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Approximation of the inverse elliptic problem with mixed boundary value conditions and overdetermination

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Abstract

In the present study, the inverse problem for a multidimensional elliptic equation with mixed boundary conditions and overdetermination is considered. The first and second orders of accuracy in t and the second order of accuracy in space variables for the approximate solution of this inverse problem are constructed. Stability, almost coercive stability, and coercive stability estimates for the solution of these difference schemes are established. For the two-dimensional inverse problems with mixed boundary value conditions, numerical results are presented in test examples.

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1 Introduction

Inverse problems for partial differential equations are frequently encountered in various branches of science (see [1–3] and the bibliography therein).

In recent years, theory and methods of solutions of inverse problems of determining the parameter of partial differential equations have been extensively studied by several researchers (see [1–25] and the references therein).

We will denote the set of natural numbers from 1 to n by $\mathbb{N}_n = \{1, 2, \dots, n\}$, and closure of set Ω by $\overline{\Omega} = \Omega \cup S$.

Let $n \in \mathbb{N}$, $\delta, T, L > 0$, $\lambda \in (0, T)$ be given numbers and $\Omega = (0, L) \times \dots \times (0, L)$ be the open cube in the n -dimensional Euclidean space with boundary $S = \partial\Omega = S_1 \cup S_2$, where

$$S = \bigcup_{i=1}^n \{x = (x_1, x_2, \dots, x_n) \mid x_i = 0 \text{ or } x_i = L, x_k \in [0, L], k \in \mathbb{N}_n \setminus \{i\}\},$$

$$S_1 = \bigcup_{i=1}^n \{x = (x_1, x_2, \dots, x_n) \mid x_i = 0, x_k \in [0, L], k \in \mathbb{N}_n \setminus \{i\}\},$$

$$S_2 = \bigcup_{i=1}^n \{x = (x_1, x_2, \dots, x_n) \mid x_i = L, x_k \in (0, L), k \in \mathbb{N}_n \setminus \{i\}\},$$

and let $a_r, \varphi, \xi, \psi : \overline{\Omega} \rightarrow \mathbb{R}, f : (0, T) \times \Omega \rightarrow \mathbb{R}$ be given smooth functions such that $a_r(x) \geq a > 0$ for $x \in \overline{\Omega}, r \in \mathbb{N}_n$. Consider the multidimensional elliptic equation

$$\begin{aligned}
 -u_{tt}(t, x) - \sum_{r=1}^n (a_r(x)u_{x_r}(t, x))_{x_r} + \delta u(t, x) &= f(t, x) + p(x), \\
 x = (x_1, \dots, x_n) \in \Omega, 0 < t < T. &
 \end{aligned}
 \tag{1.1}$$

Here $u(t, x)$ and $p(x)$ are unknown functions.

In $[0, T] \times \Omega$, consider the inverse problem of finding functions $u(t, x)$ and $p(x)$ for the multidimensional elliptic equation (1.1) with overdetermined boundary conditions

$$u(0, x) = \varphi(x), \quad u(\lambda, x) = \xi(x), \quad u(T, x) = \psi(x), \quad x \in \overline{\Omega}
 \tag{1.2}$$

and one of the following usual boundary conditions on $[0, T] \times S$:

$$u(t, x) = 0, \quad 0 \leq t \leq T, x \in S,
 \tag{1.3}$$

$$\frac{\partial u(t, x)}{\partial \vec{n}} = 0, \quad 0 \leq t \leq T, x \in S,
 \tag{1.4}$$

$$\frac{\partial u(t, x)}{\partial \vec{n}} = 0, \quad x \in S_1, \quad u(t, x) = 0, \quad x \in S_2, 0 \leq t \leq T,
 \tag{1.5}$$

$$u(t, x) = 0, \quad x \in S_1, \quad \frac{\partial u(t, x)}{\partial \vec{n}} = 0, \quad x \in S_2, 0 \leq t \leq T.
 \tag{1.6}$$

For the problem of determining the parameter of an elliptic equation in a Banach space, the exact estimates of solution in Hölder norms are established in [21]. Well-posedness of inverse problem (1.1), (1.2), (1.3) and its approximations were investigated in [18]. In [21], the third and fourth order stable difference schemes for approximate solution of this problem were presented. Stability and coercive stability estimates for the solution of the first and second order difference schemes for inverse problem (1.1), (1.2), (1.4) were established in [19]. In [23], the authors studied inverse problem (1.1), (1.2), (1.5).

Our aim in this work is to construct the first and second orders of accuracy difference schemes for an approximate solution of inverse problem (1.1), (1.2), (1.6) and to establish stability, coercive stability, almost coercive stability estimates for the solution of these difference schemes.

This paper is planned as follows. In Section 2, we give theorems on well-posedness of inverse problems with mixed boundary conditions and overdetermination. Section 3 is devoted to the construction of the first and second order difference schemes for approximate solution of problem (1.1), (1.2), (1.6). In this section, we establish stability, almost coercive, and coercive stability inequalities for the solution of difference schemes. In Section 4, we present numerical results for a two-dimensional elliptic equation. The conclusion is given in the final Section 5.

2 Well-posedness of inverse problems with mixed boundary conditions and overdetermination

Let $C_{0T}^{\alpha, \alpha}(L_2(\overline{\Omega}))$ be the space obtained by completion of the space of all smooth $L_2(\overline{\Omega})$ -valued functions g on $[0, T]$ with the norm

$$\|g\|_{C_{0T}^{\alpha, \alpha}(L_2(\overline{\Omega}))} = \|g\|_{C(L_2(\overline{\Omega}))} + \sup_{0 \leq t < t + \tau \leq T} \frac{(t + \tau)^\alpha (T - t)^\alpha \|g(t + \tau) - g(t)\|_{L_2(\overline{\Omega})}}{\tau^\alpha}.$$

Denote the differential expression generated by (1.1) [26]

$$A\rho(x) = - \sum_{r=1}^n (a_r(x)\rho_{x_r}(x))_{x_r} + \delta\rho(x), \tag{2.1}$$

where $\rho \in W_2^2(\overline{\Omega})$.

The differential expression (1.1) defines a self-adjoint positive definite operator A acting on $L_2(\overline{\Omega})$ with the domain

$$D(A) = \left\{ \rho \in W_2^2(\overline{\Omega}), \frac{\partial\rho}{\partial\vec{n}} = 0 \text{ on } S_1, \rho = 0 \text{ on } S_2 \right\}.$$

Therefore, by using abstract Theorems 2.1 and 2.2 of paper [19] for $H = L_2(\overline{\Omega})$, we get the following theorems on well-posedness of inverse problem (1.1), (1.2), (1.5).

Theorem 2.1 *Suppose that A is defined by formula (2.1), $\varphi, \xi, \psi \in D(A)$. Then, for the solutions (u, p) of inverse problem (1.1), (1.2), (1.5), the stability inequalities*

$$\|u\|_{C(L_2(\overline{\Omega}))} \leq M_1 [\|\varphi\|_{L_2(\overline{\Omega})} + \|\psi\|_{L_2(\overline{\Omega})} + \|\xi\|_{L_2(\overline{\Omega})} + \|f\|_{C(L_2(\overline{\Omega}))}], \tag{2.2}$$

$$\|A^{-1}p\|_{L_2(\overline{\Omega})} \leq M_1 [\|\varphi\|_{L_2(\overline{\Omega})} + \|\psi\|_{L_2(\overline{\Omega})} + \|\xi\|_{L_2(\overline{\Omega})} + \|f\|_{C(L_2(\overline{\Omega}))}], \tag{2.3}$$

$$\begin{aligned} \|p\|_{L_2(\overline{\Omega})} \leq M_1 & \left[\|A\varphi\|_{L_2(\overline{\Omega})} + \|A\psi\|_{L_2(\overline{\Omega})} + \|A\xi\|_{L_2(\overline{\Omega})} \right. \\ & \left. + \frac{1}{\alpha(1-\alpha)} \|f\|_{C_{0T}^{\alpha,\alpha}(L_2(\overline{\Omega}))} \right], \end{aligned} \tag{2.4}$$

are satisfied, where M_1 is independent of $\alpha, \varphi(x), \xi(x), \psi(x)$, and $f(t, x)$.

Theorem 2.2 *Assume that A is defined by formula (2.1), $\varphi, \psi, \xi \in D(A)$. Then, for the solution of inverse problem (1.1), (1.2), (1.5), the coercive stability estimate*

$$\begin{aligned} & \|u'\|_{C_{0T}^{\alpha,\alpha}(L_2(\overline{\Omega}))} + \|u\|_{C_{0T}^{\alpha,\alpha}(W_2^2(\overline{\Omega}))} + \|p\|_{L_2(\overline{\Omega})} \\ & \leq M_2 \left[\frac{1}{\alpha(1-\alpha)} \|f\|_{C_{0T}^{\alpha,\alpha}(L_2(\overline{\Omega}))} + \|\varphi\|_{W_2^2(\overline{\Omega})} + \|\psi\|_{W_2^2(\overline{\Omega})} + \|\xi\|_{W_2^2(\overline{\Omega})} \right] \end{aligned} \tag{2.5}$$

holds, where M_2 does not depend on $\alpha, \varphi(x), \xi(x), \psi(x)$, and $f(t, x)$.

Now, consider inverse problem (1.1), (1.2), (1.6).

The differential expression (1.1) defines a self-adjoint positive definite operator A acting on $L_2(\overline{\Omega})$ with the domain

$$D(A) = \left\{ \rho \in W_2^2(\overline{\Omega}), \rho = 0 \text{ on } S_1, \frac{\partial\rho}{\partial\vec{n}} = 0 \text{ on } S_2 \right\}.$$

So, in a similar manner, applying abstract Theorems 2.1 and 2.2 of paper [19] for $H = L_2(\overline{\Omega})$, we can get the theorems on well-posedness of inverse problem (1.1), (1.2), (1.6).

Theorem 2.3 *Suppose that A is defined by formula (2.1), $\varphi, \xi, \psi \in D(A)$. Then, for the solution (u, p) of inverse problem (1.1), (1.2), (1.6), the stability inequalities (2.2), (2.3), (2.4) hold.*

Theorem 2.4 *Assume that A is defined by formula (2.1), $\varphi, \psi, \xi \in D(A)$. Then, for the solution (u, p) of inverse problem (1.1), (1.2), (1.6), coercive stability estimate (2.5) is valid.*

3 Difference schemes and their well-posedness

We discretize inverse problem (1.1), (1.2), (1.6) in two steps. In the first step, define the set of grid points in space variables,

$$\begin{cases} \tilde{\Omega}_h = \{x = (h_1 m_1, \dots, h_n m_n) \mid m_i = 0, \dots, M_i, h_i M_i = L, i = 1, \dots, n\}, \\ \Omega_h = \tilde{\Omega}_h \cap \Omega, \quad S_h^1 = \tilde{\Omega}_h \cap S^1, \quad S_h^2 = \tilde{\Omega}_h \cap S^2. \end{cases}$$

Introduce the Hilbert spaces $L_{2h} = L_2(\tilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\tilde{\Omega}_h)$ of grid functions $g^h(x) = \{g(h_1 m_1, \dots, h_n m_n) \mid m_i = 0, \dots, M_i, i = 1, \dots, n\}$ defined on $\tilde{\Omega}_h$ equipped with the norms

$$\begin{aligned} \|g^h\|_{L_{2h}} &= \left(\sum_{x \in \tilde{\Omega}_h} |g^h(x)|^2 h_1 \cdots h_n \right)^{1/2}, \\ \|g^h\|_{W_{2h}^2} &= \|g^h\|_{L_{2h}} + \left(\sum_{x \in \tilde{\Omega}_h} \sum_{i=1}^n |(g^h(x))_{x_i, m_i}|^2 h_1 \cdots h_n \right)^{1/2} \\ &\quad + \left(\sum_{x \in \tilde{\Omega}_h} \sum_{i=1}^n |(g^h(x))_{x_i, \bar{x}_i, m_i}|^2 h_1 \cdots h_n \right)^{1/2}, \end{aligned}$$

respectively.

To the differential operator A (2.1) assign the difference operator A_h^x , defined by the formula

$$A_h^x u^h = - \sum_{i=1}^n (a_i(x) u_{\bar{x}_i}^h)_{x_i, m_i} + \delta u^h \tag{3.1}$$

acting in the space of grid functions $u^h(x)$ satisfying the conditions $u^h(x) = 0$ for all $x \in S_h^1$ and $D^h u^h(x) = 0$ for all $x \in S_h^2$. Here, $D^h u^h(x)$ is an approximation of $\frac{\partial u}{\partial n}$. Note that [26, 27] A_h^x is a self-adjoint positive definite operator in $L_2(\tilde{\Omega}_h)$.

In this step, by using A_h^x , for obtaining $u^h(t, x)$ functions, we arrive at the problem

$$\begin{cases} -\frac{d^2 u^h(t, x)}{dt^2} + A_h^x u^h(t, x) = f^h(t, x) + p^h(x), & 0 < t < T, x \in \Omega_h, \\ u^h(0, x) = \varphi^h(x), \quad u^h(\lambda, x) = \xi^h(x), \quad u^h(T, x) = \psi^h(x), & x \in \tilde{\Omega}_h. \end{cases} \tag{3.2}$$

In the second step, applying the approximate formula

$$u^h(\lambda, x) = u^h\left(\left[\frac{\lambda}{\tau}\right] \tau, x\right) + o(\tau)$$

for $u^h(\lambda, x) = \xi^h(x)$, we replace inverse problem (1.1), (1.2), (1.6) with the first order of accuracy difference scheme

$$\begin{cases} -\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = \theta_k^h(x) + p^h(x), \\ \theta_k^h(x) = f^h(t_k, x), t_k = k\tau, 1 \leq k \leq N - 1, x \in \Omega_h, \\ u_0^h(x) = \varphi^h(x), \quad u_N^h(x) = \psi^h(x), \quad u_l^h(x) = \xi^h(x), \quad x \in \tilde{\Omega}_h, N\tau = T. \end{cases} \tag{3.3}$$

Here, $l = [\frac{\lambda}{\tau}]$, $[\cdot]$ is a notation for the greatest integer function.

In this step, by using the approximate formula

$$u^h(\lambda, x) = u^h(l\tau, x) + \left(\frac{\lambda}{\tau} - l\right) (u^h(l\tau + \tau, x) - u^h(l\tau, x)) + o(\tau^2)$$

for $u^h(\lambda, x) = \xi^h(x)$, we construct the second order of accuracy difference scheme for inverse problem (1.1), (1.2), (1.6)

$$\begin{cases} -\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = \theta_k^h(x) + p^h(x), \\ \theta_k^h(x) = f^h(t_k, x), t_k = k\tau, 1 \leq k \leq N - 1, x \in \tilde{\Omega}_h, \\ u_0^h(x) = \varphi^h(x), \quad u_N^h(x) = \psi^h(x), \\ u_l^h(x) + \left(\frac{\lambda}{\tau} - l\right) (u_{l+1}^h(x) - u_l^h(x)) = \xi^h(x), \quad x \in \tilde{\Omega}_h, N\tau = T. \end{cases} \tag{3.4}$$

Applying

$$u_k^h(x) = v_k^h(x) - (A_h^x)^{-1} p^h(x), \quad 1 \leq k \leq N - 1, \tag{3.5}$$

we can reduce difference problems (3.3) and (3.4) to the auxiliary nonlocal difference problems

$$\begin{cases} -\frac{v_{k+1}^h(x) - 2v_k^h(x) + v_{k-1}^h(x)}{\tau^2} + A_h^x v_k^h(x) = \theta_k^h(x), \\ \theta_k^h(x) = f^h(t_k, x), t_k = k\tau, 1 \leq k \leq N - 1, x \in \Omega_h, \\ v_0^h(x) - v_l^h(x) = \varphi^h(x) - \xi^h(x), \\ v_N^h(x) - v_l^h(x) = \psi^h(x) - \xi^h(x), \quad x \in \tilde{\Omega}_h, N\tau = T \end{cases} \tag{3.6}$$

and

$$\begin{cases} -\frac{v_{k+1}^h(x) - 2v_k^h(x) + v_{k-1}^h(x)}{\tau^2} + A_h^x v_k^h(x) = \theta_k^h(x), \\ \theta_k^h(x) = f^h(t_k, x), t_k = k\tau, 1 \leq k \leq N - 1, x \in \tilde{\Omega}_h, \\ v_0^h(x) - v_l^h(x) - \left(\frac{\lambda}{\tau} - l\right) (v_{l+1}^h(x) - v_l^h(x)) = \varphi^h(x) - \xi^h(x), \\ v_N^h(x) - v_l^h(x) - \left(\frac{\lambda}{\tau} - l\right) (v_{l+1}^h(x) - v_l^h(x)) = \psi^h(x) - \xi^h(x), \\ x \in \tilde{\Omega}_h, N\tau = T, \end{cases} \tag{3.7}$$

respectively.

The difference schemes for nonlocal boundary value problems for the multidimensional elliptic equation were studied in [28, 29].

Denote

$$C = \frac{1}{2} \left(\tau A_h^x + \sqrt{4A_h^x + \tau^2 (A_h^x)^2} \right), \quad R = (I + \tau C)^{-1}.$$

Let H be the Hilbert space $L_2(\bar{\Omega})$.

Lemma 3.1 *The following estimates are satisfied [27]:*

$$\|R^k\|_{H \rightarrow H} \leq M_3 (1 + \delta^{\frac{1}{2}} \tau)^{-k}, \quad \delta > 0,$$

$$\|CR^k\|_{H \rightarrow H} \leq \frac{M_3}{k\tau}, \quad k \geq 1,$$

$$\|(I - R^{2N})^{-1}\|_{H \rightarrow H} \leq M_3,$$

where M_3 is independent of τ .

Lemma 3.2 ([19]) *For $1 \leq l \leq N - 1$ and for the operator $S = R^{2N} + R^l - R^{2N-l} + R^{N-l} - R^{N+l}$, the operator $I - S$ has an inverse $G = (I - S)^{-1}$ and the estimate*

$$\|G\|_{H \rightarrow H} \leq M_4 \tag{3.8}$$

is satisfied, where M_4 does not depend on τ .

Lemma 3.3 ([19]) *For $1 \leq l \leq N - 1$ and for the operator*

$$S_1 = R^{2N} - \left(\frac{\lambda}{\tau} - l - 1 \right) (R^l - R^{2N-l} + R^{N-l} - R^{N+l}) \\ + \left(\frac{\lambda}{\tau} - l \right) (R^{l+1} - R^{2N-l-1} + R^{N-l-1} - R^{N+l+1}),$$

the operator $I - S_1$ has an inverse

$$G_1 = \left(I - R^{2N} + \left(\frac{\lambda}{\tau} - l - 1 \right) (R^l - R^{2N-l} + R^{N-l} - R^{N+l}) \right. \\ \left. - \left(\frac{\lambda}{\tau} - l \right) (R^{l+1} - R^{2N-l-1} + R^{N-l-1} - R^{N+l+1}) \right)^{-1},$$

and the estimate

$$\|G_1\|_{H \rightarrow H} \leq M_5 \tag{3.9}$$

is valid, where M_5 does not depend on τ .

In a similar manner as [19], we can obtain the following representation formula

$$u_k^h(x) = (I - R^{2N})^{-1} \left[((R^k - R^{2N-k})v_0^h(x) + (R^{N-k} - R^{N+k})v_N^h(x)) \right. \\ \left. - (R^{N-k} - R^{N+k})(I + \tau C)(2I + \tau C)^{-1} C^{-1} \sum_{i=1}^{N-1} (R^{N-i} - R^{N+i})f_i^h(x)\tau \right]$$

$$+ (I + \tau C)(2I + \tau C)^{-1} C^{-1} \sum_{i=1}^{N-1} (R^{|k-i|} - R^{k+i}) f_i^h(x) \tau + \varphi^h(x) - v_0^h(x), \tag{3.10}$$

$$p^h(x) = A_h^x \varphi^h(x) - A_h^x v_0^h(x),$$

$$v_N^h(x) = v_0^h(x) + \psi^h(x) - \varphi^h(x),$$

with

$$\begin{aligned} v_0^h(x) = & -G(R^{N-l} - R^{N+l})(I + \tau C)(2I + \tau C)^{-1} C^{-1} \\ & \times \sum_{i=1}^{N-1} (R^{N-i} - R^{N+i}) f_i^h(x) \tau + G(I - R^{2N})(I + \tau C) \\ & \times (2I + \tau C)^{-1} C^{-1} \sum_{i=1}^{N-1} (R^{|l-i|} - R^{l+i}) f_i^h(x) \tau \\ & + G(I - R^{2N})(\varphi^h(x) - \xi^h(x)) + G(R^{N-l} - R^{N+l})(\psi^h(x) - \varphi^h(x)) \end{aligned} \tag{3.11}$$

for the solution of difference scheme (3.3) and

$$\begin{aligned} v_0^h(x) = & \left(\frac{\lambda}{\tau} - l - 1\right) G_1(R^{N-l} - R^{N+l})(I + \tau C)(2I + \tau C)^{-1} C^{-1} \\ & \times \sum_{i=1}^{N-1} (R^{N-i} - R^{N+i}) f_i^h(x) \tau \\ & - \left(\frac{\lambda}{\tau} - l - 1\right) G_1(I - R^{2N})(I + \tau C)(2I + \tau C)^{-1} C^{-1} \\ & \times \sum_{i=1}^{N-1} (R^{|l-i|} - R^{l+i}) f_i^h(x) \tau \\ & - \left(\frac{\lambda}{\tau} - l\right) G_1(R^{N-l-1} - R^{N+l+1})(I + \tau C)(2I + \tau C)^{-1} C^{-1} \\ & \times \sum_{i=1}^{N-1} (R^{N-i} - R^{N+i}) f_i^h(x) \tau \\ & + \left(\frac{\lambda}{\tau} - l\right) G_1(I - R^{2N})(I + \tau C)(2I + \tau C)^{-1} C^{-1} \\ & \times \sum_{i=1}^{N-1} (R^{|l+1-i|} - R^{l+1+i}) f_i^h(x) \tau \\ & + G_1(I - R^{2N})(\varphi^h(x) - \xi^h(x)) + \left(\left(\frac{\lambda}{\tau} - l - 1\right) G_1(R^{N-l} - R^{N+l})\right. \\ & \left. + \left(\frac{\lambda}{\tau} - l\right) G_1(R^{N-l-1} - R^{N+l+1})\right) (\psi^h(x) - \varphi^h(x)) \end{aligned} \tag{3.12}$$

for difference scheme (3.4).

Now, we consider the linear spaces of mesh functions $\theta^\tau = \{\theta_k\}_1^{N-1}$ with values in the Hilbert space H . We denote by $C([0, T]_\tau, H)$ a normed space with the norm

$$\|\{\theta_k\}_1^{N-1}\|_{C([0, T]_\tau, H)} = \max_{1 \leq k \leq N-1} \|\theta_k\|_H,$$

and by $C_{0T}^{\alpha,\alpha}([0, T]_\tau, H)$ a normed space with the norm

$$\begin{aligned} \|\{\theta_k\}_1^{N-1}\|_{C_{0T}^{\alpha,\alpha}([0, T]_\tau, H)} &= \|\{\theta_k\}_1^{N-1}\|_{C([0, T]_\tau, H)} \\ &+ \sup_{1 \leq k < k+n \leq N-1} \frac{(k\tau + n\tau)^\alpha (T - k\tau)^\alpha \|\theta_{k+n} - \theta_k\|_H}{(n\tau)^\alpha}. \end{aligned}$$

Let τ and $|h| = \sqrt{h_1^2 + \dots + h_n^2}$ be sufficiently small positive numbers.

Theorem 3.1 *The solutions $(\{u_k^h\}_1^{N-1}, p^h)$ of difference schemes (3.3) and (3.4) obey the following stability estimates:*

$$\begin{aligned} \|\{u_k^h\}_1^{N-1}\|_{C([0, T]_\tau, L_{2h})} &\leq M_6 [\|\varphi^h\|_{L_{2h}} + \|\psi^h\|_{L_{2h}} + \|\xi^h\|_{L_{2h}} + \|\{f_k^h\}_1^{N-1}\|_{C([0, T]_\tau, L_{2h})}], \\ \|p^h\|_{L_{2h}} &\leq M_6 \left[\|\varphi^h\|_{W_{2h}^2} + \|\psi^h\|_{W_{2h}^2} + \|\xi^h\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)} \|\{f_k^h\}_1^{N-1}\|_{C_{0T}^{\alpha,\alpha}([0, T]_\tau, L_{2h})} \right], \end{aligned}$$

where M_6 is independent of $\tau, \alpha, h, \varphi^h, \psi^h, \xi^h$, and $\{f_k^h\}_1^{N-1}$.

Theorem 3.2 *The solutions of difference schemes (3.3) and (3.4) obey the following almost coercive stability estimate:*

$$\begin{aligned} &\left\| \left\{ \frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2} \right\}_1^{N-1} \right\|_{C([0, T]_\tau, L_{2h})} + \|p^h\|_{L_{2h}} \\ &\leq M_7 \left(\|\varphi^h\|_{W_{2h}^2} + \|\psi^h\|_{W_{2h}^2} + \|\xi^h\|_{W_{2h}^2} + \ln \left(\frac{1}{\tau + h} \right) \|\{f_k^h\}_1^{N-1}\|_{C([0, T]_\tau, L_{2h})} \right), \end{aligned}$$

where M_7 does not depend on $\tau, \alpha, h, \varphi^h, \psi^h, \xi^h$, and $\{f_k^h\}_1^{N-1}$.

Theorem 3.3 *The solutions of difference schemes (3.3) and (3.4) obey the following coercive stability estimate:*

$$\begin{aligned} &\left\| \left\{ \frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2} \right\}_1^{N-1} \right\|_{C_{0T}^{\alpha,\alpha}([0, T]_\tau, L_{2h})} + \|\{u_k^h\}_1^{N-1}\|_{C_{0T}^{\alpha,\alpha}([0, T]_\tau, W_{2h}^2)} + \|p^h\|_{L_{2h}} \\ &\leq M_8 \left[\frac{1}{\alpha(1-\alpha)} \|\{f_k^h\}_1^{N-1}\|_{C_{0T}^{\alpha,\alpha}([0, T]_\tau, L_{2h})} + \|\varphi^h\|_{W_{2h}^2} + \|\psi^h\|_{W_{2h}^2} + \|\xi^h\|_{W_{2h}^2} \right], \end{aligned}$$

where M_8 is independent of $\tau, \alpha, h, \varphi^h, \psi^h, \xi^h$, and $\{f_k^h\}_1^{N-1}$.

The proofs of Theorems 3.1-3.3 are based on the symmetry property of operator A_h^x in L_{2h} , representation formulas (3.10), (3.11), (3.12), Lemmas 3.1-3.3 and the following theorem on the coercivity estimate for the solution of the elliptic difference problem in L_{2h} .

Theorem 3.4 ([30]) *For the solution of the elliptic difference problem*

$$\begin{cases} A_h^x u^h(x) = \omega^h(x), & x \in \tilde{\Omega}_h, \\ u^h(x) = 0, & x \in S_1^h, \quad D^h u^h(x) = 0, \quad x \in S_2^h, \end{cases}$$

the following coercivity inequality holds:

$$\sum_{q=1}^n \|(u_k^h)_{x_q, \bar{x}_q, m_q}\|_{L_{2h}} \leq M \|\omega^h\|_{L_{2h}},$$

where M does not depend on h and ω^h .

4 Numerical example

We consider the inverse problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial x}((2 + \cos x) \frac{\partial u(t,x)}{\partial x}) + u(t,x) = f(t,x) + p(x), \\ 0 < x < \pi, 0 < t < T, \\ f(t,x) = -\exp(-t)(\cos(x) - 1) + (\exp(-t) + t)(3 \cos(x) + \cos(2x) - 1), \\ u(0,x) = 2(\cos(x) - 1), \quad 0 \leq x \leq \pi, \\ u(T,x) = (\exp(-T) + T + 1)(\cos(x) - 1), \quad 0 \leq x \leq \pi, \\ u(\lambda,x) = (\exp(-\lambda) + \lambda + 1)(\cos(x) - 1), \quad 0 \leq x \leq \pi, \\ u(t,0) = u_x(t,\pi) = 0, \quad 0 \leq t \leq T, \lambda = \frac{4T}{7} \end{cases} \tag{4.1}$$

for the elliptic equation. It is easy to see that $u(t,x) = (\exp(-t) + t + 1)(\cos(x) - 1)$ and $p(x) = 3 \cos(x) + \cos(2x) - 1$ are the exact solutions of (4.1).

For this example, we have the following auxiliary nonlocal boundary value problem for an unknown function $v(t,x)$:

$$\begin{cases} -\frac{d^2 v(t,x)}{dt^2} - \frac{\partial}{\partial x}((2 + \cos x) \frac{\partial v(t,x)}{\partial x}) + v(t,x) = f(t,x), \quad 0 < x < \pi, 0 < t < T, \\ v(t,0) = v_x(t,\pi) = 0, \quad 0 \leq t \leq T, \\ v(0,x) - v(\lambda,x) = (1 - \exp(-\lambda) - \lambda)(\cos(x) - 1), \quad 0 \leq x \leq \pi, \\ v(T,x) - v(\lambda,x) = (\exp(-T) - \exp(-\lambda) + T - \lambda)(\cos(x) - 1), \quad 0 \leq x \leq \pi. \end{cases} \tag{4.2}$$

Consider the set $[0, T]_\tau \times [0, \pi]_h$ of grid points depending on the small parameters τ and h ,

$$\begin{aligned} [0, T]_\tau \times [0, \pi]_h &= \{(t_k, x_n) : t_k = k\tau, k = 1, \dots, N-1, N\tau = T, \\ &\quad x_n = nh, n = 1, \dots, M-1, Mh = \pi\}. \end{aligned}$$

We get difference schemes of the first order of accuracy in t and the second order of accuracy in x ,

$$\begin{cases} \frac{v_n^{k+1} - 2v_n^k + v_n^{k-1}}{\tau^2} + (2 + \cos(x_n)) \frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2} - \sin(x_n) \frac{v_{n+1}^k - v_{n-1}^k}{2h} - v_n^k = \theta_n^k, \\ \theta_n^k = -f(t_k, x_n), k = 1, \dots, N-1, n = 1, \dots, M-1, \\ v_0^k = 0, \quad v_M^k - v_{M-1}^k = 0, \quad k = 0, \dots, N, \\ v_n^0 - v_n^l = (1 - \exp(-\lambda) - \lambda)(\cos(x_n) - 1), \quad n = 0, \dots, M, \\ v_n^N - v_n^l = (\exp(-t_N) - \exp(-\lambda) + t_N - \lambda)(\cos(x_n) - 1), \\ n = 0, \dots, M, l = [\frac{\lambda}{\tau}] \end{cases} \tag{4.3}$$

for the approximate solutions of the nonlocal boundary value problem (4.2).

Now, it is easy to rewrite difference scheme (4.3) in the following matrix form:

$$\begin{aligned}
 A_n v_{n+1} + B_n v_n + C_n v_{n-1} &= I_{N+1} \theta_n, \quad n = 1, \dots, M-1, \\
 v_0 &= \vec{0}, \quad v_M = v_{M-1}.
 \end{aligned}
 \tag{4.4}$$

Here, θ_n is an $(N+1) \times 1$ column matrix, I_{N+1} is the $(N+1) \times (N+1)$ identity matrix, A_n, B_n, C_n are $(N+1) \times (N+1)$ square matrices

$$A_n = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & a_n & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & a_n & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_n & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & a_n & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix},
 \tag{4.5}$$

$$B_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & -1 & \dots & 0 & 0 & 0 & 0 \\ d & b_n & d & 0 & \dots & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & d & b_n & d & \dots & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots & d & b_n & d & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & d & b_n & d \\ 0 & 0 & 0 & 0 & \dots & -1 & \dots & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_n = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & c_n & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & c_n & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_n & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & c_n & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix},
 \tag{4.6}$$

$$\begin{aligned}
 a_n &= \frac{2 + \cos(x_n)}{h^2} - \frac{\sin(x_n)}{2h}, & b_n &= -\frac{2}{\tau^2} - \frac{2(2 + \cos(x_n))}{h^2} - 1, \\
 c_n &= \frac{2 + \cos(x_n)}{h^2} + \frac{\sin(x_n)}{2h}, & d &= \frac{1}{\tau^2},
 \end{aligned}
 \tag{4.7}$$

$$\theta_n = \begin{bmatrix} \theta_n^0 \\ \vdots \\ \theta_n^N \end{bmatrix},$$

$$\theta_n^k = -f(t_k, x_n), \quad k = 1, \dots, N-1, n = 1, \dots, M-1,$$

$$\theta_n^0 = (1 - \exp(-\lambda) - \lambda)(\cos(x_n) - 1),$$

$$\theta_n^N = (\exp(-t_N) - \exp(-\lambda) + t_N - \lambda)(\cos(x_n) - 1), \quad n = 1, \dots, M-1,$$

$$v_{n-1} = \begin{bmatrix} v_{n-1}^0 \\ \vdots \\ v_{n-1}^N \end{bmatrix}, \quad v_n = \begin{bmatrix} v_n^0 \\ \vdots \\ v_n^N \end{bmatrix}, \quad v_{n+1} = \begin{bmatrix} v_{n+1}^0 \\ \vdots \\ v_{n+1}^N \end{bmatrix}.$$

For numerical solving of this system equation, we use the modified Gauss elimination method [31]. Solution of (4.4) is presented by the formula

$$v_n = \alpha_{n+1}v_{n+1} + \beta_{n+1}, \quad n = M - 1, \dots, 1,$$

where α_n ($n = 1, \dots, M - 1$) are $(N + 1) \times (N + 1)$ square matrices and β_n ($n = 1, \dots, M - 1$) are $(N + 1) \times 1$ column matrices. For α_{n+1} , β_{n+1} , we have the following formulas:

$$\begin{aligned} \alpha_{n+1} &= -(B_n + C_n\alpha_n)^{-1}A_n, \\ \beta_{n+1} &= (B_n + C_n\alpha_n)^{-1}(I_{N+1}\theta_n - C_n\beta_n), \quad n = 1, \dots, M - 1, \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= 0_{(N+1) \times (N+1)}, \quad \beta_1 = 0_{(N+1) \times 1} \quad (v_0 = \vec{0}), \\ \alpha_M &= I_{N+1}, \quad \beta_M = 0_{(N+1) \times 1} \quad (v_{M-1} = v_M). \end{aligned}$$

By using the second order of accuracy in x approximation of A , we get the following values of p in grid points:

$$\begin{aligned} p_n &= -\frac{(2 + \cos(x_n))}{h^2} [(\varphi_{n+1} - v_{n+1}^0) - 2(\varphi_n - v_n^0) + (\varphi_{n-1} - v_{n-1}^0)] \\ &\quad + \frac{\sin(x_n)}{2h} ((\varphi_{n+1} - v_{n+1}^0) - (\varphi_{n-1} - v_{n-1}^0)) + (\varphi_n - v_n^0), \\ n &= 1, \dots, M - 1. \end{aligned} \tag{4.8}$$

For the approximate solution of problem (4.1), we use the formula

$$u_n^k = v_n^k + \varphi_n - v_n^0, \quad n = 0, \dots, M, k = 0, \dots, N.$$

Now, we consider again nonlocal boundary value problem (4.2).

Applying (3.7), we get the second order of accuracy in t and x difference scheme

$$\begin{cases} \frac{v_n^{k+1} - 2v_n^k + v_n^{k-1}}{\tau^2} + (2 + \cos(x_n)) \frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2} \\ \quad - \sin(x_n) \frac{v_{n+1}^k - v_{n-1}^k}{2h} - v_n^k = \theta_n^k, \\ \theta_n^k = -f(t_k, x_n), k = 1, \dots, N - 1, n = 1, \dots, M - 1, \\ v_0^k = 0, \quad -3v_M^k + 4v_{M-1}^k - v_{M-2}^k = 0, \quad k = 0, \dots, N, \\ 10v_M^k - 15v_{M-1}^k + 6v_{M-2}^k - v_{M-3}^k = 0, \\ v_n^0 + (\frac{\lambda}{\tau} - l - 1)v_n^l - (\frac{\lambda}{\tau} - l)v_n^{l+1} = (1 - \exp(-\lambda) - \lambda) \cos(x_n), \\ \quad n = 0, \dots, M, \\ v_n^N + (\frac{\lambda}{\tau} - l - 1)v_n^l - (\frac{\lambda}{\tau} - l)v_n^{l+1} \\ \quad = (\exp(-t_N) - \exp(-\lambda) + t_N - \lambda) \cos(x_n), \quad n = 0, \dots, M \end{cases} \tag{4.9}$$

for the approximate solution of nonlocal boundary value problem (4.2).

Therefore, we can rewrite difference scheme (4.9) in the matrix form

$$\begin{aligned}
 A_n v_{n+1} + B_n v_n + C_n v_{n-1} &= I_{N+1} \theta_n, \quad n = 1, \dots, M-1, \\
 v_0 &= 0, \quad -3v_M + 4v_{M-1} - v_{M-2} = 0,
 \end{aligned}
 \tag{4.10}$$

where A_n, C_n are defined by (4.5), (4.6), (4.7) and B_n is an $(N + 1) \times (N + 1)$ square matrix

$$B_n = \begin{bmatrix}
 1 & 0 & 0 & 0 & \cdots & 0 & y & z & 0 & \cdots & 0 & 0 & 0 & 0 \\
 d & b_n & d & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 0 & d & b_n & d & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 0 & 0 & d & b_n & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 \vdots & \vdots \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & b_n & d & 0 & 0 \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & d & b_n & d & 0 \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & d & b_n & d \\
 0 & 0 & 0 & 0 & \cdots & 0 & y & z & 0 & \cdots & 0 & 0 & 0 & 1
 \end{bmatrix},$$

$$b_n = -\frac{2}{\tau^2} - \frac{2(2 + \cos(x_n))}{h^2} - 1, \quad d = \frac{1}{\tau^2}, \quad y = \left(\frac{\lambda}{\tau} - l - 1\right), \quad z = -\left(\frac{\lambda}{\tau} - l\right).$$

For solution of (4.10), we use the formula

$$v_n = \alpha_n v_{n+1} + \beta_n v_{n+2} + \gamma_n, \quad n = M-2, \dots, 0,$$

where α_n, β_n ($n = 0, \dots, M-2$) are $(N + 1) \times (N + 1)$ square matrices and γ_n ($n = 0, \dots, M-2$) are $(N + 1) \times 1$ column matrices. For the coefficients $\alpha_n, \beta_n, \gamma_n$, we get formulas

$$\begin{aligned}
 \alpha_n &= -(B_n + C_n \alpha_{n-1})^{-1} (A_n + C_n \beta_{n-1}), \\
 \beta_n &= 0, \\
 \gamma_n &= (B_n + C_n \alpha_{n-1})^{-1} (I_{N+1} \theta_n - C_n \gamma_{n-1}), \quad n = 1, \dots, M-1.
 \end{aligned}$$

Here

$$\begin{aligned}
 \alpha_0 &= 0, \quad \beta_0 = 0, \quad \gamma_0 = 0 \quad (v_0 = 0), \\
 \alpha_1 &= \frac{8}{5} I_{N+1}, \quad \beta_1 = -\frac{3}{5} I_{N+1}, \\
 \alpha_{M-2} &= 4 I_{N+1}, \quad \beta_{M-2} = -3 I_{N+1}, \\
 \alpha_{M-3} &= \frac{8}{3} I_{N+1}, \quad \beta_{M-3} = -\frac{5}{3} I_{N+1},
 \end{aligned}$$

and $\gamma_0, \gamma_1, \gamma_{M-2}, \gamma_{M-3}$ are the $(N + 1) \times 1$ zero column vector. For v_M and v_{M-1} , we have the following formulas:

$$\begin{aligned}
 v_M &= (Q_{11} - Q_{12} Q_{22}^{-1} Q_{21})^{-1} (G_1 - Q_{12} Q_{22}^{-1} G_2), \\
 v_{M-1} &= Q_{22}^{-1} (G_2 - Q_{21} v_M),
 \end{aligned}$$

where

$$Q_{11} = -3A_{M-2} - 8B_{M-2} - 8C_{M-2}\alpha_{M-3} - 3C_{M-2}\beta_{M-3},$$

$$Q_{12} = 4A_{M-2} + 9B_{M-2} + 9C_{M-2}\alpha_{M-3} + 4C_{M-2}\beta_{M-3},$$

$$Q_{21} = -3B_{M-1} - 8C_{M-1},$$

$$Q_{22} = A_{M-1} + 4B_{M-1} + 9C_{M-1},$$

$$G_1 = I_{N+1}\theta_{M-2} - C_{M-2}\gamma_{M-3},$$

$$G_2 = I_{N+1}\theta_{M-1}.$$

Now, by using MATLAB programs, we present numerical results for this example (4.1). The numerical solutions are recorded for different values of N and M .

The numerical solutions are recorded for different values of $N = M$ for $T = 2$. The grid function u_n^k represents the numerical solution of difference schemes for inverse problem at (t_k, x_n) , respectively. Grid function p_n represents the numerical solutions at x_n for unknown function p . For their comparison, the errors are computed by

$$Eu_M^N = \max_{1 \leq k \leq N-1} \left(\sum_{n=1}^{M-1} |u(t_k, x_n) - u_n^k|^2 h \right)^{\frac{1}{2}},$$

$$Ep_M = \left(\sum_{n=1}^{M-1} |p(x_n) - p_n|^2 h \right)^{\frac{1}{2}}.$$

Tables 1 and 2 give the error analysis between the exact solution and solutions derived by difference schemes. Tables 1 and 2 are constructed for $N = M = 20, 40, 80,$ and 160 . Hence, the second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme.

Table 1 is the error analysis between the exact solution p and solutions derived by difference schemes in the second stage of algorithm.

Table 2 presents the error analysis between the exact solution u and solutions derived by first order and second order accuracy of difference schemes.

Table 1 Error analysis for p

	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
First order of accuracy difference scheme	0.25111	0.24195	0.11044	0.045366
Second order of accuracy difference scheme	0.024061	0.0052977	0.0012825	3.16×10^{-4}

Table 2 Error analysis for u

	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
First order of accuracy difference scheme	0.036174	0.042733	0.016978	0.0041271
Second order of accuracy difference scheme	0.019509	0.0034552	6.41×10^{-4}	1.56×10^{-4}

5 Conclusion

In this paper, the inverse problem for a multidimensional elliptic equation with Dirichlet-Neumann conditions and overdetermination is considered. The first and second orders of accuracy difference schemes for approximate solution of this problem are presented. Theorems on the stability, almost coercive stability, and coercive stability inequalities for the solutions of difference schemes for inverse problem are proved. Numerical example in a two-dimensional case is given. As it can be seen from Tables 1 and 2, the second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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