathbb{C}^2}[g](\xi)]=g(\xi)$. Hence, by (3.2), we know the general solution of system (1.2) has the form

$$ f(\xi) = \Psi(\xi) + {}^{\psi}T_{\mathbb{C}^2}[g](\xi), \tag{4.7} $$

where ${}^{\psi}D[\Psi]=0$, $\xi=z_1+z_2j$, $f(\xi)=f(z_1,z_2)=u_1(z_1,z_2)+u_2(z_1,z_2)j=u_1(z_1,z_2)+j\bar{u}_2(z_1,z_2)$, $\Psi(\xi)=\Psi(z_1,z_2)=w_1(z_1,z_2)+w_2(z_1,z_2)j=w_1(z_1,z_2)+j\bar{w}_2(z_1,z_2)$, and

$$ \begin{aligned} {}^{\psi}T_{\mathbb{C}^2}[g](\xi) = {}^{\psi}T_{\mathbb{C}^2}[g](z_1,z_2) &= \frac{1}{2\pi^2}\int_{\mathbb{C}^2}\frac{(\bar{z}_1-\bar{\varsigma}_1)+(\bar{z}_2-\bar{\varsigma}_2)j}{(|z_1-\varsigma_1|^2+|z_2-\varsigma_2|^2)^2}g(\varsigma_1,\varsigma_2)\,d_{\mathbb{C}^2_{\varsigma_1,\varsigma_2}} \\ &= {}^{\psi}T^{(1)}_{\mathbb{C}^2}[g](z_1,z_2) + {}^{\psi}T^{(2)}_{\mathbb{C}^2}[g](z_1,z_2)j = {}^{\psi}T^{(1)}_{\mathbb{C}^2}[g](z_1,z_2) + j\overline{{}^{\psi}T^{(2)}_{\mathbb{C}^2}[g]}(z_1,z_2). \end{aligned} $$

Thus

$$\bar{u}_2(z_1,z_2) = \bar{w}_2(z_1,z_2) + \overline{{}^{\psi}T^{(2)}_{\mathbb{C}^2}[g]}(z_1,z_2).$$

So the boundary condition (4.2) in Problem P can be written as

$$\bar{w}_2 = \bar{h}(t_1,t_2) - \overline{{}^{\psi}T^{(2)}_{\mathbb{C}^2}[g]}(t_1,t_2), \quad t = (t_1,t_2) \in \partial E. \tag{4.8}$$

Therefore, by Lemma 4.1, the solution to the equation ${}^{\psi}D[\Psi] = 0$ with boundary condition (4.8) can be expressed as

$$\Psi(\xi) = w_1(\xi) + w_2(\xi)j = w_1(\xi) + j\bar{w}_2(\xi),$$

where w_1, $\bar{w}_2$ are as stated in Lemma 4.1. Again, by (4.7), we have

$$u_1(z_1,z_2) = w_1(z_1,z_2) + {}^{\psi}T^{(1)}_{\mathbb{C}^2}[g](z_1,z_2).$$

From Lemma 4.1, we have

$$w_1(z_1,z_2) = \Phi(z_1,z_2) + w_0(z_1,z_2),$$

where $\Phi(z_1,z_2)$ is an arbitrary analytic function in $E = E_1 \times E_2$, w_0 is as stated in Lemma 4.1. In addition, by Chapter 7 in [11], we know $w_0 \in C_\alpha(\mathbb{C}^2,\mathbb{C})$ $(0 < \alpha < 1)$, by Remark 3.1, we know ${}^{\psi}T^{(1)}_{\mathbb{C}^2}[g] \in C_\beta(\mathbb{C}^2,\mathbb{C})$ $(0 < \beta < 1)$. So the boundary condition (4.1) in Problem P can be written as

$$\begin{aligned}\Phi^{++}(t_1,t_2) = {} & G_1(t_1,t_2)\Phi^{+-}(t_1,t_2) + G_2(t_1,t_2)\Phi^{-+}(t_1,t_2) \\ & + G_3(t_1,t_2)\Phi^{--}(t_1,t_2) + (G_1 + G_2 + G_3 - 1)\big(w_0(t_1,t_2) \\ & + {}^{\psi}T^{(1)}_{\mathbb{C}^2}[g](t_1,t_2)\big) + H(t_1,t_2), \quad t = (t_1,t_2) \in \Gamma.\end{aligned}$$

Therefore, by Lemma 4.2, we know $\Phi(z_1,z_2)$ can be expressed as (4.6) in Lemma 4.2. In conclusion, we complete the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LPW has presented the main purpose of the article. All authors read and approved the final manuscript.

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Acknowledgements
This work was supported by the National Science Foundation of China (No. 11401162, No. 11171349, No. 11301136), the Natural Science Foundation of Hebei Province (No. A2015205012, No. A2014205069, No. A2014208158) and Hebei Normal University Dr. Fund (No. L2015B03, No. L2015B04).

Received: 3 September 2014 Accepted: 22 December 2014 Published online: 30 January 2015

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