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Compatibility conditions for the existence of weak solutions to a singular elliptic equation

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Abstract

This paper deals with the existence of positive solutions to the singular elliptic boundary value problem involving p-Laplace operator

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u)=\frac{h(x)}{u^{\alpha}}+k(x)u^{\beta},\ x\in\Omega;\ \ u(x)>0,\ x\in\Omega;\ \ u(x)=0,\ x\in\partial\Omega;$$

where $\Omega\subset\mathbb{R}^N$ ($N\geq 1$) is a bounded domain with smooth boundary $\partial\Omega$, $h\in L^1(\Omega)$, $h(x)>0$ almost everywhere in Ω, $k\in L^\infty(\Omega)$ is nonnegative, $p>2$, $\alpha>1$ and $\beta\in(0,p-1)$. A compatibility condition on the couple $(h(x),\alpha)$ is given for the problem to have at least one solution. More precisely, it is shown that the problem admits a solution if and only if there exists $u_0\in H_0^1(\Omega)$ such that $\int_\Omega hu_0^{1-\alpha}\,\mathrm{d}x<\infty$.

Keywords: compatibility condition; existence; singular; p-Laplace

1 Introduction

In this paper, we consider the following quasilinear elliptic equation:

$$\begin{cases}-\Delta_p u=\frac{h(x)}{u^{\alpha}}+k(x)u^{\beta}, & x\in\Omega,\\ u(x)=0, & x\in\partial\Omega,\\ u(x)>0, & x\in\Omega,\end{cases}\tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N\geq 1$) with smooth boundary $\partial\Omega$, $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the standard p-Laplace operator with $p>2$, $h\in L^1(\Omega)$, $h(x)>0$ almost everywhere in Ω, $k\in L^\infty(\Omega)$ is nonnegative, $\alpha>1$ and $\beta\in(0,p-1)$.

In the past few decades much attention has been devoted to nonlinear elliptic equations with singularities because of their wide applications in applied sciences, for example, non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, *etc.* (see, for example, [1, 2]). When $p=2$ and the singularity is weak (*i.e.*, $0<\alpha<1$), the existence, uniqueness and multiplicity of positive solutions have been established (see [3–8] and the references therein). By a result of Shi and Yao [5] one knows that if $h\in C^{2,\gamma}(\overline{\Omega})$, $h(x)\geq 0$, then (1) with $p=2$, $0<\alpha<1$ and $k(x)\equiv\lambda$ has one and only one solution $u\in C^{2,\gamma}(\Omega)\cap C^{1,1-\alpha}(\overline{\Omega})$ for all $\lambda\geq 0$. However, when the singularity is strong (*i.e.*, $\alpha>1$), various difficulties arise and the situation becomes more complicated.

A famous result obtained by Lazer and Mckenna (see [9]) showed that if $h \in C^{\gamma}(\overline{\Omega})$, $h(x) > 0$ for all $x \in \overline{\Omega}$, then (1) with $p = 2$, $k(x) \equiv 0$ and $0 < \alpha < \infty$ admits a unique solution $u \in C^{2,\gamma}(\Omega) \cap C(\overline{\Omega})$. Furthermore, it was also shown in [9] that the solution u is not in $C^1(\overline{\Omega})$ if $\alpha > 1$ and $u \in H_0^1(\Omega)$ if and only if $\alpha < 3$. However, when $h(x)$ does not have a positive lower bound on $\overline{\Omega}$, whether or not there is an $H_0^1(\Omega)$ solution of (1) may be determined by both $h(x)$ and the parameter α. In particular, when $h(x)$ behaves as $\operatorname{dist}^{\gamma}(x, \partial\Omega)$ with $\gamma \in \mathbb{R}$ (*i.e.*, there exist $c, C > 0$ such that $c \operatorname{dist}^{\gamma}(x, \partial\Omega) \leq h(x) \leq C \operatorname{dist}^{\gamma}(x, \partial\Omega)$ in Ω), (1) with $p = 2$ and $k(x) \equiv 0$ has a solution $u \in H_0^1(\Omega)$ if and only if $\alpha - 2\gamma < 3$. Therefore, it is reasonable to conjecture that (1) admits a solution in $H_0^1(\Omega)$ if and only if the couple $(h(x), \alpha)$ satisfies some 'compatibility condition'. A positive answer to this conjecture was given by Sun in [10], where she showed that (1) with $p = 2$ admits at least one solution in $H_0^1(\Omega)$ if and only if there exists $u_0 \in H_0^1(\Omega)$ such that

$$\int_{\Omega} h(x)|u_0|^{1-\alpha}\,\mathrm{d}x < \infty. \tag{CD}$$

Note that the necessity of (CD) is obvious since if (1) has an $H_0^1(\Omega)$ solution u, then by choosing $\varphi = u$ in the definition of weak solution of (1) (see the definition below) it follows that $\int_{\Omega} h(x)|u|^{1-\alpha}\,\mathrm{d}x = \int_{\Omega} |\nabla u|^2\,\mathrm{d}x - \int_{\Omega} k(x)|u|^{1+\beta}\,\mathrm{d}x < \infty$. An additional significant paper is the paper by Crandall *et al.* [11], where the existence of solutions to the more general problem

$$Lu = g(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

was studied. Here L is a linear second order elliptic operator which satisfies the maximum principle and g is positive and becomes singular uniformly in x as $u \to 0$. Their techniques are also based on the use of sub-supersolution theorems. Another paper worth mentioning is the one by Boccardo and Orsina [12], in which they proved for $p = 2$, $k(x) \equiv 0$ and general $\alpha > 0$ that (1) has only an $H^1_{\mathrm{loc}}(\Omega)$-solution when $h(x) \in L^1(\Omega)$, $h(x) \geq\not\equiv 0$. However, whether or not the solutions are $H_0^1(\Omega)$ functions was not answered there.

When $p \neq 2$, the corresponding results are much fewer. By using a sub-supersolution approach and a mountain pass theorem, Giacomoni *et al.* [13] proved, among other things, that when $h(x) \equiv \lambda$, $k(x) \equiv 1$, $0 < \alpha < 1$ and $p - 1 < \beta < p^* - 1$ (p^* is the critical Sobolev exponent of p), problem (1) has multiple weak solutions (depending on certain value of the parameter λ). Later, Loc and Schmitt [14] investigated the following singular problem:

$$\begin{cases} -\Delta_p u = a(x)g(u) + \lambda h(u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \\ u(x) > 0, & x \in \Omega, \end{cases} \tag{2}$$

where $p > 1$, $g(u)$ is a non-monotone singular term, $a \in L^{\infty}(\Omega)$, $a(x) \geq 1$ for almost every $x \in \Omega$, λ is a nonnegative parameter and $h(u)$ is continuous. They first established a sub-supersolution theorem and then proved, under some structural conditions, that the problem had at least one weak solution in $W_0^{1,p}(\Omega)$ by constructing ordered sub- and supersolutions.

It is our purpose to investigate problem (1) with a general positive function $h \in L^1(\Omega)$. We shall show that the compatibility condition (CD) on the couple $(h(x), \alpha)$ is also optimal for the existence of weak solutions to problem (1). By weak solutions we mean that $u \in W_0^{1,p}(\Omega)$ satisfying $u(x) > 0$ a.e. in Ω and

$$\int_\Omega \big[|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi - h(x)u^{-\alpha}\varphi - k(x)u^\beta\varphi\big]\,\mathrm{d}x = 0, \quad \forall\varphi\in W_0^{1,p}(\Omega). \tag{3}$$

Since $h(x)$ does not have a positive lower bound on $\overline{\Omega}$ in general, the sub- and super-solution method used in [14] cannot be applied in our paper to prove the existence of weak solutions to (1). Instead, we will deal with (1) in variational framework and introduce the following singular energy functional ($\alpha > 1$):

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \frac{1}{\alpha-1}\int_\Omega h(x)|u|^{1-\alpha}\,\mathrm{d}x - \frac{1}{1+\beta}\int_\Omega k(x)|u|^{1+\beta}\,\mathrm{d}x.$$

As was pointed out in [10], the main difficulty that arises in such problems is the absence of integrability of $u^{-\alpha}$ for $u \in W_0^{1,p}(\Omega)$, and all the inequalities related to $u \in W_0^{1,p}(\Omega)$ will not be of much help. Therefore, one will encounter many difficulties in trying to find $W_0^{1,p}$-solutions by standard variational methods. The first one is that there is a sharp contrast between the case $0 < \alpha < 1$, for which the energy functional is continuous, and the case $\alpha > 1$, for which the energy functional is singular. In order to deal with this difficulty, we use variational techniques in the spirit of the work by Sun *et al.* [7] and consider the functional $I(u)$ under appropriate constraints to restore integrability. Another difficulty arises from the nonlinearity of the p-Laplace operator, since generally one cannot deduce from $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ the convergence $|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u$ in $L^{\frac{p}{p-1}}(\Omega,\mathbb{R}^N)$, which forces us to derive more properties of the functional $I(u)$ to deal with the weak convergence. The main results of this paper will be stated and proved in Section 2.

2 Main results

Very little can be done about $I(u)$ unless we confine ourselves to some subset of $W_0^{1,p}(\Omega)$ because of the singularity of $I(u)$, which results from that fact $\alpha > 1$. To overcome this difficulty, we first define the set of the specific choice of constraints

$$N_1 = \left\{u\in W_0^{1,p}(\Omega); \int_\Omega \big[|\nabla u|^p - h(x)|u|^{1-\alpha} - k(x)|u|^{1+\beta}\big]\,\mathrm{d}x \geq 0\right\},$$

and

$$N_2 = \left\{u\in W_0^{1,p}(\Omega); \int_\Omega \big[|\nabla u|^p - h(x)|u|^{1-\alpha} - k(x)|u|^{1+\beta}\big]\,\mathrm{d}x = 0\right\}.$$

Typically, N_2 is not a closed set anymore for $\alpha > 1$ (certainly not weakly closed). Moreover, we will equip $W_0^{1,p}(\Omega)$ with the norm $\|u\|_{1,p;\Omega} = (\int_\Omega |\nabla u|^p\,\mathrm{d}x)^{\frac{1}{p}}$ due to Poincare's inequality. The subscript will be omitted when no confusion arises.

Under the above hypothesis, our main results of this paper read as follows.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded domain with smooth boundary $\partial\Omega$, $k \in L^\infty(\Omega)$ be a nonnegative function, $h \in L^1(\Omega)$, $h(x) > 0$ a.e. in Ω (not necessarily with a positive lower bound and possibly unbounded in Ω), and let $p > 2$, $\alpha > 1$, $\beta \in (0, p-1)$.*

Then problem (1) *admits a unique $W_0^{1,p}$-solution if and only if there exists $u_0 \in W_0^{1,p}(\Omega)$ such that*

$$\int_\Omega h(x)|u_0|^{1-\alpha}\,\mathrm{d}x < +\infty. \tag{4}$$

Proof Necessity. Suppose that $u \in W_0^{1,p}(\Omega)$ is a solution of (1), by choosing $\varphi = u$ in (3), we then have

$$\int_\Omega h(x)|u|^{1-\alpha}\,\mathrm{d}x = \int_\Omega \bigl[|\nabla u|^p - k(x)|u|^{1+\beta}\bigr]\,\mathrm{d}x < +\infty.$$

Sufficiency. The key to solving (1) relies on a natural interpolation between the two constraints N_1 and N_2. For the convenience of the reader, we divide this part into three steps.

Step 1. *Properties of N_1 and N_2*.

Note that, for any $u \in W_0^{1,p}(\Omega)$ with $\int_\Omega h(x)|u|^{1-\alpha}\,\mathrm{d}x < \infty$, there exists a rectilinear curve $t \to tu$ in the positive semi-axis through N_2 at $t = t(u)$ satisfying $tu \in N_1$ for all $t \geq t(u)$, which implies that N_1 is unbounded. Therefore, we know from (4) that N_2 and N_1 are nonempty since $t(u_0)u_0 \in N_2 \subset N_1$. Moreover, along this curve,

$$I(tu) \geq I\bigl(t(u)u\bigr), \quad \forall t > 0. \tag{5}$$

The closedness of N_1 follows easily from Fatou's lemma. However, since $\int_\Omega h(x)|u|^{1-\alpha}\,\mathrm{d}x$ is not continuous even on N_2, N_2 is not anymore a closed set in $W_0^{1,p}(\Omega)$.

Finally, unbounded set N_1 lies in the exterior of $W_0^{1,p}(\Omega)$ (*i.e.*, it stays away from a ball centered at 0). Suppose on the contrary that there is a sequence $(u_n) \subset N_1$ with $u_n \to 0$ in $W_0^{1,p}(\Omega)$. Then, by using the reversed Hölder's inequality, it follows that

$$\begin{aligned}\left(\int_\Omega h^{\frac{1}{\alpha}}(x)\,\mathrm{d}x\right)^{\alpha}\left(\int_\Omega |u_n|\,\mathrm{d}x\right)^{1-\alpha} &\leq \int_\Omega h(x)|u_n|^{1-\alpha}\,\mathrm{d}x \\ &\leq \|u_n\|^p - \int_\Omega k(x)|u_n|^{1+\beta}\,\mathrm{d}x \to 0.\end{aligned}$$

Consequently, $\int_\Omega |u_n|\,\mathrm{d}x \to \infty$, since $\alpha > 1$, which is clearly impossible. Therefore, there is a positive constant c_1 such that $\|u\| \geq c_1$ for any $u \in N_1$.

Step 2. *Properties of the minimizing sequence $\{u_n\}$*.

Now we consider $\inf_{N_1} I$. By Ekeland's variational principle [15], there exists a corresponding minimizing sequence $\{u_n\} \in N_1$ satisfying:

(i) $I(u_n) \leq \inf_{N_1} I + \frac{1}{n}$;

(ii) $I(u_n) \leq I(\omega) + \frac{1}{n}\|u_n - \omega\|$, $\forall \omega \in N_1$.

We may assume that $u_n \geq 0$ in Ω since $I(|u_n|) = I(u)$. Furthermore, the fact $h(x) > 0$ a.e. in Ω, $\alpha > 1$ and the following inequality $\int_\Omega h(x)|u_n|^{1-\alpha}\,\mathrm{d}x \leq \|u_n\|^p - \int_\Omega k(x)|u_n|^{1+\beta}\,\mathrm{d}x$ (since $u_n \in N_1$) guarantee that $u_n(x) > 0$ a.e. in Ω. Moreover, one can observe from $\alpha > 1$ and $k \in L^\infty(\Omega)$ that

$$\begin{aligned}I(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \frac{1}{\alpha - 1}\int_\Omega h(x)|u|^{1-\alpha}\,\mathrm{d}x - \frac{1}{1+\beta}\int_\Omega k(x)|u|^{1+\beta}\,\mathrm{d}x \\ &\geq \frac{1}{p}\|u\|^p - C\|u\|_{\beta+1}^{\beta+1}\end{aligned}$$

for some constant $C > 0$, which, due to $\beta < p - 1$, implies the coercivity of $I(u)$ on N_1 and then the existence of a constant $c_2 > 0$ such that $\|u_n\| \le c_2$. Therefore, there exist a subsequence of $\{u_n\}$, which is still denoted by $\{u_n\}$, and $u^* \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u^*$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$, and pointwise a.e. in Ω. We will try to show that $u^* \in N_2$ by analyzing the minimizing sequence $\{u_n\}$. First, by Fatou's lemma we know that

$$\int_\Omega h(x)\big|u^*\big|^{1-\alpha}\,\mathrm{d}x \le \liminf_{n\to\infty}\int_\Omega h(x)|u_n|^{1-\alpha}\,\mathrm{d}x < \infty,$$

which implies that $u^* > 0$ a.e. in Ω since $\alpha > 1$. Furthermore, we have

$$\begin{aligned}
\inf_{N_1} I &= \lim_{n\to\infty} I(u_n) \\
&= \liminf_{n\to\infty}\left[\frac{1}{p}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x + \frac{1}{\alpha-1}\int_\Omega h(x)u_n^{1-\alpha}\,\mathrm{d}x - \frac{1}{1+\beta}\int_\Omega k(x)u_n^{1+\beta}\,\mathrm{d}x\right] \\
&\ge \liminf_{n\to\infty}\left[\frac{1}{p}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x\right] + \liminf_{n\to\infty}\left[\frac{1}{\alpha-1}\int_\Omega h(x)u_n^{1-\alpha}\,\mathrm{d}x\right] \\
&\quad - \frac{1}{1+\beta}\int_\Omega k(x)u^{*1+\beta}\,\mathrm{d}x \\
&\ge \frac{1}{p}\int_\Omega \big|\nabla u^*\big|^p\,\mathrm{d}x + \frac{1}{\alpha-1}\int_\Omega h(x)u^{*1-\alpha}\,\mathrm{d}x - \frac{1}{1+\beta}\int_\Omega k(x)u^{*1+\beta}\,\mathrm{d}x \\
&= I\big(u^*\big) \ge I\big(t\big(u^*\big)u^*\big) \\
&\ge \inf_{N_2} I \ge \inf_{N_1} I.
\end{aligned} \tag{6}$$

Thus, the above inequalities are actually equalities. Particularly, it follows that

$$\liminf_{n\to\infty}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x = \int_\Omega \big|\nabla u^*\big|^p\,\mathrm{d}x,$$

and there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x = \int_\Omega \big|\nabla u^*\big|^p\,\mathrm{d}x.$$

This together with the weak convergence of $u_n \rightharpoonup u^*$ in $W_0^{1,p}(\Omega)$ implies $u_n \to u^*$ strongly in $W_0^{1,p}(\Omega)$. Consequently, we have

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup \big|\nabla u^*\big|^{p-2}\nabla u^* \quad \text{in } L^{\frac{p}{p-1}}\big(\Omega, \mathbb{R}^N\big). \tag{s}$$

We shall distinguish two cases according to u_n belonging to $N_1 \setminus N_2$ or N_2 to show that $u^* \in N_2$.

Case 1. Suppose that $\{u_n\} \subset N_1 \setminus N_2$ for all n large. Fix $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \ge 0$ and n by now. Since $u_n \in N_1 \setminus N_2$, $h(x)$ is nonnegative in Ω and $\alpha > 1$, we have

$$\int_\Omega h(x)(u_n + t\varphi)^{1-\alpha}\,\mathrm{d}x \le \int_\Omega h(x)u_n^{1-\alpha}\,\mathrm{d}x < \|u_n\|^p - \int_\Omega k(x)u_n^{1+\beta}\,\mathrm{d}x, \quad \forall t \ge 0.$$

Subsequently, we can choose $t > 0$ sufficiently small such that

$$\int_\Omega h(x)(u_n + t\varphi)^{1-\gamma}\,\mathrm{d}x < \|u_n + t\varphi\|^p - \int_\Omega k(x)(u_n + t\varphi)^{1+\beta}\,\mathrm{d}x,$$

which implies that $u_n + t\varphi \in N_1$. Furthermore, by choosing $\omega = u_n + t\varphi$ in (ii), we obtain

$$\begin{aligned}&\frac{1}{n}\|t\varphi\| + \frac{1}{p}\big(\|u_n + t\varphi\|^p - \|u_n\|^p\big) - \frac{1}{1+\beta}\int_\Omega k(x)\big[(u_n + t\varphi)^{1+\beta} - u_n^{1+\beta}\big]\,\mathrm{d}x \\ &\quad\geq \frac{1}{1-\alpha}\int_\Omega h(x)\big[(u_n + t\varphi)^{1-\alpha} - u_n^{1-\gamma}\big]\,\mathrm{d}x.\end{aligned}$$

Dividing both sides of the above inequality by $t > 0$, passing to the liminf as $t \to 0$ and using Fatou's lemma, we have

$$\begin{aligned}&\frac{\|\varphi\|}{n} + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi\,\mathrm{d}x - \int_\Omega k(x)u_n^\beta\varphi\,\mathrm{d}x \\ &\quad\geq \int_\Omega \liminf_{t\to 0}\frac{h(x)[(u_n + t\varphi)^{1-\alpha} - u_n^{1-\alpha}]}{t(1-\alpha)}\,\mathrm{d}x \\ &\quad= \int_\Omega h(x)u_n^{-\alpha}\varphi\,\mathrm{d}x.\end{aligned}$$

By letting $n \to \infty$, using Fatou's lemma again and noticing (s), we arrive at

$$\int_\Omega \big[\big|\nabla u^*\big|^{p-2}\nabla u^* \cdot \nabla\varphi - h(x)u^{*-\alpha}\varphi\big]\,\mathrm{d}x \geq \int_\Omega k(x)u^{*\beta}\varphi\,\mathrm{d}x, \quad \forall\varphi \in W_0^{1,p}(\Omega), \varphi \geq 0. \tag{7}$$

Choosing $\varphi = u^*$ in (7) yields

$$\int_\Omega \big[\big|\nabla u^*\big|^p - h(x)u^{*1-\alpha}\big]\,\mathrm{d}x \geq \int_\Omega k(x)u^{*1+\beta}\,\mathrm{d}x, \tag{8}$$

which implies that $u^* \in N_1$. Furthermore, we know from (6) that

$$u^* \in N_2 \quad \big(\textit{i.e. } t\big(u^*\big) = 1\big). \tag{9}$$

Case 2. There exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, that belongs to N_2. Let $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \geq 0$ and n be fixed again. Then, for all $t \geq 0$, $\int_\Omega h(x)(u_n + t\varphi)^{1-\alpha}\,\mathrm{d}x \leq \int_\Omega h(x)u_n^{1-\alpha}\,\mathrm{d}x = \|u_n\|^p < +\infty$ since $\alpha > 1$, which ensures the existence of a unique positive number corresponding to $u_n + t\varphi$, denoted by $f_{n,\varphi}(t)$, such that $f_{n,\varphi}(t)(u_n + t\varphi) \in N_2$, that is, $f_{n,\varphi}(t)$ satisfies

$$f_{n,\varphi}^{p-1-\beta}(t)\|u_n + t\varphi\|^p - f_{n,\varphi}^{-\alpha-\beta}(t)\int_\Omega h(x)(u_n + t\varphi)^{1-\alpha}\,\mathrm{d}x = \int_\Omega k(x)(u_n + t\varphi)^{1+\beta}\,\mathrm{d}x.$$

It is easy to show that $f_{n,\varphi}(t)$ is continuous for all $t \geq 0$ by the dominated convergence theorem, and $f_{n,\varphi}(0) = 1$ since $u_n \in N_2$. Next, we shall show that $f_{n,\varphi}(t)$ has uniform behavior at 0 with respect to n, *i.e.*, $|f'_{n,\varphi}(0)| \leq C$ for suitable $C > 0$ independent of n. For this purpose, we assume henceforth that

$$f'_{n,\varphi}(0) = \lim_{t\to 0}\frac{f_{n,\varphi}(t) - 1}{t} \in [-\infty, +\infty].$$

If the above limit does not exist, we can choose a positive sequence $\{t_k\}$ with $t_k \to 0$ (instead of $t \to 0$) in such a way that $q_n := \lim_{k\to\infty} \frac{f_{n,\varphi}(t_k)-1}{n} \in [-\infty, +\infty]$, and replace $f'_{n,\varphi}(0)$ by q_n. We first derive the upper bound of $f'_{n,\varphi}(0)$. From $f_{n,\varphi}(t)(u_n + t\varphi)$, $u_n \in N_2$, we have

$$
\begin{aligned}
0 &= f^p_{n,\varphi}(t)\|u_n + t\varphi\|^p - f^{1-\alpha}_{n,\varphi}(t)\int_\Omega h(x)(u_n + t\varphi)^{1-\alpha}\,\mathrm{d}x - f^{1+\beta}_{n,\varphi}(t)\int_\Omega k(x)(u_n + t\varphi)^{1+\beta}\,\mathrm{d}x, \\
0 &= \|u_n\|^p - \int_\Omega h(x)u_n^{1-\alpha}\,\mathrm{d}x - \int_\Omega k(x)u_n^{1+\beta}\,\mathrm{d}x.
\end{aligned}
$$

Combining the above two equalities and noticing $\alpha > 1$, we obtain

$$
\begin{aligned}
0 \geq{}& \biggl\{ p\bigl[f_{n,\varphi}(0) + o(1)\bigr]^{p-1}\|u_n + t\varphi\|^p \\
&- (1-\alpha)\bigl[f_{n,\varphi}(0) + o(1)\bigr]^{-\alpha}\int_\Omega h(x)(u_n + t\varphi)^{1-\alpha}\,\mathrm{d}x \\
&- (1+\beta)\bigl[f_{n,\varphi}(0) + o(1)\bigr]^{\beta}\int_\Omega k(x)(u_n + t\varphi)^{1+\beta}\,\mathrm{d}x\biggr\}\bigl[f_{n,\varphi}(t) - f_{n,\varphi}(0)\bigr] \\
&+ \|u_n + t\varphi\|^p - \|u_n\|^p - \int_\Omega k(x)\bigl[(u_n + t\varphi)^{1+\beta} - u_n^{1+\beta}\bigr]\,\mathrm{d}x.
\end{aligned}
$$

Dividing by $t > 0$ on both sides of the above inequalities, letting $t \to 0$ and noticing the continuity of $f_{n,\varphi}(t)$, we arrive at

$$
\begin{aligned}
0 \geq{}& \biggl[p\|u_n\|^p + (\alpha - 1)\int_\Omega h(x)u_n^{1-\alpha}\,\mathrm{d}x - (1+\beta)\int_\Omega k(x)u_n^{1+\beta}\,\mathrm{d}x\biggr]f'_{n,\varphi}(0) \\
&+ p\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi\,\mathrm{d}x - (1+\beta)\int_\Omega k(x)u_n^\beta\varphi\,\mathrm{d}x.
\end{aligned}
$$

Thus, from $\{u_n\} \subset N_2$ we know that

$$
\begin{aligned}
0 \geq{}& \biggl[(\alpha - 1)\int_\Omega h(x)u_n^{1-\alpha}\,\mathrm{d}x + (p-1-\beta)\int_\Omega k(x)u_n^{1+\beta}\,\mathrm{d}x\biggr]f'_{n,\varphi}(0) \\
&+ p\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi\,\mathrm{d}x - (1+\beta)\int_\Omega k(x)u_n^\beta\varphi\,\mathrm{d}x.
\end{aligned}
$$

Since $\{u_n\} \subset N_2$ ($\subset N_1$) is bounded and N_1 stays away from some ball centered at 0, the above inequality necessarily implies that

$$
f'_{n,\varphi}(0) \in [-\infty, +\infty) \quad \text{and} \quad f'_{n,\varphi}(0) \leq c_3 \quad \text{uniformly for all large } n \tag{10}
$$

for some suitable constant $c_3 > 0$.

Next we shall derive the lower bound for $f'_{n,\varphi}(0)$. By applying (ii) again, one gets

$$
\begin{aligned}
&\frac{1}{n}\bigl|f_{n,\varphi}(t) - 1\bigr|\|u_n\| + \frac{1}{n}tf_{n,\varphi}(t)\|\varphi\| \\
&\quad\geq \frac{1}{n}\bigl\|u_n - f_{n,\varphi}(t)(u_n + t\varphi)\bigr\| \\
&\quad\geq I(u_n) - I\bigl(f_{n,\varphi}(t)(u_n + t\varphi)\bigr),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\frac{1}{n} t f_{n,\varphi}(t)\|\varphi\| + \left(\frac{1}{p} + \frac{1}{\alpha - 1}\right)\left[\|u_n + t\varphi\|^p - \|u_n\|^p\right] \\
&\quad \geq \left\{-\left(1 + \frac{p}{\alpha - 1}\right)\left[f_{n,\varphi}(0) + o(1)\right]^{p-1}\|u_n + t\varphi\|^p \right. \\
&\qquad + \left(1 + \frac{1+\beta}{\alpha - 1}\right)\left[f_{n,\varphi}(0) + o(1)\right]^{\beta}\int_\Omega k(x)(u_n + t\varphi)^{1+\beta}\,\mathrm{d}x \\
&\qquad \left. - \frac{\|u_n\|}{n}\operatorname{sgn}\left[f_{n,\varphi}(t) - 1\right]\right\}\left[f_{n,\varphi}(t) - 1\right] \\
&\qquad + \left(\frac{1}{1+\beta} + \frac{1}{\alpha_1}\right)\int_\Omega k(x)\left[(u_n + t\varphi)^{1+\beta} - u_n^{1+\beta}\right]\mathrm{d}x.
\end{aligned}
$$

Noticing

$$
\begin{aligned}
&-\left(1 + \frac{p}{\alpha - 1}\right)\left[f_{n,\varphi}(0) + o(1)\right]^{p-1}\|u_n + t\varphi\|^p \\
&\qquad + \left(1 + \frac{1+\beta}{\alpha - 1}\right)\left[f_{n,\varphi}(0) + o(1)\right]^{\beta}\int_\Omega k(x)(u_n + t\varphi)^{1+\beta}\,\mathrm{d}x \\
&\quad \to -\left(1 + \frac{p}{\alpha - 1}\right)\|u_n\|^p + \left(1 + \frac{1+\beta}{\alpha - 1}\right)\int_\Omega k(x)u_n^{1+\beta}\,\mathrm{d}x \quad (t \to 0) \\
&\quad = -\frac{p - 1 - \beta}{\alpha - 1}\|u_n\|^p - \left(1 + \frac{1+\beta}{\alpha - 1}\right)\int_\Omega h(x)u_n^{1-\alpha}\,\mathrm{d}x \quad (\text{since } u_n \in N_2) \\
&\quad \leq -\frac{p - 1 - \beta}{\alpha - 1}\|u_n\|^p \leq -\frac{p - 1 - \beta}{\alpha - 1}c_1^p,
\end{aligned}
$$

we see that $f'_{n,\varphi}(0)$ cannot diverge to $-\infty$ as $n \to \infty$, that is,

$$
f'_{n,\varphi}(0) \in (-\infty, +\infty) \quad \text{and} \quad f'_{n,\varphi}(0) \geq c_4 \quad \text{uniformly in all large } n \tag{11}
$$

for some suitable real number c_4.

Putting together (10) and (11), we find that

$$
f'_{n,\varphi}(0) \in (-\infty, +\infty) \quad \text{and} \quad \left|f'_{n,\varphi}(0)\right| \leq C \quad \text{uniformly in all large } n. \tag{12}
$$

Based on (12), we are now in the position to locate u^* in Case 2. Let $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \geq 0$ and n be fixed. By (ii) we have

$$
\begin{aligned}
&\frac{\|u_n\|}{n}\left|f_{n,\varphi}(t) - 1\right| + \frac{\|\varphi\|}{n} f_{n,\varphi}(t)t \\
&\quad \geq \frac{1}{n}\left\|u_n - f_{n,\varphi}(t)(u_n + t\varphi)\right\| \\
&\quad \geq I(u_n) - I\left(f_{n,\varphi}(t)(u_n + t\varphi)\right) \\
&\quad = \left\{-\left[f_{n,\varphi}(0) + o(1)\right]^{p-1}\|u_n + t\varphi\|^p + \left[f_{n,\varphi}(0) + o(1)\right]^{-\alpha}\int_\Omega h(x)(u_n + t\varphi)^{1-\alpha}\,\mathrm{d}x \right. \\
&\qquad \left. + \left[f_{n,\varphi}(0) + o(1)\right]^{\beta}\int_\Omega k(x)(u_n + t\varphi)^{1+\beta}\,\mathrm{d}x\right\}\left(f_{n,\varphi}(t) - 1\right)
\end{aligned}
$$

$$- \frac{1}{p}\left(\|u_n + t\varphi\|^p - \|u_n\|^p\right) + \frac{1}{1-\alpha}\int_\Omega h(x)\left[(u_n + t\varphi)^{1-\alpha} - u_n^{1-\alpha}\right] \mathrm{d}x$$
$$+ \frac{1}{1+\beta}\int_\Omega k(x)\left[(u_n + t\varphi)^{1+\beta} - u_n^{1+\beta}\right] \mathrm{d}x.$$

Thus, dividing by $t > 0$ on both sides of the inequality, passing to the liminf as $t \to 0$ and noticing that $u_n \in N_2$, we get

$$\begin{aligned}
&\frac{1}{n}\left[\left|f'_{n,\varphi}(0)\right|\|u_n\| + \|\varphi\|\right] \\
&\quad \ge \left[-\|u_n\|^p + \int_\Omega h(x)u_n^{1-\alpha}\,\mathrm{d}x + \int_\Omega k(x)u_n^{1+\beta}\,\mathrm{d}x\right]f'_{n,\varphi}(0) \\
&\qquad - \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega h(x)u_n^{-\alpha}\varphi\,\mathrm{d}x + \int_\Omega k(x)u_n^{\beta}\varphi\,\mathrm{d}x \\
&\quad = - \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega h(x)u_n^{-\alpha}\varphi\,\mathrm{d}x + \int_\Omega k(x)u_n^{\beta}\varphi\,\mathrm{d}x.
\end{aligned} \tag{13}$$

Since $|f'_{n,\varphi}(0)| \le C$ uniformly in all n large, we know that $h(x)u_n^{-\alpha}\varphi$ is integrable in Ω. Furthermore, letting $n \to \infty$ in (13) and using Fatou's lemma again, we obtain

$$\begin{aligned}
&\int_\Omega \left|\nabla u^*\right|^{p-2}\nabla u^*\cdot\nabla\varphi\,\mathrm{d}x \ge \int_\Omega h(x)u^{*-\alpha}\varphi\,\mathrm{d}x + \int_\Omega k(x)u^{*\beta}\varphi\,\mathrm{d}x, \\
&\quad \forall\varphi \in W_0^{1,p}(\Omega), \varphi \ge 0.
\end{aligned} \tag{14}$$

Taking $\varphi = u^*$ in (14), we get $u^* \in N_1$. By the same argument as in Case 1, we see also

$$u^* \in N_2. \tag{15}$$

Thus, from (7), (9), (14) and (15), we conclude that in either case $u^* \in N_2$ and

$$\begin{aligned}
&\int_\Omega \left|\nabla u^*\right|^{p-2}\nabla u^*\cdot\nabla\varphi\,\mathrm{d}x \ge \int_\Omega h(x)u^{*-\alpha}\varphi\,\mathrm{d}x + \int_\Omega k(x)u^{*\beta}\varphi\,\mathrm{d}x, \\
&\quad \forall\varphi \in W_0^{1,p}(\Omega), \varphi \ge 0.
\end{aligned} \tag{16}$$

The strict positivity $u^*(x) > 0$, $\forall x \in \Omega$ follows from the maximum principle [16].

Step 3. *Existence of weak solutions.*

Finally, we will prove that $u^* \in W_0^{1,p}(\Omega)$ is a solution to (1). To show this, for arbitrary $\phi \in W_0^{1,p}(\Omega)$ and $\varepsilon > 0$, set $\Omega_\varepsilon^+ = \{x \in \Omega : u^*(x) + \varepsilon\phi(x) \ge 0\}$ and $\Omega_\varepsilon^- = \{x \in \Omega : u^*(x) + \varepsilon\phi(x) < 0\}$. Taking $\varphi(x) = (u^*(x) + \varepsilon\phi(x))^+ \in W_0^{1,p}(\Omega)$ in (16) to obtain

$$\begin{aligned}
0 &\le \int_\Omega \left[\left|\nabla u^*\right|^{p-2}\nabla u^*\cdot\nabla\varphi\,\mathrm{d}x - h(x)u^{*-\alpha}\varphi - k(x)u^{*\beta}\varphi\right]\mathrm{d}x \\
&= \int_{\Omega_\varepsilon^+} \left[\left|\nabla u^*\right|^{p-2}\nabla u^*\cdot\nabla\left(u^* + \varepsilon\phi\right) - h(x)u^{*-\alpha}\left(u^* + \varepsilon\phi\right) - k(x)u^{*\beta}\left(u^* + \varepsilon\phi\right)\right]\mathrm{d}x \\
&= \int_\Omega \left[\left|\nabla u^*\right|^{p-2}\nabla u^*\cdot\nabla\left(u^* + \varepsilon\phi\right) - h(x)u^{*-\alpha}\left(u^* + \varepsilon\phi\right) - k(x)u^{*\beta}\left(u^* + \varepsilon\phi\right)\right]\mathrm{d}x \\
&\quad - \int_{\Omega_\varepsilon^-} \left[\left|\nabla u^*\right|^{p-2}\nabla u^*\cdot\nabla\left(u^* + \varepsilon\phi\right) - h(x)u^{*-\alpha}\left(u^* + \varepsilon\phi\right) - k(x)u^{*\beta}\left(u^* + \varepsilon\phi\right)\right]\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \|u^*\|^p - \int_\Omega h(x)u^{*1-\alpha}\,\mathrm{d}x - \int_\Omega k(x)u^{*1+\beta}\,\mathrm{d}x \\
&\quad + \varepsilon \int_\Omega \bigl[|\nabla u^*|^{p-2}\nabla u^*\nabla\phi - h(x)u^{*-\alpha}\phi - k(x)u^{*\beta}\phi\bigr]\,\mathrm{d}x \\
&\quad - \int_{\Omega_\varepsilon^-} \bigl[|\nabla u^*|^{p-2}\nabla u^*\nabla(u^*+\varepsilon\phi) - h(x)u^{*-\alpha}(u^*+\varepsilon\phi) - k(x)u^{*\beta}(u^*+\varepsilon\phi)\bigr]\,\mathrm{d}x \\
&= \varepsilon \int_\Omega \bigl[|\nabla u^*|^{p-2}\nabla u^*\cdot\nabla\phi - h(x)u^{*-\alpha}\phi - k(x)u^{*\beta}\phi\bigr]\,\mathrm{d}x \\
&\quad - \int_{\Omega_\varepsilon^-} \bigl[|\nabla u^*|^{p-2}\nabla u^*\nabla(u^*+\varepsilon\phi) - h(x)u^{*-\alpha}(u^*+\varepsilon\phi) - k(x)u^{*\beta}(u^*+\varepsilon\phi)\bigr]\,\mathrm{d}x \\
&\le \varepsilon \int_\Omega \bigl[|\nabla u^*|^{p-2}\nabla u^*\nabla\phi - h(x)u^{*-\alpha}\phi - k(x)u^{*\beta}\phi\bigr]\,\mathrm{d}x - \varepsilon\int_{\Omega_\varepsilon^-} |\nabla u^*|^{p-2}\nabla u^*\nabla\phi\,\mathrm{d}x.
\end{aligned}
$$

Since the measure of Ω_ε^- tends to zero as $\varepsilon \to 0$, dividing the above expression by $\varepsilon > 0$ and letting $\varepsilon \to 0$, we obtain

$$
\int_\Omega \bigl[|\nabla u^*|^{p-2}\nabla u^*\nabla\phi - h(x)u^{*-\alpha}\phi - k(x)u^{*\beta}\phi\bigr]\,\mathrm{d}x \ge 0, \quad \forall\phi \in W_0^{1,p}(\Omega).
$$

Replacing ϕ by $-\phi$ we obtain the reserved inequality, and hence u^* actually satisfies the following:

$$
\int_\Omega \bigl[|\nabla u^*|^{p-2}\nabla u^*\nabla\phi - h(x)u^{*-\alpha}\phi - k(x)u^{*\beta}\phi\bigr]\,\mathrm{d}x = 0, \quad \forall\phi \in W_0^{1,p}(\Omega),
$$

which means that u^* is a $W_0^{1,p}$-solution to (1). The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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