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Singularly perturbed second order semilinear boundary value problems with interface conditions

Hongxu Lin and Feng Xie*

*Correspondence: fxie@dhu.edu.cn
Department of Applied Mathematics, Donghua University, Shanghai, 201620, P.R. China

Abstract

In this paper we study a class of singularly perturbed interface boundary value problems with discontinuous source terms. We first establish a lemma of lower-upper solutions by using the Schauder fixed point theorem. By the method of boundary functions and the lemma of lower-upper solutions we obtain the existence, asymptotic estimates, and uniqueness of the solution with boundary and interior layers for the proposed problem.

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Keywords: singular perturbation; interface conditions; lower and upper solutions; asymptotic estimates

1 Introduction

We consider the interface boundary value problem with discontinuous source terms

$$\varepsilon^2 u'' = f(x,u), \quad x \in [a,d) \cup (d,b], \tag{1.1}$$

$$\left[\varepsilon u'\right](d) = c_1, \qquad [u](d) = c_2, \tag{1.2}$$

$$u(a) = A, \qquad u(b) = B, \tag{1.3}$$

where ε is a small and positive parameter, A and B being given constants, and

$$f(x,u) = \begin{cases} f_1(x,u), & x \in [a,d), \\ f_2(x,u), & x \in (d,b]. \end{cases}$$

The functions f_1 and f_2 are smooth enough on $[a,d] \times \mathbb{R}$ and on $[d,b] \times \mathbb{R}$, respectively, and $f_1(d,u) \neq f_2(d,u)$ for $u \in \mathbb{R}$. We denote by $[w](d)$ the jump of a function w at the point d, *i.e.* $[w](d) = w(d^+) - w(d^-)$. The solution of the problem and its first order derivative have a jump at $x = d \in (a,b)$. The original problem can be regarded as the coupling of the left problem and the right problem.

In recent years, the boundary value problems with interface conditions have appeared in applications such as heat transfer in layers composite material [1], the one dimensional metal-oxide-semiconductor structure [2], and population genetics [3]. The study of boundary value problems with interface conditions has attracted much attention, especially in numerical aspects; see [2, 4–8] and references therein. For instance, de Falco and O'Riordan [2] considered singularly perturbed reaction-diffusion equations with discontinuous data and interface conditions, where one of the problems was given as follows: Find $u_\varepsilon \in C^0[0,1] \cap C^2((0,d) \cup (d,1))$ such that

$$\begin{aligned}
&-\big(\varepsilon(x)u_\varepsilon'\big)' + r(x)u_\varepsilon = f, \quad x \in (0,d) \cup (d,1),\\
&u_\varepsilon(0) = B_0, \qquad u_\varepsilon(1) = B_1,\\
&-\big[\varepsilon u_\varepsilon'\big](d) = Q_1',\\
&[f](d) = Q_2, \qquad [r](d) = Q_3,\\
&r(x) \geq r_0 > 0, \qquad \varepsilon(x) > 0, \quad x \in (0,d) \cup (d,1),
\end{aligned}$$

with the diffusion coefficient

$$\varepsilon(x) = \begin{cases} \varepsilon_1 p(x), & x < d, \\ \varepsilon_2 p(x), & x > d, \end{cases} \qquad \frac{r(x)}{p(x)} > \beta > 0, \quad x \in (0,d) \cup (d,1).$$

They suggested a parameter-uniform method based on piecewise-uniform Shishkin meshes to solve the problem above.

In this paper, we are devoted to the study of the existence, uniqueness, and asymptotics of the singularly perturbed interface boundary value problem (1.1)-(1.3), whose solution exhibits an interior layer due to the discontinuity of the source term. By using the Schauder fixed point theorem, we first establish a lower and upper solutions lemma which is an extension of classical theory of lower and upper solutions (see [9], for instance). By the method of boundary functions (see [10, 11], for example) and the lemma of lower-upper solutions we obtain the existence, asymptotic estimates, and uniqueness of the solution with boundary and interior layers for the proposed problem.

The remainder of this paper is organized as follows. In Section 2 we establish the lemma of lower-upper solutions for a class of two-point boundary value problems with interface conditions by the Schauder fixed point theorem, which will be used to prove our main result. With the asymptotic expansions and the lemma of lower and upper solutions established in Section 2, the asymptotic estimates, existence, and uniqueness of the solution for the problem (1.1)-(1.3) are obtained in Section 3.

2 Basic lemmas

For $j = 1, 2$, we define the vector spaces $Q^j[a,b]$ of functions by

$$Q^j[a,b] \equiv \Big\{u(x) \in C^j\big([a,b] \setminus \{d\}\big) \mid \lim_{x \to d^\pm} u(x) = u\big(d^\pm\big) \in \mathbb{R}, \lim_{x \to d^\pm} u'(x) = u'\big(d^\pm\big) \in \mathbb{R}\Big\}.$$

We define $\widetilde{Q}^2[a,b]$ by

$$\widetilde{Q}^2[a,b] \equiv \big\{u(x) \in C^2\big([a,b] \setminus \Lambda\big) \mid u\big(d^\pm\big), u\big(x_i^\pm\big), u'\big(d^\pm\big), u'\big(x_i^\pm\big) \in \mathbb{R}, i = 1, 2, \ldots, k\big\},$$

where $\Lambda = \{d, x_1, x_2, \ldots, x_k\}$ and k is a given positive integer.

It is easily verified that $Q^1[a,b]$ is a Banach space endowed with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$. The following lemma is a generalized Arzelà-Ascoli theorem on families of functions in $Q^1[a,b]$.

Lemma 2.1 *Assume that a bounded set E in $Q^1[a,b]$ is piecewise equicontinuous, that is, $E|_{[a,d)} = \{u|_{[a,d)} : u \in E\}$ is equicontinuous on $[a,d)$ and $E|_{(d,b]} = \{u|_{(d,b]} : u \in E\}$ is equicontinuous on $(d,b]$. Then E is a relatively compact subset of $Q^1[a,b]$.*

Proof The proof follows almost the same lines as that of the classical Arzelà-Ascoli theorem (see, for instance, [12]), noting that for $u \in Q^1[a,b]$, u, and u' both have left and right limits at $x = d$. Thus, details are omitted here. □

Let us consider the two-point boundary value problem with interface conditions

$$u'' = f(x,u), \quad x \in [a,d) \cup (d,b], \tag{2.1}$$

$$\left[u'\right](d) = c_1, \qquad [u](d) = c_2, \tag{2.2}$$

$$u(a) = A, \qquad u(b) = B, \tag{2.3}$$

where the constants A, B, c_1, c_2, and the function $f(x,u)$ are given in (1.1)-(1.3). Obviously, all solutions to (2.1)-(2.3) belong to $Q^2[a,b]$.

Definition 2.1 We say that a function $\alpha \in \widetilde{Q}^2[a,b]$ is a lower solution of the problem (2.1)-(2.3) if

$$\begin{aligned}
&\alpha''(x) \ge f\bigl(x,\alpha(x)\bigr), \quad x \in (a,b) \setminus \Lambda,\\
&\alpha'\bigl(d^+\bigr) - \alpha'\bigl(d^-\bigr) > c_1,\\
&\alpha\bigl(d^+\bigr) - \alpha\bigl(d^-\bigr) = c_2,\\
&\alpha'\bigl(x_i^-\bigr) \le \alpha'\bigl(x_i^+\bigr), \quad i = 1,2,\dots,k,\\
&\alpha(a) \le A, \qquad \alpha(b) \le B.
\end{aligned} \tag{2.4}$$

We say that a function $\beta \in \widetilde{Q}^2[a,b]$ is an upper solution of the problem (2.1)-(2.3) if

$$\begin{aligned}
&\beta''(x) \le f\bigl(x,\beta(x)\bigr), \quad x \in (a,b) \setminus \Lambda,\\
&\beta'\bigl(d^+\bigr) - \beta'\bigl(d^-\bigr) < c_1,\\
&\beta\bigl(d^+\bigr) - \beta\bigl(d^-\bigr) = c_2,\\
&\beta'\bigl(x_i^-\bigr) \ge \beta'\bigl(x_i^+\bigr), \quad i = 1,2,\dots,k,\\
&\beta(a) \ge A, \qquad \beta(b) \ge B.
\end{aligned} \tag{2.5}$$

Lemma 2.2 *Assume that α and β are lower and upper solutions of the problem* (2.1)-(2.3) *such that $\alpha \le \beta$. Then the problem* (2.1)-(2.3) *has at least one solution $u \in Q^1[a,b]$ such that for all $x \in [a,d) \cup (d,b]$,*

$$\alpha(x) \le u(x) \le \beta(x).$$

Proof Let us consider the modified problem

$$u'' - u = f\big(x, \gamma(x,u)\big) - \gamma(x,u), \quad x \in [a,d) \cup (d,b], \tag{2.6}$$

$$\big[u'\big](d) = c_1, \qquad [u](d) = c_2, \tag{2.7}$$

$$u(a) = A, \qquad u(b) = B, \tag{2.8}$$

where $\gamma(x,u)$ is defined by

$$\gamma(x,u) = \begin{cases} \beta(x), & \text{if } u > \beta(x), \\ u, & \text{if } \alpha(x) \le u \le \beta(x), \\ \alpha(x), & \text{if } u < \alpha(x). \end{cases} \tag{2.9}$$

The homogeneous problem of (2.6) and (2.8) can be divided into the following two initial value problems:

$$u_1'' - u_1 = 0, \quad x \in [a,d), \tag{2.10}$$

$$u_1(a) = A \tag{2.11}$$

and

$$u_2'' - u_2 = 0, \quad x \in (d,b], \tag{2.12}$$

$$u_2(b) = B. \tag{2.13}$$

Obviously, the problems (2.10)-(2.11) and (2.12)-(2.13) have general solutions

$$v_1(x) = p_1 \exp(x-a) - p_2 \exp(a-x), \quad x \in [a,d) \tag{2.14}$$

and

$$v_2(x) = p_3 \exp(x-b) - p_4 \exp(b-x), \quad x \in (d,b], \tag{2.15}$$

respectively.

Consider the modified problem (2.6)-(2.8). Let us write the boundary value problem as an integral equation

$$u(x) = v(x) + \int_a^b G(x,s)\big[f\big(s, \gamma\big(s, u(s)\big)\big) - \gamma\big(s, u(s)\big)\big]\, ds,$$

where

$$v(x) = \begin{cases} v_1(x), & a \le x < d, \\ v_2(x), & d < x \le b, \end{cases}$$

$$G(x,s) = \frac{1}{W(s)} \begin{cases} (\exp(x-a) - \exp(a-x)) \\ \quad \times (\exp(s-b) - \exp(b-s)), & a \le x \le s \le b, \\ (\exp(s-a) - \exp(a-s)) \\ \quad \times (\exp(x-b) - \exp(b-x)), & a \le s \le x \le b \end{cases} \tag{2.16}$$

and

$$W(x) = \begin{vmatrix} \exp(x-a) - \exp(a-x) & \exp(x-b) - \exp(b-x) \\ \exp(x-a) + \exp(a-x) & \exp(x-b) + \exp(b-x) \end{vmatrix}.$$

If $x \in [a, d)$, the integral equation becomes

$$\begin{aligned} u(x) &= v_1(x) + \int_a^b G(x,s)\bigl[f\bigl(s, \gamma\bigl(s, u(s)\bigr)\bigr) - \gamma\bigl(s, u(s)\bigr)\bigr]\,ds \\ &= v_1(x) + \int_a^x \frac{(\exp(s-a) - \exp(a-s))(\exp(x-b) - \exp(b-x))[f(s,\gamma) - \gamma]}{W(s)}\,ds \\ &\quad + \int_x^d \frac{(\exp(x-a) - \exp(a-x))(\exp(s-b) - \exp(b-s))[f(s,\gamma) - \gamma]}{W(s)}\,ds \\ &\quad + \int_d^b \frac{(\exp(x-a) - \exp(a-x))(\exp(s-b) - \exp(b-s))[f(s,\gamma) - \gamma]}{W(s)}\,ds. \end{aligned}$$

If $x \in (d, b]$, the integral equation becomes

$$\begin{aligned} u(x) &= v_2(x) + \int_a^b G(x,s)\bigl[f\bigl(s, \gamma\bigl(s, u(s)\bigr)\bigr) - \gamma\bigl(s, u(s)\bigr)\bigr]\,ds \\ &= v_2(x) + \int_a^d \frac{(\exp(s-a) - \exp(a-s))(\exp(x-b) - \exp(b-x))[f(s,\gamma) - \gamma]}{W(s)}\,ds \\ &\quad + \int_d^x \frac{(\exp(s-a) - \exp(a-s))(\exp(x-b) - \exp(b-x))[f(s,\gamma) - \gamma]}{W(s)}\,ds \\ &\quad + \int_x^b \frac{(\exp(x-a) - \exp(a-x))(\exp(s-b) - \exp(b-s))[f(s,\gamma) - \gamma]}{W(s)}\,ds. \end{aligned}$$

In order that the solutions of the two problems satisfy the interface conditions (2.7) at $x = d$, we must have

$$u'\bigl(d^+\bigr) - u'\bigl(d^-\bigr) = c_1, \tag{2.17}$$

$$u\bigl(d^+\bigr) - u\bigl(d^-\bigr) = c_2. \tag{2.18}$$

Substituting the integral equations into (2.17) and (2.18), and considering (2.11) and (2.13), we must have

$$\begin{aligned} &p_1 - p_2 = A, \\ &p_3 - p_4 = B, \\ &p_3 \exp(d-b) + p_4 \exp(b-d) - p_1 \exp(d-a) - p_2 \exp(a-d) = c_1, \\ &p_3 \exp(d-b) - p_4 \exp(b-d) - p_1 \exp(d-a) + p_2 \exp(a-d) = c_2. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} p_1 &= \frac{-(c_1 + c_2)\exp(b-d) + (c_1 - c_2)\exp(d-b) - 2A\exp(a-b) + 2B}{2(\exp(b-a) - \exp(a-b))}, \\ p_2 &= \frac{-(c_1 + c_2)\exp(b-d) + (c_1 - c_2)\exp(d-b) - 2A\exp(b-a) + 2B}{2(\exp(b-a) - \exp(a-b))}, \end{aligned}$$

$$p_3 = \frac{-(c_1 + c_2)\exp(a-d) + (c_1 - c_2)\exp(d-a) + 2B\exp(b-a) - 2A}{2(\exp(b-a) - \exp(a-b))},$$
$$p_4 = \frac{-(c_1 + c_2)\exp(a-d) + (c_1 - c_2)\exp(d-a) + 2B\exp(a-b) - 2A}{2(\exp(b-a) - \exp(a-b))}.$$

Define an operator $T : Q^1[a,b] \to Q^1[a,b]$ as follows:

$$(Tu)(x) = v(x) + \int_a^b G(x,s)\big[f\big(s, \gamma\big(s, u(s)\big)\big) - \gamma\big(s, u(s)\big)\big]\,ds.$$

Observe that $f(x, \gamma(x,u)) - \gamma(x,u) : ([a,d) \cup (d,b]) \times \mathbb{R} \to \mathbb{R}$ is uniformly bounded in $u \in Q^1[a,b]$. It follows that from Lemma 2.1 that the set $T(Q^1[a,b])$ is a relatively compact subset of $Q^1[a,b]$. Moreover, T is continuous. Hence, it follows from the Schauder fixed point theorem that T has at least one fixed point $u(x) \in Q^1[a,b]$.

We are now ready to prove that each solution $u(x)$ of the problem (2.6)-(2.8) satisfies $\alpha(x) \le u(x) \le \beta(x)$ for $x \in [a,d) \cup (d,b]$.

Let us first prove that $u(x) \le \beta(x)$. Suppose, on the contrary, that the function $h(x) = u(x) - \beta(x)$ has a positive maximum at some $x_0 \in [a,d) \cup (d,b]$. First, if $x_0 \ne d^{\pm}, x_i^{\pm}$, we have $h(x_0) > 0$, $h'(x_0) = 0$, $h''(x_0) \le 0$, then

$$\begin{aligned} h''(x_0) &= u''(x_0) - \beta''(x_0) \\ &\ge u(x_0) + f\big(x_0, \gamma\big(x_0, u(x_0)\big)\big) - \gamma\big(x_0, u(x_0)\big) - f\big(x_0, \beta(x_0)\big) \\ &= u(x_0) - \gamma\big(x_0, u(x_0)\big) + f\big(x_0, \gamma\big(x_0, u(x_0)\big)\big) - f\big(x_0, \beta(x_0)\big) \\ &> 0, \end{aligned}$$

which yields a contradiction. Second, if $x_0 = d^-$ or $x_0 = d^+$, we have

$$h\big(d^+\big) - h\big(d^-\big) = u\big(d^+\big) - \beta\big(d^+\big) - u\big(d^-\big) + \beta\big(d^-\big) = 0,$$

so we can see that the function $h(x) = u(x) - \beta(x)$ is continuous. Then we have

$$h'\big(d^+\big) - h'\big(d^-\big) \le 0.$$

But

$$\begin{aligned} h'\big(d^+\big) - h'\big(d^-\big) &= u'\big(d^+\big) - \beta'\big(d^+\big) - u'\big(d^-\big) + \beta'\big(d^-\big) \\ &= u'\big(d^+\big) - u'\big(d^-\big) - \big(\beta'\big(d^+\big) - \beta'\big(d^-\big)\big) \\ &> 0, \end{aligned}$$

which yields also a contradiction. Third, if $x_0 = x_i^-$ or $x_0 = x_i^+$, we can similarly reach a contradiction. Therefore, we prove that $u(x) \le \beta(x)$ for all $x \in [a,d) \cup (d,b]$.

In a similar way, we can prove that $\alpha(x) \le u(x)$ for all $x \in [a,d) \cup (d,b]$.

Therefore, the solution of (2.6)-(2.8) is also that of (2.1)-(2.3) and satisfies $\alpha(x) \le u(x) \le \beta(x)$ for $x \in [a,d) \cup (d,b]$. The proof of the lemma is completed. □

Lemma 2.3 *Assume that function $f(x,u)$ is given in* (1.1)-(1.3), *and $f(x,u)$ is strictly increasing with respect to u. Then the problem* (2.1)-(2.3) *has at most one solution.*

Proof Assume u and w are distinct solutions of the given problem (2.1)-(2.3). Without loss of generality, we can assume that there is some $x \in [a,d) \cup (d,b]$ such that $u(x) > w(x)$. Let

$$h(x) = \begin{cases} u(x) - w(x), & x \in [a,d) \cup (d,b], \\ u(d^+) - w(d^+), & x = d. \end{cases}$$

Since

$$\begin{aligned} h(d^+) - h(d^-) &= u(d^+) - w(d^+) - u(d^-) + w(d^-) \\ &= u(d^+) - u(d^-) - \bigl(w(d^+) - w(d^-)\bigr) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} h'(d^+) - h'(d^-) &= u'(d^+) - w'(d^+) - u'(d^-) + w'(d^-) \\ &= u'(d^+) - u'(d^-) - \bigl(w'(d^+) - w'(d^-)\bigr) \\ &= 0, \end{aligned}$$

we have $h(x) \in C^1[a,b]$. Owing to $h(a) = h(b) = 0$, $h(x)$ achieves the positive maximum at some $x_0 \in (a,b)$.

First, if $x_0 \neq d$, we have

$$h(x_0) > 0, \qquad h'(x_0) = 0, \qquad h''(x_0) \leq 0.$$

However, on the other hand,

$$\begin{aligned} h''(x_0) &= u''(x_0) - w''(x_0) \\ &= f\bigl(x_0, u(x_0)\bigr) - f\bigl(x_0, w(x_0)\bigr) \\ &> 0. \end{aligned}$$

This is a contradiction.

Second, if $x_0 = d$, noting that $h(a) = 0$ and $h(d) > 0$, there exists some $x^* \in [a,d)$ such that $h(x) > 0$ for $x \in [x^*, d]$ and $h'(x^*) \geq 0$. However, on the other hand,

$$0 \geq h'(d) - h'(x^*) = \int_{x^*}^{d} \bigl[u''(x) - w''(x)\bigr]\,dx > 0,$$

which yields a contradiction. Then the proof is complete. □

3 Main results

In this section, we are interested in the asymptotic behavior of solution with respect to the small parameter ε, as well as the existence and uniqueness for the problem (1.1)-(1.3). For the sake of simplicity, we only consider the approximation of zero order.

Because of discontinuity at $x = d$, the original problem (1.1)-(1.3) can be viewed as the coupling of the left problem

$$\varepsilon^2 u_L'' = f_1(x, u_L), \quad a < x < d, \tag{3.1}$$

$$u_L(a) = A, \qquad u_L(d) = \gamma_L(\varepsilon), \tag{3.2}$$

and the right problem

$$\varepsilon^2 u_R'' = f_2(x, u_R), \quad d < x < b, \tag{3.3}$$

$$u_R(d) = \gamma_R(\varepsilon), \qquad u_R(b) = B, \tag{3.4}$$

where $\gamma_L(\varepsilon)$, $\gamma_R(\varepsilon)$ are constants dependent on ε which will be determined later.

We first make the following basic assumptions.

(H1) The functions $f_1 \in C^2([a,d] \times \mathbb{R})$, $f_2 \in C^2([d,b] \times \mathbb{R})$, and

$$f_1(d,u) \neq f_2(d,u), \quad \text{for } u \in \mathbb{R},$$
$$\frac{\partial f_1}{\partial u}(x,u) \geq \sigma_0 > 0, \quad \text{for } (x,u) \in [a,d] \times \mathbb{R},$$
$$\frac{\partial f_2}{\partial u}(x,u) \geq \sigma_0 > 0, \quad \text{for } (x,u) \in [d,b] \times \mathbb{R}.$$

(H2) The reduced problem $f(x,u) = 0$ has a solution

$$u(x) = \begin{cases} \varphi(x), & x \in [a,d), \\ \psi(x), & x \in (d,b], \end{cases}$$

such that $\varphi(x) \in C^2[a,d]$ and $\psi(x) \in C^2[d,b]$.

Let us construct the formal asymptotic solution of the left problem (3.1)-(3.2). We seek the formal solution of the form

$$u_L(x,\varepsilon) = U_L(x) + Q_L(\tau) + V_L(\eta), \quad \tau = (x-a)/\varepsilon, \eta = (x-d)/\varepsilon, \tag{3.5}$$

where

$$U_L(x) = u_0^{(-)}(x) + \varepsilon u_1^{(-)}(x) + \varepsilon^2 u_2^{(-)}(x) + \cdots,$$
$$Q_L(\tau) = Q_0^{(-)}(\tau) + \varepsilon Q_1^{(-)}(\tau) + \varepsilon^2 Q_2^{(-)}(\tau) + \cdots,$$
$$V_L(\eta) = V_0^{(-)}(\eta) + \varepsilon V_1^{(-)}(\eta) + \varepsilon^2 V_2^{(-)}(\eta) + \cdots,$$
$$\gamma_L(\varepsilon) = \gamma_0^{(-)} + \varepsilon\gamma_1^{(-)} + \varepsilon^2\gamma_2^{(-)} + \cdots.$$

Putting (3.5) into (3.1) and equating the coefficients of like powers of ε we can obtain a series of recursive equations determining these coefficients. From (H2) it follows that $u_0^{(-)}(x) = \varphi(x)$, $x \in [a,d)$. The left boundary layer term $Q_0^{(-)}(\tau)$ solves

$$\frac{d^2 Q_0^{(-)}}{d\tau^2} = f_1\bigl(a, \varphi(a) + Q_0^{(-)}(\tau)\bigr), \tag{3.6}$$

subject to the boundary conditions

$$Q_0^{(-)}(0) = A - \varphi(a), \qquad \lim_{\tau\to+\infty} Q_0^{(-)}(\tau) = 0. \tag{3.7}$$

The right boundary layer term $V_0^{(-)}(\eta)$ solves

$$\frac{d^2V_0^{(-)}}{d\eta^2} = f_1\bigl(d, \varphi(d) + V_0^{(-)}(\eta)\bigr), \tag{3.8}$$

subject to the boundary conditions

$$V_0^{(-)}(0) = \gamma_0^{(-)} - \varphi(d), \qquad \lim_{\eta\to-\infty} V_0^{(-)}(\eta) = 0. \tag{3.9}$$

Therefore, the formal approximation of zero order for the left problem (3.1)-(3.2) reads

$$\tilde{u}_L(x,\varepsilon) = \varphi(x) + Q_0^{(-)}(\tau) + V_0^{(-)}(\eta), \quad x \in [a,d].$$

In a completely similar way, we can get the formal approximation of zero order for the right problem (3.3)-(3.4), where $Q_0^{(+)}(\eta)$ solves

$$\frac{d^2Q_0^{(+)}}{d\eta^2} = f_2\bigl(d, \psi(d) + Q_0^{(+)}(\eta)\bigr), \tag{3.10}$$

$$Q_0^{(+)}(0) = \gamma_0^{(+)} - \psi(d), \qquad \lim_{\eta\to+\infty} Q_0^{(+)}(\eta) = 0, \tag{3.11}$$

and $V_0^{(+)}(\xi)$ solves

$$\frac{d^2V_0^{(+)}}{d\xi^2} = f_2\bigl(b, \psi(b) + V_0^{(+)}(\xi)\bigr), \tag{3.12}$$

$$V_0^{(+)}(0) = B - \psi(b), \qquad \lim_{\xi\to-\infty} V_0^{(+)}(\xi) = 0. \tag{3.13}$$

The following two lemmas are concerned with the asymptotic behavior of the boundary layer terms for the left problem, whose proofs are essentially similar to that of Lemma 3.2 in [13] and thus are omitted here.

Lemma 3.1 *Under the assumptions* (H1) *and* (H2), *for sufficiently small* $\varepsilon > 0$, *the problem* (3.6) *and* (3.7) *has a solution* $Q_0^{(-)}(\tau)$ *satisfying the following estimate*:

$$\bigl|Q_0^{(-)}(\tau)\bigr| \le \bigl|A - \varphi(a)\bigr| e^{-\sqrt{\sigma_0}\tau}, \quad \tau > 0.$$

Lemma 3.2 *Under the assumptions* (H1) *and* (H2), *for sufficiently small* $\varepsilon > 0$ *the problem* (3.8) *and* (3.9) *has a solution* $V_0^{(-)}(\eta)$ *satisfying the following estimate*:

$$\bigl|V_0^{(-)}(\eta)\bigr| \le \bigl|\gamma_0^{(-)} - \varphi(d)\bigr| e^{\sqrt{\sigma_0}\eta}, \quad \eta < 0.$$

In a similar way, we can prove that $Q_0^{(+)}(\eta)$ and $V_0^{(+)}(\xi)$ satisfy the following estimates:

$$\left|Q_0^{(+)}(\eta)\right| \le \left|\gamma_0^{(+)} - \psi(d)\right| \mathrm{e}^{-\sqrt{\sigma_0}\eta}, \quad \eta > 0$$

and

$$\left|V_0^{(+)}(\xi)\right| \le \left|B - \psi(b)\right| \mathrm{e}^{\sqrt{\sigma_0}\xi}, \quad \xi < 0.$$

In order that the solutions of the two problems satisfy the interface conditions (1.2) at $x = d$, we must have

$$\varepsilon \frac{du_R}{dx}(d,\varepsilon) - \varepsilon \frac{du_L}{dx}(d,\varepsilon) = c_1, \tag{3.14}$$

$$u_R(d) - u_L(d) = c_2. \tag{3.15}$$

By substituting the formal solutions into (3.14) we have

$$\begin{aligned} &\left.\frac{dQ_0^{(+)}}{d\eta}\right|_{\eta=0} - \left.\frac{dV_0^{(-)}}{d\eta}\right|_{\eta=0} = c_1, \\ &\left.\frac{du_0^{(+)}}{dx}\right|_{x=d} + \left.\frac{dQ_1^{(+)}}{d\eta}\right|_{\eta=0} = \left.\frac{du_0^{(-)}}{dx}\right|_{x=d} + \left.\frac{dV_1^{(-)}}{d\eta}\right|_{\eta=0}, \\ &\ldots. \end{aligned} \tag{3.16}$$

In view of (3.9), multiplying (3.8) by $2\frac{dV_0^{(-)}}{d\eta}$ and integrating from $-\infty$ to 0 we obtain

$$\begin{aligned} \left.\left(\frac{dV_0^{(-)}}{d\eta}\right)^2\right|_{\eta=0} &= 2\int_{-\infty}^{0} f_1\bigl(d, \varphi(d) + V_0^{(-)}(\eta)\bigr)\left(\frac{dV_0^{(-)}}{d\eta}\right) d\eta \\ &= 2\int_{-\infty}^{0} \frac{\partial f_1}{\partial u}\bigl(d, \varphi(d)\bigr) V_0^{(-)}(\eta)\left(\frac{dV_0^{(-)}}{d\eta}\right) d\eta + \Delta_1 \\ &= \frac{\partial f_1}{\partial u}\bigl(d, \varphi(d)\bigr)\bigl(\gamma_0^{(-)} - \varphi(d)\bigr)^2 + \Delta_1, \end{aligned} \tag{3.17}$$

where

$$\Delta_1 = \int_{-\infty}^{0} \frac{\partial^2 f_1}{\partial u^2}\bigl(d, \varphi(d) + \vartheta_1 V_0^{(-)}(\eta)\bigr)\bigl(V_0^{(-)}(\eta)\bigr)^2\left(\frac{dV_0^{(-)}}{d\eta}\right) d\eta, \quad 0 < \vartheta_1 < 1.$$

Analogously, it follows from (3.10)-(3.11) that

$$\begin{aligned} \left.\frac{dQ_0^{(+)}}{d\eta}\right|_{\eta=0} &= \int_{+\infty}^{0} f_2\bigl(d, \psi(d) + Q_0^{(+)}(\eta)\bigr) d\eta \\ &= \frac{\partial f_2}{\partial u}\bigl(d, \psi(d)\bigr)\bigl(\gamma_0^{(+)} - \psi(d)\bigr)^2 + \Delta_2, \end{aligned} \tag{3.18}$$

where

$$\Delta_2 = \int_{+\infty}^{0} \frac{\partial^2 f_2}{\partial u^2}\bigl(d, \psi(d) + \vartheta_2 Q_0^{(+)}(\eta)\bigr)\bigl(Q_0^{(+)}(\eta)\bigr)^2\left(\frac{dQ_0^{(+)}}{d\eta}\right) d\eta, \quad 0 < \vartheta_2 < 1.$$

Noting that $V_0^{(-)}(\eta)\frac{dV_0^{(-)}}{d\eta} > 0$ for $\eta \le 0$ and $Q_0^{(+)}(\eta)\frac{dQ_0^{(+)}}{d\eta} < 0$ for $\eta \ge 0$ we have from (3.17) and (3.18)

$$\left.\frac{dV_0^{(-)}}{d\eta}\right|_{\eta=0} = \operatorname{sgn}\bigl(\gamma_0^{(-)} - \varphi(d)\bigr)\sqrt{\frac{\partial f_1}{\partial u}\bigl(d,\varphi(d)\bigr)\bigl(\gamma_0^{(-)} - \varphi(d)\bigr)^2 + \Delta_1}, \tag{3.19}$$

$$\left.\frac{dQ_0^{(+)}}{d\eta}\right|_{\eta=0} = \operatorname{sgn}\bigl(\psi(d) - \gamma_0^{(+)}\bigr)\sqrt{\frac{\partial f_2}{\partial u}\bigl(d,\psi(d)\bigr)\bigl(\gamma_0^{(+)} - \psi(d)\bigr)^2 + \Delta_2}. \tag{3.20}$$

Putting (3.19)-(3.20) into (3.14) yields

$$\begin{aligned}&\operatorname{sgn}\bigl(\psi(d) - \gamma_0^{(+)}\bigr)\sqrt{\frac{\partial f_2}{\partial u}\bigl(d,\psi(d)\bigr)\bigl(\gamma_0^{(+)} - \psi(d)\bigr)^2 + \Delta_2}\\ &\quad - \operatorname{sgn}\bigl(\gamma_0^{(-)} - \varphi(d)\bigr)\sqrt{\frac{\partial f_1}{\partial u}\bigl(d,\varphi(d)\bigr)\bigl(\gamma_0^{(-)} - \varphi(d)\bigr)^2 + \Delta_1} = c_1.\end{aligned} \tag{3.21}$$

Substituting the formal solutions into (3.15) we have

$$\gamma_0^{(+)} - \gamma_0^{(-)} = c_2. \tag{3.22}$$

Noting that $\gamma_0^{(+)}$, $\gamma_0^{(-)}$ can be determined by (3.21) and (3.22). We suppose that:
(H3) The system of equations (3.21) and (3.22) has a solution $(\gamma_0^{(+)}, \gamma_0^{(-)})$.
Likewise, $\gamma_i^{(\pm)}$ $(i \ge 1)$ can be determined recursively.

Thus, we obtain the formal asymptotic solution of the problem (1.1)-(1.3). Now we are in a position to state our main result.

Theorem 3.1 *Let the conditions* (H1)-(H3) *hold. Then for sufficiently small* $\varepsilon > 0$ *the boundary value problem* (1.1)-(1.3) *has a unique solution* $u(x,\varepsilon) \in Q^1[a,b]$ *satisfying*:

$$u(x,\varepsilon) = \tilde{u}(x,\varepsilon) + O(\varepsilon),$$

where

$$\tilde{u}(x,\varepsilon) = \begin{cases} \tilde{u}_L(x,\varepsilon) = \varphi(x) + Q_0^{(-)}(\tau) + V_0^{(-)}(\eta), & x \in [a,d), \\ \tilde{u}_R(x,\varepsilon) = \psi(x) + Q_0^{(+)}(\eta) + V_0^{(+)}(\xi), & x \in (d,b]. \end{cases}$$

Proof In order to prove our main result we need to construct suitable upper and lower solutions. To this end, we need to make some necessary preparations.

Firstly, from the construction of asymptotic solution and the assumption (H1)-(H3) we know that there is a positive constant M such that

$$\begin{aligned}&\bigl|\varphi''(x)\bigr| \le M, \quad x \in [a,d),\\ &\bigl|f_1\bigl(a+\varepsilon\tau, \varphi(a+\varepsilon\tau) + Q_0^{(-)}\bigr) - f_1\bigl(a, \varphi(a) + Q_0^{(-)}\bigr)\bigr| \le M\varepsilon\tau \mathrm{e}^{-\sqrt{\sigma_0}\tau}, \quad x \in [a,d),\\ &\bigl|f_1\bigl(d+\varepsilon\eta, \varphi(d+\varepsilon\eta) + V_0^{(-)}\bigr) - f_1\bigl(d, \varphi(d) + V_0^{(-)}\bigr)\bigr| \le M\varepsilon\eta \mathrm{e}^{\sqrt{\sigma_0}\eta}, \quad x \in [a,d),\\ &\bigl|\psi''(x)\bigr| \le M, \quad x \in (d,b],\\ &\bigl|f_2\bigl(d+\varepsilon\eta, \psi(d+\varepsilon\eta) + Q_0^{(+)}\bigr) - f_2\bigl(d, \psi(d) + Q_0^{(+)}\bigr)\bigr| \le M\varepsilon\eta \mathrm{e}^{-\sqrt{\sigma_0}\eta}, \quad x \in (d,b],\\ &\bigl|f_2\bigl(b+\varepsilon\xi, \psi(d+\varepsilon\xi) + V_0^{(+)}\bigr) - f_2\bigl(b, \psi(b) + V_0^{(+)}\bigr)\bigr| \le M\varepsilon\xi \mathrm{e}^{\sqrt{\sigma_0}\xi}, \quad x \in (d,b],\end{aligned}$$

and, moreover,

$$\varepsilon\tilde{u}_R'(d,\varepsilon) - \varepsilon\tilde{u}_L'(d,\varepsilon) = c_1, \qquad \tilde{u}_R(d,\varepsilon) - \tilde{u}_L(d,\varepsilon) = c_2. \tag{3.23}$$

Now we select the barrier functions as follows:

$$\alpha(x) = \begin{cases} \tilde{u}_L(x,\varepsilon) - \varepsilon Q_1^{(-)}(\tau) - \Gamma_{L1}(x)\varepsilon - \lambda_1\varepsilon, & x \in [a,x_1), \\ \tilde{u}_L(x,\varepsilon) - \varepsilon V_1^{(-)}(\eta) - \Gamma_{L2}(x)\varepsilon - \lambda_2\varepsilon, & x \in [x_1,d), \\ \tilde{u}_R(x,\varepsilon) - \varepsilon Q_1^{(+)}(\eta) - \Gamma_{R1}(x)\varepsilon - \lambda_3\varepsilon, & x \in (d,x_2], \\ \tilde{u}_R(x,\varepsilon) - \varepsilon V_1^{(+)}(\xi) - \Gamma_{R2}(x)\varepsilon - \lambda_4\varepsilon, & x \in (x_2,b], \end{cases}$$

$$\beta(x) = \begin{cases} \tilde{u}_L(x,\varepsilon) + \varepsilon Q_1^{(-)}(\tau) + \Gamma_{L1}(x)\varepsilon + \lambda_1\varepsilon, & x \in [a,x_1), \\ \tilde{u}_L(x,\varepsilon) + \varepsilon V_1^{(-)}(\eta) + \Gamma_{L2}(x)\varepsilon + \lambda_2\varepsilon, & x \in [x_1,d), \\ \tilde{u}_R(x,\varepsilon) + \varepsilon Q_1^{(+)}(\eta) + \Gamma_{R1}(x)\varepsilon + \lambda_3\varepsilon, & x \in (d,x_2], \\ \tilde{u}_R(x,\varepsilon) + \varepsilon V_1^{(+)}(\xi) + \Gamma_{R2}(x)\varepsilon + \lambda_4\varepsilon, & x \in (x_2,b], \end{cases}$$

where

$$\Gamma(x) = \begin{cases} \Gamma_{L1}(x) = \frac{M+1}{4\sigma_0^2}\mathrm{e}^{\frac{\sigma_0(x-x_1)}{\varepsilon}}, & x \in [a,x_1), \\ \Gamma_{L2}(x) = \frac{M+1}{4\sigma_0^2}[\mathrm{e}^{\frac{\sigma_0(x-d)}{\varepsilon}} + \mathrm{e}^{\frac{\sigma_0(x_1-x)}{\varepsilon}}], & x \in [x_1,d), \\ \Gamma_{R1}(x) = \frac{M+1}{4\sigma_0^2}[\mathrm{e}^{\frac{\sigma_0(d-x)}{\varepsilon}} + \mathrm{e}^{\frac{\sigma_0(x-x_2)}{\varepsilon}}], & x \in (d,x_2], \\ \Gamma_{R2}(x) = \frac{M+1}{4\sigma_0^2}\mathrm{e}^{\frac{\sigma_0(x_2-x)}{\varepsilon}}, & x \in (x_2,b] \end{cases}$$

and

$$Q_1^{(-)}(\tau) = \frac{M\tau(\sigma_0\tau + \sqrt{\sigma_0})\mathrm{e}^{-\sqrt{\sigma_0}\tau}}{4\sigma_0^{3/2}}, \quad x \in [a,d).$$

$$V_1^{(-)}(\eta) = \frac{M\eta(\sigma_0\eta - \sqrt{\sigma_0})\mathrm{e}^{\sqrt{\sigma_0}\eta}}{4\sigma_0^{3/2}}, \quad x \in [a,d),$$

$$Q_1^{(+)}(\eta) = \frac{M\eta(\sigma_0\eta + \sqrt{\sigma_0})\mathrm{e}^{-\sqrt{\sigma_0}\eta}}{4\sigma_0^{3/2}}, \quad x \in (d,b],$$

$$V_1^{(+)}(\xi) = \frac{M\xi(\sigma_0\xi - \sqrt{\sigma_0})\mathrm{e}^{\sqrt{\sigma_0}\xi}}{4\sigma_0^{3/2}}, \quad x \in (d,b],$$

with $x_1 = \frac{a+d}{2}$ and $x_2 = \frac{b+d}{2}$, and positive constants λ_i ($i = 1,2,3,4$) to be determined later.

It is easily verified that $Q_1^{(-)}(\tau)$, $V_1^{(-)}(\eta)$, $Q_1^{(+)}(\eta)$, and $V_1^{(+)}(\xi)$ have the following properties:

(i) $Q_1^{(-)}(\tau)$, $V_1^{(-)}(\eta)$, $Q_1^{(+)}(\eta)$ and $V_1^{(+)}(\xi)$ are nonnegative functions;

(ii)

$$\left.\frac{dV_1^{(-)}}{d\eta}\right|_{\eta=0} = -\frac{M}{4\sigma_0} < 0, \qquad \left.\frac{dQ_1^{(+)}}{d\eta}\right|_{\eta=0} = \frac{M}{4\sigma_0} > 0;$$

(iii) $V_1^{(-)}(\eta)$ and $V_1^{(+)}(\xi)$ are solutions of the equation

$$-\frac{d^2y(t)}{dt^2} + \sigma_0 y(t) + Mt\mathrm{e}^{\sqrt{\sigma_0}t} = 0,$$

and $Q_1^{(-)}(\tau)$, $Q_1^{(+)}(\eta)$ are solutions of the equation

$$-\frac{d^2y(t)}{dt^2}+\sigma_0y(t)-Mte^{-\sqrt{\sigma_0}t}=0.$$

In order to make sure the functions α and β are continuous at x_1, x_2 and meet

$$\alpha(d^+)-\alpha(d^-)=c_2,\qquad \beta(d^+)-\beta(d^-)=c_2,$$

the constants λ_i $(i=1,2,3,4)$ that we select must satisfy

$$\begin{aligned}
&\lambda_1=\lambda_2+\frac{M+1}{4\sigma_0^2}e^{\frac{\sigma_0(x_1^+-d)}{\varepsilon}}=\lambda_2+O(\varepsilon^2),\\
&\lambda_4=\lambda_3+\frac{M+1}{4\sigma_0^2}e^{\frac{\sigma_0(d-x_2^+)}{\varepsilon}}=\lambda_3+O(\varepsilon^2),\\
&\lambda_2+\frac{M+1}{4\sigma_0^2}e^{\frac{\sigma_0(x_1^+-d)}{\varepsilon}}=\lambda_3+\frac{M+1}{4\sigma_0^2}e^{\frac{\sigma_0(d-x_2^+)}{\varepsilon}}.
\end{aligned}$$

With (3.23) and the definitions of α and β, we easily verify that

$$\begin{aligned}
&\alpha(x)<\beta(x),\quad x\in[a,d)\cup(d,b],\\
&\varepsilon\alpha'(d^+)-\varepsilon\alpha'(d^-)\ge c_1,\qquad \varepsilon\beta'(d^+)-\varepsilon\beta'(d^-)\le c_1,\\
&\alpha(d^+)-\alpha(d^-)=c_2,\qquad \beta(d^+)-\beta(d^-)=c_2,\\
&\alpha'(x_i^-)<\alpha'(x_i^+),\qquad \beta'(x_i^-)>\beta'(x_i^+),\quad i=1,2,\\
&\alpha(a)\le A\le\beta(a),\qquad \alpha(b)\le B\le\beta(b).
\end{aligned}$$

Using the above properties (i)-(iii) and the assumptions (H1)-(H3), if $x\in(a,x_1)$ we have

$$\begin{aligned}
&\varepsilon^2\alpha''(x)-f_1(x,\alpha(x))\\
&\quad=\varepsilon^2\varphi''(x)+\frac{d^2Q_0^{(-)}}{d\tau^2}+\frac{d^2V_0^{(-)}}{d\eta^2}-\varepsilon\frac{d^2Q_1^{(-)}}{d\tau^2}-\varepsilon^3\Gamma_{L1}''(x)-f_1(x,\alpha(x))\\
&\quad=\varepsilon^2\varphi''(x)+\frac{d^2V_0^{(-)}}{d\eta^2}+f_1(a,\varphi(a)+Q_0^{(-)}(\tau))-f_1(a+\varepsilon\tau,\varphi(a+\varepsilon\tau)+Q_0^{(-)}(\tau))\\
&\qquad+f_1(x,\varphi(x)+Q_0^{(-)}(\tau))-f_1(x,\alpha(x))-\varepsilon\frac{d^2Q_1^{(-)}}{d\tau^2}-\varepsilon^3\Gamma_{L1}''(x)\\
&\quad\ge-M\varepsilon^2+\frac{\partial f_1}{\partial u}(a,\varphi(a)+\theta_1V_0^{(-)}(\eta))V_0^{(-)}(\eta)-M\varepsilon\tau e^{-\sqrt{\sigma_0}\tau}-\varepsilon\frac{d^2Q_1^{(-)}}{d\tau^2}-\varepsilon^3\Gamma_{L1}''(x)\\
&\qquad+\frac{\partial f_1}{\partial u}(x,\varphi(x)+Q_0^{(-)}(\tau)+\theta_2\vartheta_3)(\lambda_1\varepsilon+\varepsilon\Gamma_{L1}+\varepsilon Q_1^{(-)}(\tau)-|V_0^{(-)}(\eta)|)\\
&\quad\ge-M\varepsilon^2-\sigma_1|\gamma_0^{(-)}-\varphi(d)|e^{\sqrt{\sigma_0}\eta}-M\varepsilon\tau e^{-\sqrt{\sigma_0}\tau}-\varepsilon\frac{d^2Q_1^{(-)}}{d\tau^2}-\varepsilon^3\Gamma_{L2}''(x)\\
&\qquad+\sigma_0(\lambda_1\varepsilon+\varepsilon\Gamma_{L1}+\varepsilon Q_1^{(-)}(\tau)-|\gamma_0^{(-)}-\varphi(d)|e^{\sqrt{\sigma_0}\eta})\\
&\quad\ge(\lambda_1+\Gamma_{L1}(x)-M\varepsilon-\varepsilon^2\Gamma_{L1}''(x))\varepsilon-(\sigma_0+\sigma_1)|\gamma_0^{(-)}-\varphi(d)|e^{\sqrt{\sigma_0}\eta}
\end{aligned}$$

$$
\begin{aligned}
&= \bigl(\lambda_1 + \bigl(1-\sigma_0^2\bigr)\Gamma_{L1}(x) - M\varepsilon\bigr)\varepsilon - O\bigl(\varepsilon^2\bigr) \\
&\geq 0,
\end{aligned}
$$

provided ε is small enough and λ_1 is large enough, where $0 \geq \theta_1, \theta_2 \geq 1$, and

$$\vartheta_3 = \lambda_1\varepsilon + \varepsilon\Gamma_{L1} + \varepsilon Q_1^{(-)}(\tau) - V_0^{(-)}(\eta).$$

If $x \in (x_1, d)$ we have

$$
\begin{aligned}
&\varepsilon^2\alpha''(x) - f_1\bigl(x, \alpha(x)\bigr) \\
&= \varepsilon^2\varphi''(x) + \frac{d^2Q_0^{(-)}}{d\tau^2} + \frac{d^2V_0^{(-)}}{d\eta^2} - \varepsilon\frac{d^2V_1^{(-)}}{d\eta^2} - \varepsilon^3\Gamma_{L2}''(x) - f_1\bigl(x, \alpha(x)\bigr) \\
&= \varepsilon^2\varphi''(x) + \frac{d^2Q_0^{(-)}}{d\tau^2} + f_1\bigl(d, \varphi(d) + V_0^{(-)}(\eta)\bigr) - f_1\bigl(d + \varepsilon\eta, \varphi(d+\varepsilon\eta) + V_0^{(-)}(\eta)\bigr) \\
&\quad + f_1\bigl(x, \varphi(x) + V_0^{(-)}(\eta)\bigr) - f_1\bigl(x, \alpha(x)\bigr) - \varepsilon\frac{d^2V_1^{(-)}}{d\eta^2} - \varepsilon^3\Gamma_{L2}''(x) \\
&\geq -M\varepsilon^2 + \frac{\partial f_1}{\partial u}\bigl(d, \varphi(d) + \theta_3 Q_0^{(-)}(\tau)\bigr)Q_0^{(-)}(\tau) + M\varepsilon\eta \mathrm{e}^{\sqrt{\sigma_0}\eta} - \varepsilon\frac{d^2V_1^{(-)}}{d\eta^2} - \varepsilon^3\Gamma_{L2}''(x) \\
&\quad + \frac{\partial f_1}{\partial u}\bigl(x, \varphi(x) + V_0^{(-)}(\eta) + \theta_4\vartheta_4\bigr)\bigl(\lambda_2\varepsilon + \varepsilon\Gamma_{L2} + \varepsilon V_1^{(-)}(\eta) - \bigl|Q_0^{(-)}(\tau)\bigr|\bigr) \\
&\geq -M\varepsilon^2 - \sigma_1\bigl|A - \varphi(a)\bigr|\mathrm{e}^{-\sqrt{\sigma_0}\tau} + M\varepsilon\eta\mathrm{e}^{\sqrt{\sigma_0}\eta} - \varepsilon\frac{d^2V_1^{(-)}}{d\eta^2} - \varepsilon^3\Gamma_{L2}''(x) \\
&\quad + \sigma_0\bigl(\lambda_2\varepsilon + \varepsilon\Gamma_{L2} + \varepsilon V_1^{(-)}(\eta) - \bigl|A - \varphi(a)\bigr|\mathrm{e}^{-\sqrt{\sigma_0}\tau}\bigr) \\
&\geq \bigl(\lambda_2 + \Gamma_{L2}(x) - M\varepsilon - \varepsilon^2\Gamma_{L2}''(x)\bigr)\varepsilon - (\sigma_0 + \sigma_1)\bigl|A - \varphi(a)\bigr|\mathrm{e}^{-\sqrt{\sigma_0}\tau} \\
&= \bigl(\lambda_2 + \bigl(1-\sigma_0^2\bigr)\Gamma_{L2}(x) - M\varepsilon\bigr)\varepsilon - O\bigl(\varepsilon^2\bigr) \\
&\geq 0,
\end{aligned}
$$

provided ε is small enough and λ_2 is large enough, where $0 \geq \theta_3, \theta_4 \geq 1$, and

$$\vartheta_4 = \lambda_2\varepsilon + \varepsilon\Gamma_{L2} + \varepsilon V_1^{(-)}(\eta) - Q_0^{(-)}(\tau).$$

Similarly, we have for ε small enough

$$\varepsilon^2\alpha''(x) - f_2\bigl(x, \alpha(x)\bigr) \geq 0, \quad x \in (d, b) \setminus x_2.$$

We can prove in a similar way that for sufficiently small $\varepsilon > 0$,

$$\varepsilon^2\beta''(x) - f\bigl(x, \beta(x)\bigr) \leq 0, \quad x \in (a, b) \setminus \{d, x_1, x_2\}.$$

It follows from Lemma 2.2 and Lemma 2.3 that the boundary value problem (1.1)-(1.3) has a unique solution $u(x, \varepsilon) \in Q^1[a, b]$ such that

$$\alpha(x) \leq u(x, \varepsilon) \leq \beta(x), \quad x \in [a, d) \cup (d, b].$$

The proof is completed. □

Finally, we present an example to illustrate our results. Consider an interface two-point boundary value problem:

$$\begin{aligned}
&\varepsilon^2 u''(x) = f(x,u), \quad x \in [0,1) \cup (1,2],\\
&\varepsilon u'(1^+) - \varepsilon u'(1^-) = 1,\\
&u(1^+) - u(1^-) = 1,\\
&u(0) = 1, \qquad u(2) = 2,
\end{aligned}$$

where

$$f(x,u) = \begin{cases} f_1(x,u) = u - x, & x \in [0,1),\\ f_2(x,u) = u + x, & x \in (1,2]. \end{cases}$$

The left reduced problem $f_1(x,u) = u - x = 0$ has a unique smooth solution

$$\varphi(x) = x, \quad x \in [0,1),$$

and the right reduced problem $f_2(x,u) = u + x = 0$ has a unique smooth solution

$$\psi(x) = -x, \quad x \in (1,2].$$

The problems (3.6)-(3.7), (3.8)-(3.9), (3.10)-(3.11), and (3.12)-(3.13) for the boundary layer terms have the solutions

$$Q_0^{(-)}(\tau) = \mathrm{e}^{-\tau}, \tag{3.24}$$

$$V_0^{(-)}(\eta) = \left(\gamma_0^{(-)} - 1\right)\mathrm{e}^{\eta}, \tag{3.25}$$

$$Q_0^{(+)}(\eta) = \left(\gamma_0^{(+)} + 1\right)\mathrm{e}^{-\eta}, \tag{3.26}$$

$$V_0^{(+)}(\xi) = 4\mathrm{e}^{\xi}. \tag{3.27}$$

Putting (3.24)-(3.27) into (3.21) and (3.16), we can see that

$$\gamma_0^{(-)} = -1, \qquad \gamma_0^{(+)} = 0.$$

By Theorem 3.1 the above problem has the unique solution

$$u(x,\varepsilon) = \tilde{u}(x,\varepsilon) + O(\varepsilon),$$

where

$$\tilde{u}(x,\varepsilon) = \begin{cases} x + \mathrm{e}^{-\frac{x}{\varepsilon}} - 2\mathrm{e}^{\frac{x-1}{\varepsilon}}, & x \in [0,1),\\ -x + \mathrm{e}^{-\frac{x-1}{\varepsilon}} + 4\mathrm{e}^{\frac{x-2}{\varepsilon}}, & x \in (1,2]. \end{cases}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Mikhailov, M, Özisik, M: Unified Analysis and Solutions of Heat and Mass Diffusion. Dover, New York (1994)
2. de Falco, C, O'Riordan, E: Interior layers in a reaction-diffusion equation with a discontinuous diffusion coefficient. Int. J. Numer. Anal. Model. **7**, 444-461 (2010)
3. Chen, CK: A fixed interface boundary value problem for differential equations: a problem arising from population genetics. Dyn. Partial Differ. Equ. **3**, 199-208 (2006)
4. Aitbayev, R: Existence and uniqueness for a two-point interface boundary value problem. Electron. J. Differ. Equ. **2013**, 242 (2013)
5. Farrell, PA, Hegarty, AF, Miller, JJH, O'Riordan, E, Shishkin, GI: Global maximum norm parameter-uniform numerical method for a singularly perturbed convection-diffusion problem with discontinuous convection coefficient. Math. Comput. Model. **40**, 1375-1392 (2004)
6. Farrell, PA, O'Riordan, E, Shishkin, GI: A class of singularly perturbed quasilinear differential equations with interior layers. Math. Comput. **78**, 103-127 (2009)
7. Huang, Z: Tailored finite point method for the interface problem. Netw. Heterog. Media **4**, 91-106 (2009)
8. Loubenets, A, Ali, T, Hanke, M: Highly accurate finite element method for one-dimensional elliptic interface problems. Appl. Numer. Math. **59**, 119-134 (2009)
9. Coster, CD, Habets, P: Two-Point Boundary Value Problems: Lower and Upper Solutions. Elsevier, New York (2006)
10. Vasiléva, AB, Butuzov, VF, Kalachev, LV: The Boundary Function Method for Singular Perturbation Problems. SIAM Studies in Applied Mathematics. SIAM, Philadelphia (1995)
11. de Jager, EM, Jiang, F: The Theory of Singular Perturbations. North-Holland, Amsterdam (1996)
12. Zeidler, E: Applied Functional Analysis: Applications to Mathematical Physics. Springer, New York (1995)
13. Xie, F: On a class of singular boundary value problems with singular perturbation. J. Differ. Equ. **252**, 2370-2387 (2012)