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Global attractors for nonlinear wave equations with linear dissipative terms

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Abstract
An initial boundary value problem of the semilinear wave equation of which the source term $f(x,u)$ is without variational structure in a bounded domain is considered. Firstly, we prove that it has a unique globally weak solution $(u,u_t) \in C^0([0,\infty), H_0^1(\Omega) \times L^2(\Omega))$ by using our previous results (Pan *et al.* in Bound. Value Probl. 2012:42, 2012). Secondly, we obtain the existence of global attractors in $H_0^1(\Omega) \times L^2(\Omega)$ by using the ω-limit compactness condition (Ma *et al.* in Indiana Univ. Math. J. 5(6):1542-1558, 2002), rather than the traditional method.
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1 Introduction

In this paper we are concerned with the existence of global attractors for nonlinear wave equations with linear dissipative terms in a bounded domain Ω in R^n:

$$\begin{cases} u_{tt} + 2ku_t = \triangle u - |u|^{p-1}u + f(x,u) & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, $\triangle = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, $x = (x_1,\ldots,x_n)$; the sourcing terms are $-|u|^{p-1}u + f(x,u)$, $1 < p < \frac{n}{n-2}$, $n \geq 3$; $1 < p < \infty$, $n = 1,2$; and $f(x,u)$ satisfies

$$\bigl|f(x,z)\bigr| \leq C|z|^q + g(x), \quad q \leq \frac{p+1}{2}, g \in L^2(\Omega). \tag{1.2}$$

The attractor is an important concept describing the asymptotic properties of dynamical systems. A great deal of work has been devoted to the existence of global attractors of dynamical systems (see, *e.g.* [1–9] and references therein). The existence of a global attractor (1.1) with a source term only containing f was proved by Hale [7] for f satisfying for $n \geq 3$ the growth condition $f(u) \leq C_0(|u|^\gamma + 1)$, with $1 \leq \gamma < \frac{n}{n-2}$. For the case $n = 2$, Hale and Raugel [10] proved the existence of the attractor under an exponential growth condition of the type $|f(u)| \leq \exp\theta(u)$ (such a condition previously appeared in the work of Gallouët [11]). The existence of the attractor in the critical case $\gamma = \frac{n}{n-2}$ was first proved by Babin and Vishik [1], and then more generally by Arrieta *et al.* [12]. For other treatments see Chepyzhov and Vishik [3], Ladyzhenskaya [13], Raugel [14] and Temam [9]. When Ω

is bounded and u is subjected to suitable boundary conditions, the general result is that the dynamical system associated with the problem possesses a global attractor in the natural energy space $H_0^1(\Omega) \times L^2(\Omega)$ if nonlinear term f has a subcritical or critical exponent, because there exist typical parabolic-like flows with an inherent smoothing mechanism. By the traditional method (see [15] for examples), in order to obtain the existence of global attractors for semilinear wave equations, one needs to verify the uniform compactness of the semigroup by getting the boundedness in a more regular function space. However, in some cases it is difficult to obtain the uniform compactness of the semigroup. Fortunately, a new method for obtaining the global attractors has been developed in [16]. With this method, one only needs to verify a necessary compactness condition (ω-limit compactness) with the same type of energy estimates as those for establishing the absorbing sets. In this paper, we use this method to obtain the existence of global attractors for problem (1.1) with the general condition where the source term $f(x,u)$ is without variational structure.

This paper is organized as follows:

- in Section 2 we recall some preliminary tools, definitions and our previous results;
- in Section 3 we obtain the existence and uniqueness of weak solution by using our previous results [17] and the various conditions can also be found [18];
- in Section 4 we obtain our main results for problem (1.1) by using the new method (ω-compactness condition).

2 Preliminaries

Consider the abstract nonlinear evolution equation defined on X, given by

$$\begin{cases} \frac{d^2u}{dt^2} + k\frac{du}{dt} = G(u), \quad k > 0, \\ u(x,0) = \varphi(x), \\ u_t(x,0) = \psi(x), \end{cases} \tag{2.1}$$

where $G : X_2 \times \mathbf{R}^+ \to X_1{}^*$ is a mapping, $X_2 \subset X_1$, X_1, X_2 are Banach spaces and X_1^* is the dual space of X_1, $\mathbf{R}^+ = [0,\infty)$, $u = u(x,t)$ is an unknown function.

First we introduce a sequence of function spaces:

$$\begin{cases} X \subset H_2 \subset X_2 \subset X_1 \subset H, \\ X_2 \subset H_1 \subset H, \end{cases} \tag{2.2}$$

where H, H_1, H_2 are Hilbert spaces, X is a linear space, X_1, X_2 are Banach spaces and all inclusions are dense embeddings.

Suppose that

$$\begin{cases} L : X \to X_1 \text{ is a one to one dense linear operator}, \\ \langle Lu, v\rangle_H = \langle u, v\rangle_{H_1}, \quad \forall u, v \in X. \end{cases} \tag{2.3}$$

In addition, the operator L has an eigenvalue sequence

$$Le_k = \lambda_k e_k \quad (k = 1, 2, \ldots) \tag{2.4}$$

such that $\{e_k\} \subset X$ is the common orthogonal basis of H and H_2.

Definition 2.1 [17] Set $(\varphi,\psi)\in X_2\times H_1$, $u\in W^{1,\infty}_{\mathrm{loc}}((0,\infty),H_1)\cap L^{\infty}_{\mathrm{loc}}((0,\infty),X_2)$ is called a globally weak solution of (2.1), if $\forall v\in X_1$, we have

$$\langle u_t,v\rangle_H+k\langle u,v\rangle_H=\int_0^t\langle Gu,v\rangle\,dt+k\langle\varphi,v\rangle_H+\langle\psi,v\rangle_H. \tag{2.5}$$

Definition 2.2 [17] Let Y_1, Y_2 be Banach spaces, the solution $u(t,\varphi,\psi)$ of (2.1) is called uniformly bounded in $Y_1\times Y_2$, if for any bounded domain $\Omega_1\times\Omega_2\subset Y_1\times Y_2$, there exists a constant C which only depends on the domain $\Omega_1\times\Omega_2$, such that

$$\|u\|_{Y_1}+\|u_t\|_{Y_2}\le C,\quad \forall(\varphi,\psi)\in\Omega_1\times\Omega_2 \text{ and } t\ge 0.$$

Suppose that $G=A+B:X_2\times\mathbf{R}^+\to {X_1}^*$. Throughout this paper, we assume that:

(i) There exists a functional $F\in C^1:X_2\to\mathbf{R}^1$ such that

$$\langle Au,Lv\rangle=\bigl\langle -DF(u),v\bigr\rangle,\quad \forall u,v\in X. \tag{2.6}$$

(ii) The functional F is coercive, *i.e.*

$$F(u)\to\infty\quad\Leftrightarrow\quad\|u\|_{X_2}\to\infty. \tag{2.7}$$

(iii) There exist constants $C_1>0$ and $C_2>0$ such that

$$\bigl|\langle Bu,Lv\rangle\bigr|\le C_1F(u)+C_2\|v\|^2_{H_1},\quad\forall u,v\in X. \tag{2.8}$$

Lemma 2.1 [17] *Set $G:X_2\times\mathbf{R}^+\to {X_1}^*$ to be weakly continuous*, $(\varphi,\psi)\in X_2\times H_1$, *then we obtain the following results*:

(1) *If $G=A$ satisfies the assumptions* (i) *and* (ii), *then there exists a globally weak solution of* (2.1),

$$u\in W^{1,\infty}_{\mathrm{loc}}\bigl((0,\infty),H_1\bigr)\cap L^{\infty}_{\mathrm{loc}}\bigl((0,\infty),X_2\bigr),$$

and u is uniformly bounded in $X_2\times H_1$.

(2) *If $G=A+B$ satisfies the assumptions* (i), (ii) *and* (iii), *then there exists a globally weak solution of* (2.1),

$$u\in W^{1,\infty}_{\mathrm{loc}}\bigl((0,\infty),H_1\bigr)\cap L^{\infty}_{\mathrm{loc}}\bigl((0,\infty),X_2\bigr).$$

(3) *Furthermore, if $G=A+B$ satisfies*

$$\bigl|\langle Gu,v\rangle\bigr|\le\frac{1}{2}\|v\|^2_H+CF(u)+g(t) \tag{2.9}$$

for some $g\in L^1_{\mathrm{loc}}(0,\infty)$, then $u\in W^{2,2}_{\mathrm{loc}}((0,\infty),H)$.

A family of operators $S(t):X\to X$ $(t\ge 0)$ is called a semigroup generated by (2.1) if it satisfies the following properties:

(1) $S(t): X \to X$ is a continuous map for any $t \geq 0$,
(2) $S(0) = \mathrm{id}: X \to X$ is the identity,
(3) $S(t+s) = S(t) \cdot S(s)$, $\forall t, s \geq 0$. Then the solution of (2.1) can be expressed as

$$u(t, u_0) = S(t)u_0.$$

Introducing the expression of the abstract semilinear wave equation:

$$\begin{cases} \frac{d^2u}{dt^2} + 2k\frac{du}{dt} = Lu + T(u), \quad k \geq 0, \\ u(x,0) = \varphi(x), \\ u_t(x,0) = \psi(x), \end{cases} \tag{2.10}$$

where X_1, X are Banach spaces, $X_1 \subset X$ is a dense inclusion, $L: X_1 \to X$ is a sectorial linear operator, and $T: X_1 \to X$ is a nonlinear bounded operator.

Lemma 2.2 [19] *Set $L: X_1 \to X$, a sectorial linear operator and $T: X_1 \to X$, a nonlinear bounded operator, $\mathcal{L} = L + k^2 I$, then the solution of* (2.9) *can be expressed as follows*:

$$\begin{aligned} u &= e^{-kt}\biggl[\cos t(-\mathcal{L})^{\frac{1}{2}}\varphi + k(-\mathcal{L})^{-\frac{1}{2}}\sin(-\mathcal{L})^{\frac{1}{2}}\varphi + (-\mathcal{L})^{-\frac{1}{2}}\sin t(-\mathcal{L})^{\frac{1}{2}}\psi \\ &\quad + \int_0^t e^{-k(t-\tau)}(-\mathcal{L})^{-\frac{1}{2}}\sin(t-\tau)(-\mathcal{L})^{\frac{1}{2}}T(u)\,d\tau\biggr], \\ u_t &= -ku + e^{-kt}\biggl[-(-\mathcal{L})^{\frac{1}{2}}\sin t(-\mathcal{L})^{\frac{1}{2}}\varphi + k\cos t(-\mathcal{L})^{\frac{1}{2}}\varphi + \cos t(-\mathcal{L})^{\frac{1}{2}}\psi \\ &\quad + \int_0^t e^{-k(t-\tau)}\cos(t-\tau)(-\mathcal{L})^{\frac{1}{2}}T(u)\,d\tau\biggr]. \end{aligned}$$

Next, we introduce the concepts and definitions of invariant sets, global attractors, and ω-limit compactness sets for the semigroup $S(t)$.

Definition 2.3 Let $S(t)$ be a semigroup defined on X. A set $\Sigma \subset X$ is called an invariant set of $S(t)$ if $S(t)\Sigma = \Sigma$, $\forall t \geq 0$. An invariant set Σ is an attractor of $S(t)$ if Σ is compact, and there exists a neighborhood $U \subset X$ of Σ such that, for any $u_0 \in U$,

$$\inf_{v \in \Sigma} \bigl\| S(t)u_0 - v \bigr\|_X \to 0, \quad \text{as } t \to 0.$$

In this case, we say that Σ attracts U. Especially, if Σ attracts any bounded set of X, Σ is called a global attractor of $S(t)$ in X.

Definition 2.4 Let X be an infinite dimensional Banach space and A be a bounded subset of X. The measure of noncompactness $\gamma(A)$ of A is defined by

$$\gamma(A) = \inf\{\delta > 0 \mid \text{for } A \text{ there exists a finite cover by sets whose diameter} \leq \delta\}.$$

Lemma 2.3 [11] *If $A_n \subset X$ is a sequence bounded and closed sets, $A_n \neq \emptyset$, $A_{n+1} \subset A_n$, and $\gamma(A_n) \to 0$ $(n \to \infty)$, then the set $A = \bigcap_{n=1}^{\infty} A_n$ is a nonempty compact set.*

Definition 2.5 [16] A semigroup $S(t): X \to X$ $(t \geq 0)$ in X is called ω-limit compact, if for any bounded set $B \subset X$ and $\forall \varepsilon > 0$, there exists t_0 such that

$$\gamma\left(\bigcup_{t \geq t_0} S(t)B\right) \leq \varepsilon,$$

where γ is a noncompact measure in X.

For a set $D \subset X$, we define the ω-limit set of D as follows:

$$\omega(D) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)D},$$

where the closure is taken in the X-norm.

Lemma 2.4 [19] *Let $S(t)$ be a semigroup in X, then $S(t)$ has a global attractor $\mathcal{A}$ in X if and only if*

(1) *$S(t)$ has ω-limit compactness, and*
(2) *there is a bounded absorbing set $B \subset X$.*

In addition, the ω-limit set of B is the attractor $\mathcal{A} = \omega(B)$.

Remark 2.1 Although the lemma has been proved partly in [19], we still give a proof here. Our proof is different from that in [20] but is similar to that in [16]. We adopt and present the proof also because we will use the same method to obtain the existence of the global attractor.

Proof Step 1. To prove the sufficiency of Lemma 2.4.

(a) $S(t)$ has ω-limit compactness, *i.e.*, for any bounded set $B \subset X$ and $\forall \varepsilon > 0$, there exists a t_0, such that

$$\gamma\left(\bigcup_{t \geq t_0} S(t)B\right) \leq \varepsilon.$$

So, we know that $\omega(B) = \bigcap_{t_0=0}^{\infty} \overline{\bigcup_{t \geq t_0} S(t)B}$ is a compact set from Lemma 2.3.

(b) $\omega(B)$ is nonempty.

For $B \neq \emptyset$, so $\overline{\bigcup_{t \geq s} S(t)B} \neq \emptyset$, $\forall s \geq 0$, and

$$\overline{\bigcup_{t \geq s_1} S(t)B} \subset \overline{\bigcup_{t \geq s_2} S(t)B}, \quad \forall s_1 \geq s_2,$$

we can obtain

$$\omega(B) = \bigcap_{s \geq 0}^{\infty} \overline{\bigcup_{t \geq s} S(t)B} \neq \emptyset.$$

(c) $\omega(B)$ is invariant.

For $x \in \omega(B) \Leftrightarrow$ there exist $\{x_n\} \in B$ and $t_n \to \infty$, such that $S(t_n)x_n \to x$.

If $y \in S(t)\omega(B)$, then for some $x \in \omega(B)$, $y = S(t)x$.

Hence, there exist $\{x_n\} \subset B$, $t_n \to \infty$, such that

$$S(t)S(t_n)x_n = S(t+t_n)x_n \to S(t)x = y.$$

In conclusion, $y \in \omega(B)$, $S(t)\omega(B) \in \omega(B)$, $\forall t \geq 0$.

If $x \in \omega(B)$, fix $\{x_n\} \subset B$ and t_n, such that

$$S(t)x_n \to x, \quad \text{as } t_n \to \infty, n \to \infty.$$

$S(t)$ is ω-limit compact, *i.e.*, there exists a $y \in H$, such that

$$S(t) \bigcap_{t_n \geq 0} \overline{\bigcup_{t \geq t_n} S(t_n)x_n} \to y, \quad n \to \infty.$$

Therefore $y \in \omega(B)$.

For

$$\bigcap_{t_n \geq 0} \overline{\bigcup_{t \geq t_n} S(t_n)x_n} = \bigcap_{t_n \geq 0} \overline{\bigcup_{t \geq t_n} S(t)S(t_n - t)x_n} \to \bigcap_{t_n \geq 0} \overline{\bigcup_{t \geq t_n} S(t)y}$$

and

$$S(t_n)x_n \to x \in \omega(B),$$

which implies that

$$S(t)y \to x, \qquad \omega(B) \subset S(t)\omega(B).$$

In conclusion, combining (a)-(c) and condition (2), Step 1 has been proved.

Step 2. To prove the necessity of Lemma 2.4.

If $\mathcal{A}$ is a global attractor, then the ε-neighborhood $U_\varepsilon(\mathcal{A}) \subset X$ is an absorbing set. So we need only to prove $S(t)$ has ω-limit compactness.

Since $U_\varepsilon(\mathcal{A})$ is an absorbing set, for any bounded set $B \subset X$ and $\varepsilon > 0$, there exists a time $t_\varepsilon(B) > 0$ such that

$$\bigcup_{t \geq t_\varepsilon(B)} S(t)B \subset U_{\frac{\varepsilon}{4}}(\mathcal{A}) = \left\{ x \in X \,\middle|\, \operatorname{dist}(x, \mathcal{A}) < \frac{\varepsilon}{4} \right\}.$$

On the other hand, $\mathcal{A}$ is a compact set, and there exist finite elements $x_1, x_2, \ldots, x_n \in X$ such that

$$\mathcal{A} \subset \bigcup_{k=1}^{n} U\left(x_k, \frac{\varepsilon}{4}\right).$$

Then

$$U_{\frac{\varepsilon}{2}}(\mathcal{A}) \subset \bigcup_{k=1}^{n} U\left(x_k, \frac{\varepsilon}{2}\right),$$

which implies that

$$\gamma\left(\bigcup_{t\geq t_\varepsilon(B)} S(t)B\right) \leq \gamma\left(U_{\frac{\varepsilon}{4}}(\mathcal{A})\right) \leq \varepsilon.$$

Hence, Lemma 2.4 has been proved. □

3 Existence and uniqueness of globally weak solution

Now, in this section, we begin to prove that problem (1.1) has a unique globally weak solution $(u, u_t) \in C^0([0,\infty), H_0^1 \times L^2(\Omega))$.

Theorem 3.1 (Existence) *If* $\forall(\varphi,\psi) \in H_0^1(\Omega) \times L^2(\Omega)$, f *satisfies condition* (1.2) *and* $1 < p < \frac{n}{n-2}$, $n \geq 3$; $1 < p < \infty$, $n = 1, 2$, *then* (1.1) *has a globally weak solution*

$$u \in W_{\mathrm{loc}}^{1,\infty}\left((0,\infty), L^2(\Omega)\right) \cap L_{\mathrm{loc}}^{\infty}\left((0,\infty), H_0^1(\Omega)\right).$$

Remark 3.1 Divide the operator $G(u)$ in Lemma 2.1 into two parts: A and B, where A has a variational structure and B has a non-variational structure. Then we obtain the globally weak solution by applying our result (2) in Lemma 2.1.

Proof Fix spaces as follows:

$$X_2 = X_1 = H_0^1(\Omega) \cap L^{p+1}(\Omega), \tag{3.1}$$

$$X = C_0^\infty(\Omega), \qquad H_1 = H = L^2(\Omega). \tag{3.2}$$

In problem (1.1), set $G(u) = \triangle u - |u|^{p-1}u + f(x,u)$.

Define the map $G(u) = A + B : X_1 \to X_1^*$ as

$$\langle Au, v\rangle = -\int_\Omega \left[\nabla u \cdot \nabla v + |u|^{p-1}u \cdot v\right] dx, \tag{3.3}$$

$$\langle Bu, v\rangle = \int_\Omega f(x,u)v\, dx. \tag{3.4}$$

Note the functional $I : X_1 \to R^1$,

$$I[u] = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{p+1}|u|^{p+1}\right] dx. \tag{3.5}$$

Obviously, we obtain

$$\langle Au, v\rangle = -\langle DI[u], v\rangle, \quad \forall u, v \in X \tag{3.6}$$

and

$$I[u] \to \infty \quad \Leftrightarrow \quad \|u\|_{X_1} \to \infty, \tag{3.7}$$

which implies that conditions (1) and (2) in Lemma 2.1 hold.

From the growth restriction condition (1.2), we get

$$\begin{aligned}
\left|\langle Bu, v\rangle\right| &= \left|\int_\Omega f(x,u)v\,dx\right| \\
&\leq \int_\Omega \left|f(x,u)\right||v|\,dx \\
&\leq \frac{1}{2}\int_\Omega |v|^2\,dx + \frac{1}{2}\int_\Omega \left|f(x,u)\right|^2\,dx \\
&\leq \frac{1}{2}\int_\Omega |v|^2\,dx + C\int_\Omega \left[|u|^{2q} + g^2(x)\right]dx \\
&\leq \frac{1}{2}\|v\|_{H_1}^2 + C_1\int_\Omega |u|^{p+1}\,dx + C_2 \\
&\leq \frac{1}{2}\|v\|_{H_1}^2 + C_1 I[u] + C_2,
\end{aligned}$$

where $C, C_1, C_2 > 0$. It implies that condition (3) in Lemma 2.1 holds.

In conclusion, we see that problem (1.1) has a globally weak solution

$$u \in W^{1,\infty}_{\text{loc}}\big((0,\infty), L^2(\Omega)\big) \cap L^\infty_{\text{loc}}\big((0,\infty), H^1_0(\Omega)\big)$$

from the second result in Lemma 2.1. □

Next, we prove the uniqueness of the globally weak solution to problem (1.1).

Theorem 3.2 *If* $u \in W^{1,\infty}_{\text{loc}}((0,\infty), L^2(\Omega)) \cap L^\infty_{\text{loc}}((0,\infty), H^1_0(\Omega))$ *is a weak solution of problem* (1.1), *then the solution* u *is unique.*

Remark 3.2 From the formula of the wave equation in Lemma 2.2 and using the Gronwall inequality, we obtain the uniqueness of the globally weak solution.

Proof Set $u_1, u_2 \in W^{1,\infty}_{\text{loc}}((0,\infty), L^2(\Omega)) \cap L^\infty_{\text{loc}}((0,\infty), H^1_0(\Omega))$ as the solutions of problem (1.1), then from Lemma 2.2, we get $u_i \in C^0([0,\infty), H^1_0(\Omega))$, $i = 1, 2$, and

$$\begin{aligned}
\|u_1 - u_2\|_{H^1_0} &= \left\|\left(-\triangle^{\frac{1}{2}}\right)(u_1 - u_2)\right\|_{L^2} \\
&\leq C\int_0^t \left\|\left[|u_2|^{p-1}u_2 - |u_1|^{p-1}u_1\right] + \left[f(x,u_1) - f(x,u_2)\right]\right\|_{L^2}\,d\tau \\
&\leq C_1\int_0^t \left[\left(\left\||\tilde{u}|^{p-1}\right\| + \left\|Df(x,\tilde{u})\right\|\right)\cdot\|u_1 - u_2\|_{H^1_0}\right]d\tau;
\end{aligned}$$

by using the Gronwall inequality, we easily obtain

$$\|u_1 - u_2\|_{H^1_0} \leq 0,$$

where $\tilde{u}$ is the mean value between u_1 and u_2.

It implies that

$$\|u_1 - u_2\|_{H^1_0} \leq 0 \quad \Rightarrow \quad u_1 = u_2.$$

□

4 Existence of global attractor

In this section, we proved the existence of global attractor to problem (1.1).

Theorem 4.1 *For any* $(\varphi, \psi) \in (H_0^1(\Omega) \times L^2(\Omega))$, *the sourcing term* f *satisfies the growth restriction* (1.2) *and the exponent of* p *satisfies* $1 < p < \frac{n}{n-2}$, $n \geq 3$ *or* $1 < p < \infty$, $n = 1, 2$; *then problem* (1.1) *has a global attractor* $\mathcal{A}$ *in* $(H_0^1(\Omega) \times L^2(\Omega))$.

Remark 4.1 Comparing Remark 3.1, we divide the operator $G(u)$ of (2.1) into two parts: L and T, where L is a linear operator, while T is a nonlinear operator. We obtain the global attractor of problem (1.1) by using Lemma 2.4.

Proof According to Lemma 2.4, we prove Theorem 4.1 in the following three steps.

Step 1. Problem (1.1) has a globally unique weak solution.

Step 2. $S(t)$ has a bounded absorbing set in $H_0^1(\Omega) \times L^2(\Omega)$.

From Theorems 3.1 and 3.2, we see that problem (1.1) has a globally unique weak solution $(u, u_t) \in C^0([0, \infty), H_0^1 \times L^2)$. Equation (1.1) generates a semigroup:

$$S(t) : H_{\frac{1}{2}} \times H \to H_{\frac{1}{2}} \times H.$$

Fix the spaces as follows:

$$\begin{aligned} &H = L^2(\Omega), \qquad H_1 = H^2(\Omega) \cap H_0^1(\Omega), \\ &L : H_1 \to H, \qquad T : H_1 \to H. \end{aligned}$$

Note that

$$Lu = \triangle u, \tag{4.1}$$

$$Tu = -|u|^{p-1}u + f(x, u), \tag{4.2}$$

and L generates the fractional space, $H_{\frac{1}{2}} = H_0^1(\Omega)$.

Obviously, there exists a C^1 functional $F : H_{\frac{1}{2}} \to R^1$ such that

$$F(u) = \frac{1}{p+1}|u|^{p+1} - \int_0^t f(x, u)\, d\tau, \tag{4.3}$$

and we easily get

$$T(u) = -DF(u), \quad \forall u \in H_1. \tag{4.4}$$

Since

$$\left|f(x, z)\right| \leq C|z|^q + g(x), \quad q \leq \frac{p+1}{2}, g \in L^2(\Omega),$$

then we get

$$F(u) \geq -C_1 \tag{4.5}$$

and

$$\langle DF(u), u\rangle_H - k\langle u, v\rangle_H \geq -\frac{1}{2}\|v\|_H^2 - C_2, \quad C_2 > 0. \tag{4.6}$$

Equation (1.1) is equivalent to the equations that follow:

$$\begin{cases} \frac{\partial u}{\partial t} = -ku + v, \quad k \geq 0, \\ \frac{\partial v}{\partial t} = Lu + k^2 u - kv - |u|^{p-1}u + f(x,u). \end{cases} \tag{4.7}$$

Multiply (4.7) by $(-Lu, v)$ and take the inner product in H:

$$\left\langle \frac{\partial u}{\partial t}, -Lu \right\rangle_H = -k\langle u, -Lu\rangle_H + \langle -Lu, v\rangle_H, \tag{4.8}$$

$$\left\langle \frac{\partial v}{\partial t}, v \right\rangle_H = \langle Lu, v\rangle_H + \langle k^2 u, v\rangle_H - k\langle v, v\rangle_H + \langle T(u), v\rangle_H. \tag{4.9}$$

Summing (4.8) and (4.9), it follows that

$$\begin{aligned} &\left\langle \frac{\partial u}{\partial t}, -Lu \right\rangle_H + \left\langle \frac{\partial v}{\partial t}, v \right\rangle_H \\ &\quad = -k\langle u, -Lu\rangle_H - k\langle v, v\rangle_H + k^2\langle u, v\rangle_H + \langle Tu, v\rangle_H. \end{aligned} \tag{4.10}$$

Furthermore,

$$\langle -Lu, \omega\rangle_H = \langle (-L^{\frac{1}{2}})u, (-L^{\frac{1}{2}})\omega\rangle_H, \quad \forall u, \omega \in H_{\frac{1}{2}}. \tag{4.11}$$

From (4.4) and (4.7), we get

$$\begin{aligned} \langle Tu, v\rangle_H &= \left\langle Tu, \frac{\partial u}{\partial t} + ku \right\rangle_H \\ &= \left\langle -DF(u), \frac{\partial u}{\partial t} + ku \right\rangle_H \\ &= -\left\langle DF(u), \frac{\partial u}{\partial t} \right\rangle_H - k\langle DF(u), u\rangle_H \\ &= -\frac{dF(u)}{dt} - k\langle DF(u), u\rangle_H. \end{aligned}$$

Integrating (4.10) over $[0, t]$ with respect to time t and combining the two formulas, we get

$$\begin{aligned} &\frac{1}{2}\|u\|_{H^2_{\frac{1}{2}}} + \frac{1}{2}\|v\|_H^2 - \frac{1}{2}\|\varphi\|_{H^2_{\frac{1}{2}}} - \frac{1}{2}\|\psi\|_{H^2} \\ &\quad = \int_0^t \left[\left\langle \frac{\partial u}{\partial t}, -Lu \right\rangle_H + \left\langle \frac{\partial v}{\partial t}, v \right\rangle_H \right] d\tau \\ &\quad = -k\int_0^t \left[\langle u, -Lu\rangle_H + \langle v, v\rangle_H - k\langle u, v\rangle_H \right] d\tau + \int_0^t \langle Tu, v\rangle_H \, d\tau \end{aligned}$$

$$\begin{aligned}
&= -k\int_0^t \bigl[\bigl\langle(-L^{\frac{1}{2}})u, (-L^{\frac{1}{2}})u\bigr\rangle_H + \|v\|_H^2 - k\langle u, v\rangle_H\bigr]\,d\tau \\
&\quad + \int_0^t \biggl[-\frac{dF(u)}{dt} - k\bigl\langle DF(u), u\bigr\rangle_H\biggr]\,d\tau \\
&= -k\int_0^t \bigl[\|u\|_{H^2_{\frac{1}{2}}} + \|v\|_H^2 - k\langle u, v\rangle_H\bigr]\,d\tau - F\bigl(u(t)\bigr) + F\bigl(u(0)\bigr) \\
&\quad - k\int_0^t \bigl\langle DF(u), u\bigr\rangle_H\,d\tau \\
&= -k\int_0^t \bigl[\|u\|_{H^2_{\frac{1}{2}}} + \bigl\langle DF(u), u\bigr\rangle_H - k\langle u, v\rangle_H\bigr]\,d\tau - F(u) + F(\varphi);
\end{aligned}$$

combining with (4.6), it follows that

$$\|u\|_{H^2_{\frac{1}{2}}} + \|v\|_H^2 \le -k\int_0^t \bigl[\|u\|_{H^2_{\frac{1}{2}}} + \|v\|_H^2\bigr]\,d\tau + f(\varphi, \psi) + Ct, \quad C > 0.$$

Applying the Gronwall inequality, we get

$$\|u\|_{H^2_{\frac{1}{2}}} + \|v\|_H^2 \le f(\varphi, \psi)e^{-kt} + C_1\bigl(1 - e^{-t}\bigr). \tag{4.12}$$

It implies that $S(t)$ has a bounded absorbing set in $H_{\frac{1}{2}} \times H$.

Step 3. $S(t)$ has ω-limit compactness.

From the formula in Lemma 2.2, the solution of problem (1.1) can be expressed as follows:

$$u = e^{-kt}\bigl[\cos t(-\triangle)^{\frac{1}{2}}\varphi + k(-\triangle)^{-\frac{1}{2}}\sin t(-\triangle)^{\frac{1}{2}}\varphi + (-\triangle)^{-\frac{1}{2}}\sin t(-\triangle)^{\frac{1}{2}}\psi\bigr]$$

$$\quad + \int_0^t \bigl[e^{-k(t-\tau)}(-\triangle)^{-\frac{1}{2}}\sin(t-\tau)(-\triangle)^{\frac{1}{2}}\bigl(-|u|^{p-1}u + f\bigr)\bigr]\,d\tau, \tag{4.13}$$

$$u_t = -ku + e^{-kt}\bigl[-(-\triangle)^{\frac{1}{2}}\sin t(-\triangle)^{\frac{1}{2}}\varphi + k\cos t(-\triangle)^{\frac{1}{2}}\varphi + \cos t(-\triangle)^{\frac{1}{2}}\psi\bigr]$$

$$\quad + \int_0^t \bigl[e^{-k(t-\tau)}\cos(t-\tau)(-\triangle)^{\frac{1}{2}}\bigl(-|u|^{p-1}u + f\bigr)\bigr]\,d\tau. \tag{4.14}$$

Since the linear operator

$$L = \triangle : H^2(\Omega) \times H_0^1(\Omega) \to L^2(\Omega)$$

is a symmetrical sector operator, it has the eigenvalue sequence:

$$0 > \lambda_1 \ge \lambda_2 \ge \cdots, \quad \lambda_k \to -\infty, k \to \infty.$$

Then

$$\sin t(-\triangle)^{\frac{1}{2}}v = \sum_{j=1}^{\infty} v_j \sin\sqrt{-\lambda_j}te_j, \tag{4.15}$$

$$\cos t(-\triangle)^{\frac{1}{2}}v = \sum_{j=1}^{\infty} v_j \cos\sqrt{-\lambda_j}te_j. \tag{4.16}$$

For any $v = \sum_{j=1}^{\infty} v_j e_j \in L^2(\Omega)$ and $-\lambda_j > 0$ ($j \geq 1$), the operator

$$\sin t(-\triangle)^{\frac{1}{2}}, \cos t(-\triangle)^{\frac{1}{2}} : L^2(\Omega) \to L^2(\Omega)$$

is uniformly bounded, *i.e.*

$$\left\| \sin t(-\triangle)^{\frac{1}{2}} \right\|_{L^2}, \left\| \cos t(-\triangle)^{\frac{1}{2}} \right\|_{L^2} \leq 1, \quad \forall t \geq 0. \tag{4.17}$$

Furthermore, (u, u_t) contains two parts:

degenerative term

$$\begin{pmatrix} u^1 \\ u^1_t \end{pmatrix} = e^{-kt} \begin{pmatrix} \cos(-\triangle)^{\frac{1}{2}} + k(-\triangle)^{-\frac{1}{2}} \sin t(-\triangle)^{\frac{1}{2}} & (-\triangle)^{-\frac{1}{2}} \sin t(-\triangle)^{\frac{1}{2}} \\ k \cos t(-\triangle)^{\frac{1}{2}} - (-\triangle)^{\frac{1}{2}} \sin t(-\triangle)^{\frac{1}{2}} & \cos t(-\triangle)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix};$$

integral term

$$\begin{pmatrix} u^2 \\ u^2_t \end{pmatrix} = \begin{pmatrix} \int_0^t e^{-k(t-\tau)}(-\triangle)^{-\frac{1}{2}} \sin(t-\tau)(-\triangle)^{\frac{1}{2}}(-|u|^{p-1}u + f)\, d\tau \\ \int_0^t e^{-k(t-\tau)} \cos(t-\tau)(-\triangle)^{\frac{1}{2}}(-|u|^{p-1}u + f)\, d\tau \end{pmatrix}.$$

From the uniformly bounded condition (4.17), we get

$$\lim_{t\to\infty} \left(u_1, u^1_t\right) = 0 \quad \text{in } H^1_0(\Omega) \times L^2(\Omega); \tag{4.18}$$

and for any $(\varphi, \psi) \in B$,

$$\bigcup_{t\geq 0} \left(u^2, u^2_t\right) \text{ is a compact set in } H^1_0(\Omega) \times L^2(\Omega), \tag{4.19}$$

where $B \subset H^1_0(\Omega) \times L^2(\Omega)$ is a bounded set.

From (1.2) and $H^1_0(\Omega) \hookrightarrow L^{2p}(\Omega)$ ($p < \frac{n}{n-2}$), we get

$$T : H^1_0(\Omega) \to L^2(\Omega) \text{ is a compact map,}$$

Hence, combining (4.18) and (4.19), for the noncompact measure γ we get

$$\begin{aligned} &\gamma\left(\bigcup_{t\geq t_0} S(t)B\right) \\ &\quad = \gamma\left(\bigcup_{t\geq t_0} \left(u(t,B), u_t(t,B)\right)\right) \\ &\quad \leq \gamma\left(\bigcup_{t\geq t_0} \left(u^1, -ku^1 + u^1_t\right)\right) + \gamma\left(\bigcup_{t\geq t_0} \left(u^2, -ku^2 + u^2_t\right)\right) \\ &\quad = \gamma\left(\bigcup_{t\geq t_0} \left(u^1, -ku^1 + u^1_t\right)\right) \\ &\quad \to 0 \quad (t_0 \to \infty), \end{aligned} \tag{4.20}$$

it implies that

$$S(t) = \big(u(t,\cdot), u_t(t,\cdot)\big) \text{ has } \omega\text{-limit compactness.}$$

Finally, combining Step 2 and Step 3, applying Lemma 2.4, problem (1.1) has a global attractor $\mathcal{A}$ in $H_0^1(\Omega) \times L^2(\Omega)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
DY and QZ discussed with ZP the paper who helped to check and prove the whole paper. All authors read and approved the final manuscript.

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