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Regularity of weak solutions of the Cauchy problem to a fractional porous medium equation

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Abstract
This paper concerns the regularity of the weak solutions of the Cauchy problem to a fractional porous medium equation with a forcing term. In the recent work (Fan *et al.* in Comput. Math. Appl. 67:145-150, 2014), the authors established the existence of the weak solution and the uniqueness of the weak energy solution. In this paper, we show that the every nonnegative bounded weak energy solution is indeed a strong solution.

MSC: 26A33; 35K57

Keywords: fractional diffusion; porous medium equation; weak solution; strong solution

1 Introduction

This is a sequel of the previous work [1]. We continue to investigate the Cauchy problem to a fractional porous medium equation with a forcing term

$$\begin{cases} \frac{\partial u}{\partial t} + \sqrt{-\Delta}(|u|^{m-1}u) = f(x,t), & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where the integer $m > 0$, the forcing term $f(x,t) \in C(0,\infty; L^1(\mathbb{R}^N))$. Furthermore, we assume that $u_0(x)$ is a bounded and integrable function.

Recently, the fractional porous medium equation $u_t + (-\Delta)^{\frac{\alpha}{2}} u^m = 0$ ($m > 0$, $\alpha > 0$) has been investigated by Pablo *et al.* in [2] for $\alpha = 1$ and in [3] for general case $0 < \alpha < 2$. Systematic and satisfactory results on the weak solutions to the Cauchy problem to the fractional porous medium problem have been obtained, including the existence, uniqueness, comparison principle, and regularity to the suitable weak solutions.

The interest in studying the fractional diffusion in modeling diffusive processes has a wide literature, especially studying the long-range diffusive interaction in porous medium type propagation and infinitesimal generators of stable Lévy processes [4, 5]. We would like to refer to the survey papers [6] and [7] for the fractional operators and to [8–10] for fractional partial differential equations. For instance, a method based on the Jacobi-tau approximation for solving multi-term time-space-fractional partial differential equations. A spectral-tau algorithm is based on the Jacobi operational matrix for a numerical solution of time fractional diffusion-wave equations. (See [11] for details.) On the other side, there is much literature on the porous medium equations (see the classical book [12] and references [1, 10, 13–18] and references therein).

Before we state the main results in this paper, we present some definitions as regards the fractional operator and the weak solutions to the problem (1.1). The nonlocal operator square root Laplacian operator can be illustrated in the following three items.

The first one is that

$$(-\Delta)^{\frac{1}{2}}g(x) = \big(|\xi|\hat{g}(\xi)\big)^{\vee}(x)$$

using the Fourier transform for any function g in the Schwartz class.

The second one is via the Riesz potential (see [19] and [20]):

$$(-\Delta)^{\frac{1}{2}}g(x) = C(N)P.V.\int_{\mathbb{R}^N}\frac{g(x)-g(y)}{|x-y|^{N+1}}\,dy, \tag{1.2}$$

where $C(N) = \pi^{-\frac{N+1}{2}}\Gamma(\frac{N+1}{2})$ is the normalization constant, $\Gamma(s)$ is the gamma function.

Thirdly, we can define the nonlocal operator $(-\Delta)^{\frac{1}{2}}$ using the so-called Dirichlet to Neumann operator, which is introduced in [21] by Caffarelli and Silvestre. For any given smooth bounded function $g(x)$, which is defined in $\mathbb{R}^N$, we consider the harmonic extension $v = v(x,y)$ in $\mathbb{R}^{N+1}_+$, denoting $v = E(g)$, which is the unique smooth bounded solution to the following problem:

$$\begin{cases} \Delta_{x,y}v = 0, & x\in\mathbb{R}^N, y>0,\\ v(x,0) = g(x), & x\in\mathbb{R}^N. \end{cases} \tag{1.3}$$

Then we claim that $(-\Delta_x)^{\frac{1}{2}}$ as the operator

$$T: g\mapsto -v_y(x,0), \tag{1.4}$$

where Δ_x is the Laplacian acting only on the x variables, and $\Delta_{x,y}$ acts on all the variables (x,y). In fact,

$$T\big(T(g)\big)(x) = T\big(-v_y(x,0)\big)(x) = v_{yy}(x,0) = -\Delta_x u_0(x).$$

Furthermore, it is easy to check that T is indeed a positive operator by integration by parts argument.

The third way to define the square root of Laplacian developed in [21] was proved to be powerful in dealing with the nonlocal operator. For any given function $|u|^{m-1}u$, we consider its harmonic extension and denote $w = E(|u|^{m-1}u)$; then the extension function satisfies the following problem:

$$\begin{cases} \Delta w = 0 & \text{for } \bar{x}\in\Omega, t>0,\\ \frac{\partial\Phi^{-1}(w)}{\partial t} - \frac{\partial w}{\partial y} = -f(x,t) & \text{on } \Gamma, t>0,\\ w(x,0,0) = u_0^m(x) & \text{on } \Gamma, \end{cases} \tag{1.5}$$

where Ω is the upper half-space $\mathbb{R}^{N+1}_+$, $\bar{x} = (x_1,x_2,\ldots,x_n,y)$, $y>0$, and we denote the boundary of Ω by $\Gamma = \mathbb{R}^N\times\{0\}$.

We focus on the quasi-stationary problem (1.5) with a dynamical boundary condition. Once w is solved, the solution u to the problem (1.1) can be understand as the trace of $|w|^{\frac{1}{m}-1}w$.

Fan *et al.* [1] established the existence of the weak solution and the uniqueness of the weak energy solution to the problem (1.1). The weak solution and the weak energy solution of the Cauchy problem are defined as follows.

Multiplying the equation $\Delta w = 0$ in (1.5) by a test function $\varphi \in C_0^1(\bar{\Omega} \times [0, T))$ and integrating by parts with respect to $\bar{x}$, we get

$$-\int_0^T \int_\Omega \langle \nabla w, \nabla \varphi \rangle \, d\bar{x}\, ds + \int_0^T \int_\Gamma u \frac{\partial \varphi}{\partial t} \, dx\, ds + \int_0^T \int_\Gamma f(x,s)\varphi \, dx\, ds = 0, \tag{1.6}$$

with $u = \operatorname{Tr}(|w|^{\frac{1}{m}-1}w)$.

Then a pair (u, w) of functions is called a weak solution to the problem (1.5) if $w \in L^1((0,T); W^{1,1}_{\text{loc}}(\Omega))$, $u = \operatorname{Tr}(|w|^{\frac{1}{m}-1}w) \in L^1(\Gamma \times (0,T))$, and $u_0(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, the equality (1.6) holds for any $\varphi \in C_0^1(\bar{\Omega} \times [0, T))$.

Furthermore, if a pair weak solution pair satisfies $w \in L^2([0,T]; H^1(\Omega))$, we call the weak solution a weak energy solution.

In a previous work [1], Fan *et al.* obtained the following existence and uniqueness results.

Theorem 1.1 *Assume that* $m > 0, f \in C(0,\infty; L^1(\mathbb{R}^N))$ *and the data* $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, *then there exists a unique weak energy solution* (w, u) *to the problem* (1.4), *and* $u \in C([0,\infty); L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^N \times [0,\infty))$.

We are ready to announce the main result, proved in the next section.

Theorem 1.2 *Every nonnegative bounded weak energy solution to the problem* (1.1) *is a strong solution. Moreover, for every* $t \in (0, T]$, *we have*

$$\left\| \frac{\partial u}{\partial t}(\cdot, t) \right\|_{L^1(\mathbb{R}^N)} \le \frac{C}{t} \tag{1.7}$$

for all $m \ge 1$, *where* $C = C(T, m, \|u_0\|_{L^1(\mathbb{R}^N)})$.

2 Proof of the main result

In this section, we prove Theorem 1.2. Namely, it suffices to show that the time partial derivative of u is an L^1 function. As the first step, we will show that the time-increment quotients are bounded in $L^1(\Gamma)$, where $\Gamma = \mathbb{R}^N \times \{0\}$, and thus the limit is still in $L^1(\mathbb{R}^N)$.

2.1 Time-increment quotients of the solution are bounded in $L^1(\mathbb{R}^N)$

Proposition 2.1 *If* u *is the weak solution to the problem* (1.1) *established in Theorem* 1.1, *then*

$$h^{-1}\big(u(\cdot, t+h) - u(\cdot, t)\big) \in L^1\big(\mathbb{R}^N\big)$$

for every $t > 0$ *and* $h > 0$.

Proof In fact, if follows from [22] that we have

$$\sup_{0\le h\le t\le T}\frac{t}{h}\big\|u(\cdot,t+h)-u(\cdot,t)\big\|\le C\big(T,m,\|u_0\|_{L^1(\mathbb{R}^N)}\big). \tag{2.1}$$

The same type of estimate can be obtained for the time-increment quotients of u^m under the condition of $m\ge 1$. Since $u\in L^\infty$, we have

$$\big|u^m(\cdot,t+h)-u^m(\cdot,t)\big|\le C\big|u(\cdot,t+h)-u(\cdot,t)\big|,$$

where $C=m\cdot\max\{u^{m-1}(\cdot,t+h),u^{m-1}(\cdot,t)\}$.

Define $\delta^h u^m=\frac{1}{h}(u^m(\cdot,t+h)-u^m(\cdot,t))$, we get

$$\delta^h u^m\in L^1\big(\mathbb{R}^N\big)$$

for $m\ge 1$. □

2.2 Regularity of time derivative of solution

Proposition 2.2 *If u is a weak solution to the problem* (1.1), *then*

$$\partial_t u^{(m+1)/2}\in L^2_{\mathrm{loc}}\big((0,\infty);L^2\big(\mathbb{R}^N\big)\big). \tag{2.2}$$

Proof Due to the lack of regularity of u in time, we also use the Steklov average of u instead of u itself.

Step 1. Assume $\partial_t u\in L^1(\mathbb{R}^N)$, then the following identity can be obtained by multiplying the equation (1.1) by a test function φ and integrating in $\mathbb{R}^N\times(0,\infty)$,

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^N}\frac{\partial u}{\partial t}\varphi\,dx\,dt\\&\quad=-\int_0^\infty\int_{\mathbb{R}^N}(-\Delta)^{1/4}\big(u^m\big)(-\Delta)^{1/4}\varphi\,dx\,dt+\int_0^\infty\int_{\mathbb{R}^N}f(x,t)\varphi\,dx\,dt.\end{aligned} \tag{2.3}$$

Thus, we can rewrite the weak formulation (1.6) into the following form via the Steklov average u^h:

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^N}\frac{\partial u^h}{\partial t}\varphi\,dx\,dt\\&\quad=-\int_0^\infty\int_{\mathbb{R}^N}(-\Delta)^{1/4}\big(u^m\big)^h(-\Delta)^{1/4}\varphi\,dx\,dt+\int_0^\infty\int_{\mathbb{R}^N}f(x,t)\varphi\,dx\,dt.\end{aligned} \tag{2.4}$$

Step 2. Let $\varphi=\zeta\,\partial_t(u^m)^h$, where $\zeta\in C_0^\infty((0,\infty))$, $0\le\zeta\le 1$, $\zeta(t)=1$, for $t\in[t_1,t_2]$ is a cutoff function. Inserting φ as a test function in (2.4) yields

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^N}\zeta\frac{\partial u^h}{\partial t}\frac{\partial(u^m)^h}{\partial t}\,dx\,dt\\&\quad=-\frac{1}{2}\int_0^\infty\int_{\mathbb{R}^N}\zeta\frac{\partial}{\partial t}\big|(-\Delta)^{1/4}\big(u^m\big)^h\big|^2\,dx\,dt+\int_0^\infty\int_{\mathbb{R}^N}\zeta f(x,t)\partial_t\big(u^m\big)^h\,dx\,dt\\&\quad=\frac{1}{2}\int_0^\infty\int_{\mathbb{R}^N}\zeta'\big|(-\Delta)^{1/4}\big(u^m\big)^h\big|^2\,dx\,dt+\int_0^\infty\int_{\mathbb{R}^N}\zeta f(x,t)\partial_t\big(u^m\big)^h\,dx\,dt.\end{aligned}$$

It follows from the inequality

$$\delta^h u^m \delta^h u \ge C\bigl(\delta^h\bigl(u^{\frac{m+1}{2}}\bigr)\bigr)^2$$

in [2] and the facts $\delta^h u^m \in L^1(\mathbb{R}^N)$, $u^m \in L^2_{\mathrm{loc}}((0,\infty); H_0^{1/2})$ that

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\bigl(\delta^h\bigl(u^{\frac{m+1}{2}}\bigr)\bigr)^2\,dx\,dt \le C.$$

Then the proof is completed by passing to the limit $h \to 0$. □

2.3 Proof of Theorem 1.2

The estimate (2.1) implies

$$\liminf_{h\to 0}\frac{1}{h}\int_{\tau}^{T-h}\bigl\|u(t+h)-u(t)\bigr\|\,dt \le C$$

for all $0 < \tau \le t \le T < \infty$. Furthermore, we can deduce that $u \in BV((\tau,T); L^1(\mathbb{R}^N))$.

It follows from the fact in [23] that $u \in W^{1,1}((\tau,T); L^1(\mathbb{R}^N))$, provided that

$$u = \int_0^{v} p(r)\,dr \in BV\bigl((\tau,T); L^1\bigl(\mathbb{R}^N\bigr)\bigr)$$

for some $v \in W^{1,1}((\tau,T); L^1(\mathbb{R}^N))$, $p \in L^1_{\mathrm{loc}}(\mathbb{R})$.

Let $v = u^{\frac{m+1}{2}}$, we claim that $u^{\frac{m+1}{2}} \in W^{1,1}((\tau,T); L^1(\mathbb{R}^N))$.

In fact, since $\partial_t u^{(m+1)/2} \in L^2_{\mathrm{loc}}((0,\infty); L^2(\mathbb{R}^N))$, we can get $\partial_t u^{(m+1)/2} \in L^1(K \times (\tau,T))$ for any compact set $K \subset\subset \mathbb{R}^N$.

In addition, we have the estimate

$$\frac{1}{h}\int_{\tau}^{T}\int_{\mathbb{R}^n}\bigl|u^{\frac{m+1}{2}}(t+h)-u^{\frac{m+1}{2}}(t)\bigr|\,dx\,dt \le C$$

for $m \ge 1$. Hence, we have $\partial_t u^{(m+1)/2} \in L^1((\tau,T); L^1(\mathbb{R}^N))$, and therefore $\partial_t u \in L^1((\tau,T); L^1(\mathbb{R}^N))$ for all $0 < \tau < T < \infty$. Hence, (1.7) can be obtained by passing $h \to 0$ in (2.1). We complete the proof of Theorem 1.2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The article is a joint work of two authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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