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Optimal control problem for a generalized sixth order Cahn-Hilliard type equation with nonlinear diffusion

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Abstract

In this paper, we study the initial-boundary-value problem for a generalized sixth order Cahn-Hilliard type equation, which describes the separation properties of oil-water mixtures when a substance enforcing the mixing of the phases is added. The optimal control under boundary condition is given and the existence of optimal solution is proved.

MSC: Primary 49J20; secondary 35K35; 35K55

Keywords: sixth order Cahn-Hilliard; existence; optimal control

1 Introduction

We consider the equation

$$u_t = D^2\left[\gamma D^4 u - a(u)D^2 u - \frac{a'(u)}{2}|Du|^2 + f(u) + ku_t - \gamma_2 D^2 u_t\right], \tag{1.1}$$

in $\Omega \times (0,T)$, where $\Omega = (0,1)$, $\gamma > 0$, $k > 0$, and $\gamma_2 > 0$ with the initial and boundary conditions

$$u(x,0) = u_0, \quad \text{in } \Omega, \tag{1.2}$$

$$u(x,t) = D^2 u(x,t) = D^4 u(x,t) = 0, \quad \text{on } \partial\Omega. \tag{1.3}$$

The function $f(u)$ stands for the derivative of a potential $F(u)$ with $F(u)$, $a(u)$ approximated, respectively, by a sixth and a second order polynomial

$$F(u) = \int_0^u f(s)\,ds = (u+1)^2\left(u^2 + h_0\right)(u-1)^2, \tag{1.4}$$

$$a(u) = a_2 u^2 + a_0, \tag{1.5}$$

where $a_2 > 0$.

The free energy functional proposed by Gompper *et al.* [1–4] has the form

$$\psi(u) = \int_\Omega \varphi(u, \nabla u, \Delta u)\,dx,$$

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with the density given by

$$\varphi(u, \nabla u, \Delta u) = f(u) + \frac{1}{2}a(u)|\nabla u|^2 + \frac{1}{2}\gamma(\Delta u)^2.$$

Here u is the scalar order parameter, which is proportional to the local difference between oil and water concentrations. The properties of the amphiphile and its concentration enter model (1.1) implicitly via (1.4) and (1.5). $F(u)$ has three minima at $u = -1$, $u = 1$, and $u = 0$, which describe the oil, water and disordered microemulsion phases. In [2–4], the coefficient $a(u)$ is approximated by the quadratic function (1.5) with constants a_0 of arbitrary sign and a_2 positive.

Like in the classical Cahn-Hilliard the theory the order parameter u is a conserved quantity. Thus it satisfies the conservation law

$$u_t + \nabla j = 0, \tag{1.6}$$

with the mass flux j given by the constitutive equation

$$-j = \frac{\partial \mathcal{D}}{\partial \nabla \mu} = M\nabla \mu, \tag{1.7}$$

and μ representing the chemical potential

$$\mu = \frac{\delta \psi}{\delta u} + \frac{\delta \mathcal{D}}{\delta u_t}, \tag{1.8}$$

where $\mathcal{D} \geq 0$, the dissipation potential, has the form

$$\mathcal{D}(u_t, \nabla u_t, \nabla \mu) = \frac{1}{2}k(u_t)^2 + \frac{1}{2}\gamma_2|\nabla u_t|^2 + \frac{1}{2}M|\nabla \mu|^2, \tag{1.9}$$

and M is the mobility, k, γ_2 are the viscosity coefficients corresponding to the rate of the order parameter and its spatial gradient.

The first variation $\frac{\delta \psi}{\delta u}$ is defined by the condition that

$$\frac{d}{d\lambda}\int_\Omega \psi(u + \lambda\zeta, \nabla u + \lambda\nabla\zeta, \Delta u + \lambda\triangle\zeta)\,dx|_{\lambda=0} =: \int_\Omega \frac{\delta\psi}{\delta u}\zeta\,dx \tag{1.10}$$

most hold for all test functions $\zeta \in C_0^\infty(\Omega)$. In the case of free energy this leads to the following expressions:

$$\frac{\delta \psi}{\delta u} = f(u) - a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + \gamma\Delta^2 u, \tag{1.11}$$

$$\frac{\delta \mathcal{D}}{\delta u_t} = ku_t - \gamma_2\Delta u_t. \tag{1.12}$$

From the above discussions we know that

$$\mu = f(u) - a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + \gamma\Delta^2 u + ku_t - \gamma_2\Delta u_t. \tag{1.13}$$

Combining (1.6)-(1.13) we get the following conserved evolution system:

$$u_t - \nabla(M\nabla\mu) = 0,$$
$$\mu = f(u) - a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + \gamma\Delta^2 u + ku_t - \gamma_2\Delta u_t,$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with the boundary $\partial\Omega$, occupied by the ternary mixture, and $(0,T)$ is the time interval. We endow this system with the initial and boundary condition (1.2) and (1.3), in this paper we consider the one-dimensional case with $M = 1$.

Schimperna and Pawłow [5] studied (1.1) when $\gamma_2 = 0$ with logarithmic potential

$$F(r) = (1-r)\log(1-r) + (1+r)\log(1+r) - \frac{\sigma}{2}r^2, \quad \sigma > 0.$$

They investigated the behavior of the solutions to the sixth order system as the parameter γ tended to 0, the uniqueness and regularization properties of the solutions have been discussed.

Pawłow and Zajączkowski [6] proved that the problem (1.1)-(1.5) with $k = \gamma_2 = 0$ under consideration is well posed in the sense that it admits a unique global smooth solution which depends continuously on the initial datum.

In past decades, the optimal control of distributed parameter system had received much attention in the academic field. A wide spectrum of problems in applications can be solved by methods of optimal control, such as chemical engineering and vehicle dynamics. Modern optimal control theories and applied models are not only represented by ODEs, but also by PDEs. Kunisch and Volkwein solved open-loop and closed-loop optimal control problems for the Burgers equation [7], Armaou and Christofides studied the feedback control of Kuramto-Sivashing equation [8].

Recently, many authors studied the optimal control problem for the pseudo-parabolic equation, such as Tian *et al.* [9–11], Zhao and Liu [12].

In this paper, we consider the optimal control problem for the following equation:

$$\begin{aligned}&\left(u - kD^2u + \gamma_2 D^4 u\right)_t - \frac{\gamma}{\gamma_2}D^2\left(u - kD^2u + \gamma_2 D^4u\right) + \frac{\gamma}{\gamma_2}D^2u \\ &\quad + D^2\left(\left(a(u) - \frac{\gamma k}{\gamma_2}\right)D^2u + \frac{a'(u)}{2}|Du|^2\right) = D^2 f(u) + B^*\overline{\omega},\end{aligned} \tag{1.14}$$

with (1.2)-(1.5).

When $y = u - kD^2u + \gamma_2 D^4u$, we take the distributed optimal control problem

$$\begin{cases}\min \mathcal{J}(y,\overline{\omega}) = \frac{1}{2}\|Cy - z\|_S^2 + \frac{\delta}{2}\|\overline{\omega}\|^2_{L^2(0,T;Q_0)}, \\ \quad \text{s.t. } y_t - \frac{\gamma}{\gamma_2}D^2y + \frac{\gamma}{\gamma_2}D^2u \\ \qquad\qquad + D^2((a(u) - \frac{\gamma k}{\gamma_2})D^2u + \frac{a'(u)}{2}|Du|^2) - D^2f(u) = B^*\overline{\omega}, \\ y(x,0) = y_0 = u_0 - kD^2u(x,0) + \gamma_2 D^4u(x,0), \\ u(x,t) = D^2u(x,t) = D^4u(x,t) = 0.\end{cases} \tag{1.15}$$

For fixed $T > 0$, we set $\Omega = (0,1)$ and $Q = \Omega \times (0,T)$. Let $Q_0 \subset Q$ be an open set with positive measure.

Let $V = H_0^1(0,1)$, $H = L^2(0,1)$; $V^* = H^{-1}(0,1)$, and $H^* = L^2(0,1)$ are dual spaces, respectively, and we have

$$V \hookrightarrow H = H^* \hookrightarrow V^*.$$

The extension operator $B^* \in L(L^2(0,T;Q_0), L^2(0,T;V^*))$ is given by

$$B^* q = \begin{cases} q, & q \in Q_0, \\ 0, & q \in Q/Q_0. \end{cases} \tag{1.16}$$

The space $W(0,T;V)$ is defined by

$$W(0,T;V) = \{y, y \in L^2(0,T;V), y_t \in L^2(0,T;V^*)\},$$

which is a Hilbert space endowed with the common inner product.

The plan of the paper is as follows. In Section 2, we prove the existence of the weak solution in a special space. The optimal control is discussed in Section 3, and the existence of an optimal solution is proved.

2 Existence of weak solution

Consider the following the sixth order Cahn-Hilliard type equation:

$$\begin{aligned} &(u - kD^2u + \gamma_2 D^4 u)_t - \frac{\gamma}{\gamma_2} D^2(u - kD^2 u + \gamma_2 D^4 u) + \frac{\gamma}{\gamma_2} D^2 u \\ &\quad + D^2\left(\left(a(u) - \frac{\gamma k}{\gamma_2}\right) D^2 u + \frac{a'(u)}{2}|Du|^2\right) = D^2 f(u) + B^*\overline{\omega}, \end{aligned} \tag{2.1}$$

under the initial value

$$u(x,0) = u_0,$$

and boundary condition

$$u(x,t) = D^2 u(x,t) = D^4 u(x,t) = 0,$$

where $B^*\overline{\omega} \in L^2(0,T;V^*)$ and the control item $\overline{\omega} \in L^2(0,T;Q_0)$.

Let $y = u - kD^2u + \gamma_2 D^4 u$; the above problem is rewritten as

$$\begin{cases} y_t - \frac{\gamma}{\gamma_2} D^2 y + \frac{\gamma}{\gamma_2} D^2 u + D^2((a(u) - \frac{\gamma k}{\gamma_2}) D^2 u + \frac{a'(u)}{2}|Du|^2) - D^2 f(u) = B^*\overline{\omega}, \\ y(x,0) = y_0 = u_0 - kD^2 u_0 + \gamma_2 D^4 u_0, \\ u(x,t) = D^2 u(x,t) = D^4 u(x,t) = 0, \end{cases} \tag{2.2}$$

with (1.3)-(1.5).

Now, we give the definition of the weak solution to the problem (2.2) in the space $W(0,T;V)$.

Definition 2.1 A function $y(x,t) \in W(0,T;V)$ is called a weak solution to problem (2.2), if

$$\frac{d}{dt}(y,\phi) + \frac{\gamma}{\gamma_2}(Dy,D\phi) - \frac{\gamma}{\gamma_2}(Du,D\phi) \\ - \left(D\left(a(u) - \frac{\gamma k}{\gamma_2}\right)D^2u + \frac{a'(u)}{2}|Du|^2, D\phi\right) + \left(Df(u), D\phi\right) = \left(B^*\overline{\omega},\phi\right)_{V^*,V},$$

for all $\phi \in V$, a.e. $t \in [0,T]$ and $y_0 \in H$ are valid.

Theorem 2.1 *The problem* (2.2) *admits a weak solution* $y(x,t) \in W(0,T;V)$ *in the interval* $[0,T]$, *if* $B^*\overline{\omega} \in L^2(0,T;V^*)$ *and* $y_0 \in H$.

Proof Employ the standard Galerkin method.

The differential operator $A = -\partial_x^2$ is a linear unbounded self-adjoint operator in H with $D(A)$ dense in H, where H is a Hilbert space with a scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

There exists an orthogonal basis $\{\psi_i\}$ of H. Let $\{\psi_i\}_{i=1}^{\infty}$ be the eigenfunctions of the operator $A = -\partial_x^2$ with

$$A\psi_j = \lambda_j\psi_j, \quad 0 < \lambda_1 \le \lambda_2 \le \cdots, \text{as } j \to \infty.$$

For $n \in \mathbb{N}$, we define the discrete ansatz space by

$$V_n = \operatorname{span}\{\psi_1, \psi_2, \ldots, \psi_n\} \subset V.$$

Set $y_n(t) = y_n(x,t) = \sum_{i=1}^{n} y_i^n(t)\psi_i(x)$ and require $y_n(0,\cdot) \mapsto y_0$ in H holds true.

To prove the existence of a unique weak solution to the problem (2.2), we are going to analyze the limiting behavior of sequences of smooth functions $\{y_n\}$ and $\{u_n\}$.

Performing the Galerkin procedure for the problem (2.2), we have

$$\begin{cases} y_{n,t} - \frac{\gamma}{\gamma_2}D^2y_n + \frac{\gamma}{\gamma_2}D^2u_n \\ \quad + D^2((a(u_n) - \frac{\gamma k}{\gamma_2})D^2u_n + \frac{a'(u_n)}{2}|Du_n|^2) - D^2f(u_n) = B^*\overline{\omega}, \\ y_n(x,0) = y_{n,0} = u_{n,0} - kD^2u_n(x,0) + \gamma_2D^4u_n(x,0), \\ u_n(x,t) = D^2u_n(x,t) = D^4u_n(x,t) = 0. \end{cases} \tag{2.3}$$

According to ODE theory, there is a unique solution to (2.3) in the interval $[0,t_n]$. We should show that the solution is uniformly bounded when $t_n \to T$.

As a first step, multiplying the first equation of (2.3) by

$$\mu_n = \gamma D^4u_n - a(u_n)D^2u_n - \frac{a'(u_n)}{2}|Du_n|^2 + f(u_n) + ku_{n,t} - \gamma_2D^2u_{n,t},$$

and integrating with respect to x, we obtain

$$\frac{d}{dt}E(u_n) + \|D\mu_n\|^2 + k\|u_{n,t}\|^2 + \gamma_2\|Du_{n,t}\|^2 = \left(B^*\overline{\omega},\mu_n\right)_{V^*,V}, \tag{2.4}$$

where

$$E(u_n) = \int_0^1 \left(\frac{\gamma}{2} \left| D^2 u_n \right|^2 + \frac{a(u_n)}{2} |Du_n|^2 + F(u_n) \right) dx \tag{2.5}$$

and

$$F(u_n) = \left(u_n^6 + (h_0 - 2)u_n^4 + (1 - 2h_0)u_n^2 + h_0 \right). \tag{2.6}$$

Applying a simple calculation, we have

$$F(u_n) \geq C_1 u_n^6 - C_0, \tag{2.7}$$

where $C_1 > 0$ and $C_0 \geq 0$.

Since $B^*\overline{\omega} \in L^2(0, T; V^*)$ is a control item, we assume

$$\left\| B^*\overline{\omega} \right\|_{V^*} \leq M. \tag{2.8}$$

Taking into account (2.4), (2.7), (2.8), (1.4), and integrating (2.4) with respect to time from 0 to t, we know

$$\begin{aligned}
&\int_0^1 \left(\frac{\gamma}{2} \left| D^2 u_n \right|^2 + \frac{a_2}{2} u_n^2 |Du_n|^2 + C_1 u_n^6 \right) dx \\
&\quad + \int_0^t \|D\mu_n\|^2 \, dt + k \int_0^t \|u_{n,t}\|^2 \, dt + \gamma_2 \int_0^t \|Du_{n,t}\|^2 \, dt \\
&\leq \int_0^1 \frac{|a_0|}{2} |Du_n|^2 \, dx + E(u_{n,0}) + C_0 + \int_0^t \left| \left(B^*\overline{\omega}, \mu_n \right)_{V^*,V} \right| dt \\
&\leq \varepsilon_1 \int_0^1 \frac{|a_0|}{2} \left| D^2 u_n \right|^2 dx + C(\varepsilon_1) \int_0^1 u_n^2 \, dx \\
&\quad + E(u_{n,0}) + C_0 + \int_0^t \left\| B^*\overline{\omega} \right\|_{V^*} \|\mu_n\|_V \, dt \\
&\leq \varepsilon_1 \int_0^1 \frac{|a_0|}{2} \left| D^2 u_n \right|^2 dx + C(\varepsilon_1)\varepsilon_2 \int_0^1 u_n^6 \, dx + C(\varepsilon_2) \\
&\quad + E(u_{n,0}) + C_0 + C(\varepsilon) \int_0^t \left\| B^*\overline{\omega} \right\|_{V^*}^2 dt + \varepsilon \int_0^t \left\| D^2 \mu_n \right\|^2 dt \\
&= \varepsilon_1 \int_0^1 \frac{|a_0|}{2} \left| D^2 u_n \right|^2 dx + C(\varepsilon_1)\varepsilon_2 \int_0^1 u_n^6 \, dx + C(\varepsilon_2) \\
&\quad + E(u_{n,0}) + C_0 + C(\varepsilon) \int_0^t \left\| B^*\overline{\omega} \right\|_{V^*}^2 dt + \varepsilon \int_0^t \|u_{n,t}\|^2 \, dt.
\end{aligned}$$

Choosing ε_1, ε_2, and ε sufficiently small, from the above inequality and the Poincaré inequality, we have

$$\int_0^1 \left| D^2 u_n \right|^2 dx \leq C, \tag{2.9}$$

$$\int_0^1 |Du_n|^2 \, dx \leq C, \tag{2.10}$$

$$\int_0^1 u_n^6\,dx \le C, \tag{2.11}$$

$$\iint_{Q_T} |u_{n,t}|^2\,dx\,dt \le C. \tag{2.12}$$

From (2.11), we know

$$\int_0^1 u_n^2\,dx \le C. \tag{2.13}$$

By virtue of (2.9), (2.10), and (2.13), we obtain

$$\|u_n\|_{H^2} \le C. \tag{2.14}$$

By Sobolev's imbedding theorem it follows from (2.14) that

$$\|u_n\|_{L^\infty} \le C, \qquad \|Du_n\|_{L^\infty} \le C. \tag{2.15}$$

As a second step, multiplying (1.1) by D^2u_n and integrating with respect to x, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_0^1 |Du_n|^2\,dx + k\int_0^1 \left|D^2u_n\right|^2 dx + \gamma_2\int_0^1 \left|D^3u_n\right|^2 dx\right) + \gamma\int_0^1 \left|D^4u_n\right|^2 dx \\
&\quad = -\int_0^1 D^2f(u_n)D^2u_n\,dx + \int_0^1 a(u_n)D^2u_nD^4u_n\,dx \\
&\qquad + \int_0^1 \frac{a'(u_n)}{2}|Du_n|^2D^4u_n\,dx - \left(B^*\overline{\omega}, D^2u_n\right)_{V^*,V}.
\end{aligned} \tag{2.16}$$

From a simple calculation, we have

$$a'(u_n) = 2a_2u_n, \tag{2.17}$$

$$D^2f(u_n) = f'(u_n)D^2u_n + f''(u_n)(Du_n)^2, \tag{2.18}$$

where

$$f'(u_n) = \left(30u_n^4 + 12(h_0-2)u_n^2 + 2(1-2h_0)\right) \ge -C_2, \quad C_2 > 0, \tag{2.19}$$

$$f''(u) = 120u_n^3 + 24(h_0-2)u_n. \tag{2.20}$$

Thus it follows from (2.14), (2.18), and (2.19) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_0^1 (Du_n)^2\,dx + k\int_0^1 \left|D^2u_n\right|^2 dx + \gamma_2\int_0^1 \left|D^3u_n\right|^2 dx\right) + \gamma\int_0^1 \left|D^4u_n\right|^2 dx \\
&\quad \le -\int_0^1 \left(f'(u_n)D^2u_n + f''(u_n)|Du_n|^2\right)D^2u_n\,dx \\
&\qquad + \int_0^1 \left(a_2u_n^2 + a_0\right)D^2u_nD^4u_n\,dx \\
&\qquad + \int_0^1 \frac{a'(u_n)}{2}|Du_n|^2D^4u_n\,dx + \left\|B^*\overline{\omega}\right\|_{V^*}\left\|D^2u_n\right\|_V
\end{aligned}$$

$$
\begin{aligned}
&\le C_2\int_0^1 \left|D^2u_n\right|^2dx + C\left(\|u_n\|_{L^\infty}^3 + \|u_n\|_{L^\infty}\right)\|Du_n\|_{L^\infty}\int_0^1 Du_nD^2u_n\,dx \\
&\quad + |a_2|\|u_n\|_{L^\infty}^2\int_0^1 D^2u_nD^4u_n\,dx + |a_0|\int_0^1 D^2u_nD^4u_n\,dx \\
&\quad + |a_2|\int_0^1 u_n|Du_n|^2D^4u_n\,dx + C(\varepsilon)\left\|B^*\overline{\omega}\right\|_{V^*}^2 + \varepsilon\int_0^1\left|D^4u_n\right|^2dx \\
&\le \frac{\gamma}{2}\int_0^1\left|D^4u_n\right|^2dx + C,
\end{aligned}
\tag{2.21}
$$

where ε is sufficiently small.

By the Gronwall inequality, (2.21) implies

$$
\iint_{Q_T}\left|D^4u_n\right|^2dx\,dt \le C, \tag{2.22}
$$

$$
\int_\Omega\left|D^3u_n\right|^2dx \le C. \tag{2.23}
$$

As a third step, multiplying (1.1) by D^4u_n and integrating with respect to x, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_\Omega\left|D^2u_n\right|^2dx + \delta\int_\Omega\left|D^3u_n\right|^2dx + \gamma_2\int_\Omega\left|D^4u_n\right|^2dx\right) + \gamma\int_\Omega\left|D^5u_n\right|^2dx \\
&= -\int_\Omega f'(u_n)Du_nD^5u_n\,dx \\
&\quad - \int_\Omega D^2\left(a(u_n)\left|D^2u_n\right|\right)D^4u_n\,dx - \int_\Omega D^2\left(\frac{a'(u_n)}{2}|Du_n|^2\right)D^4u_n\,dx \\
&= -\int_\Omega f'(u_n)Du_nD^5u_n\,dx \\
&\quad + \int_\Omega D\left(a(u_n)\left|D^2u_n\right|\right)D^5u_n\,dx + \int_\Omega D\left(\frac{a'(u_n)}{2}|Du_n|^2\right)D^5u_n\,dx \\
&= I_1 + I_2 + I_3.
\end{aligned}
\tag{2.24}
$$

On account of (2.15) and (2.10), we know

$$
I_1 \le C_2\left(C(\varepsilon_3)\int_\Omega|Du_n|^2dx + \varepsilon_3\int_\Omega\left|D^5u_n\right|^2dx\right).
$$

On the other hand, by the Nirenberg inequality, we have

$$
\left\|D^3u_n\right\|_\infty \le \left\|D^5u_n\right\|^{\frac{5}{8}}\|Du_n\|^{\frac{3}{8}}. \tag{2.25}
$$

Hence, by the Hölder and Young inequalities, we obtain

$$
\begin{aligned}
|I_2| &= \left|\int_\Omega D\left(a_2u_n^2 + a_0\right)\left(D^2u_n\right)D^5u_n\,dx\right| \\
&\le 2a_2\|u_n\|_\infty\|Du_n\|_\infty\left(\int_\Omega\left(D^2u_n\right)^2dx\right)^{\frac{1}{2}}\left(\int_\Omega\left|D^5u_n\right|^2dx\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + C\left\|D^3 u\right\|_\infty \|u_n\|_\infty \left(\int_\Omega u_n^2\,dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|D^5 u_n\right|^2 dx\right)^{\frac{1}{2}} \\
&\le C(\varepsilon_3) C \int_\Omega \left|D^5 u_n\right|^2 dx.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|I_3| &= \left|\int_\Omega D\left(\frac{a'(u_n)}{2}|Du_n|^2\right) D^5 u_n\,dx\right| \\
&\le a_2 \|Du_n\|_\infty^2 \left(\int_\Omega |Du_n|^2\,dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|D^5 u_n\right|^2 dx\right)^{\frac{1}{2}} \\
&\quad + 2a_2 \|u_n\|_\infty \|Du_n\|_\infty \left(\int_\Omega \left(D^2 u_n\right)^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|D^5 u_n\right|^2 dx\right)^{\frac{1}{2}} \\
&\le C(\varepsilon_3) C \int_\Omega \left|D^5 u_n\right|^2 dx + C,
\end{aligned}
$$

the ε_3 is sufficiently small.

Therefore, by the Gronwall inequality, we have

$$
\sup_{0<t<T} \int_\Omega \left(D^4 u_n\right)^2 dx \le C, \tag{2.26}
$$

$$
\iint_{Q_T} \left|D^5 u_n\right|^2 dx\,dt \le C. \tag{2.27}
$$

From a simple calculation, we have

$$
\begin{aligned}
\|y_n\|_V &= \left\|u_n - kD^2 u_n + \gamma_2 D^4 u_n\right\|_V^2 \\
&\le C\left(\|u_n\| + \|Du_n\| + \left\|D^2 u_n\right\| + \left\|D^3 u_n\right\| + \left\|D^4 u_n\right\| + \left\|D^5 u_n\right\|\right).
\end{aligned}
$$

From (2.14), (2.15), (2.26), and (2.27), we obtain

$$
\|y_n\|_{L^2(0,T;V)} \le C. \tag{2.28}
$$

As a fourth step, from (2.2), (2.14), (2.15), and the Sobolev embedding theorem, we have

$$
\begin{aligned}
\|y_{n,t}\|_{V^*} &\le \left\|B^*\overline{\omega}\right\|_{V^*} + \left\|D^5 u_n\right\| + \left\|D\left(\left(a(u_n) - \frac{\gamma k}{\gamma_2}\right) D^2 u_n + \frac{a'(u_n)}{2}|Du_n|^2\right)\right\| \\
&\quad + \left\|Df(u_n)\right\| \\
&\le \left\|B^*\overline{\omega}\right\|_{V^*} + \left\|D^5 u_n\right\| + C\|u_n\|_{L^\infty}^2 \left\|D^3 u_n\right\| + C\|u_n\|_{L^\infty} \left\|D^2 u_n\right\|^2 \\
&\quad + C\|u_n\|_{L^\infty}^5 \|Du_n\| + C \\
&\le C\left\|D^5 u_n\right\| + C.
\end{aligned}
$$

Then

$$
\|y_{n,t}\|_{L^2(0,T;V^*)} \le C.
$$

Thus, we have:

(i) For every $t \in [0, T]$, the sequence $\{y_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0, T; H)$ as well as in $L^2(0, T; V)$, which is independent of the dimension of the ansatz space n.

(ii) For every $t \in [0, T]$, the sequence $\{y_{n,t}\}_{n\in\mathbb{N}}$ is bounded in $L^2(0, T; V^*)$, which is independent of the dimension of the ansatz space n.

Hence, we get $\{y_{n,t}\}_{n\in\mathbb{N}} \subset W(0, T; V)$, and $\{y_{n,t}\}_{n\in\mathbb{N}}$ weak in $W(0, T; V)$, weak star in $L^\infty(0, T; H)$ and strong in $L^2(0, T; H)$ to a function $y(x, t) \in W(0, T; V)$. Obviously, the uniqueness of the solution is easy to obtain [13]. We omit it here. □

To ensure that the norm of weak solution in the space $W(0, T; V)$ can be controlled by the initial value and the control item, we need the following theorem.

Theorem 2.2 *If $B^*\overline{\omega} \in L^2(0, T; V^*)$ and $y_0 \in H$, then there exist constants $C_3 > 0$ and $C_4 > 0$, such that*

$$\|y\|^2_{W(0,T;V)} \leq C_3\big(\|y_0\|^2_H + \|\overline{\omega}\|^2_{L^2(0,T;Q_0)}\big) + C_4. \tag{2.29}$$

Proof Similar to the proof of Theorem 2.1, we obtain

$$\|u\| \leq C, \qquad \|Du\| \leq C, \qquad \|u\|_V \leq C, \qquad \big\|D^3u\big\| \leq C. \tag{2.30}$$

Multiplying the equation by y and integrating the equation with respect to x, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|y\|^2_H + \frac{\gamma}{\gamma_2}\|Dy\|^2_H \\
&\quad = \frac{\gamma}{\gamma_2}\int_0^1 DyDu\,dx + \int_0^1 D\bigg(\bigg(a(u) - \frac{\gamma k}{\gamma_2}\bigg)D^2u + \frac{a'(u)}{2}|Du|^2\bigg)Dy\,dx \\
&\qquad - \int_0^1 DyDf(u)\,dx + \big(B^*\overline{\omega}, y\big)_{V^*,V}.
\end{aligned} \tag{2.31}$$

From the Hölder and Young inequalities, we have

$$\frac{\gamma}{\gamma_2}\int_0^1 DyDu\,dx \leq C(\varepsilon)\|Du\|^2 + \varepsilon\|Dy\|^2. \tag{2.32}$$

From (2.30), we have

$$\begin{aligned}
&\int_0^1 D\bigg(\bigg(a(u) - \frac{\gamma k}{\gamma_2}\bigg)D^2u + \frac{a'(u)}{2}|Du|^2\bigg)Dy\,dx \\
&\quad \leq \big(\|u\|^2_{L^\infty} + C\big)\|Dy\|\,\big\|D^3u\big\| + C\|u\|^2_{L^\infty}\|Du\|_{L^\infty}\big\|D^2u\big\|^2_{L^\infty}\|Dy\| \\
&\qquad + \|Du\|^3\|Dy\| \leq \varepsilon\|Dy\|^2 + C
\end{aligned}$$

and

$$-\int_0^1 DyDf(u)\,dx \leq C_2\int_0^1 Dy\,dx \leq C\|Dy\| + C \leq C. \tag{2.33}$$

Note that

$$\left(B^*\overline{\omega}, y\right)_{V^*,V} \le \left\|B^*\overline{\omega}\right\|_{V^*}\|y\|_V. \tag{2.34}$$

From (2.31)-(2.34), we have

$$\frac{1}{2}\frac{d}{dt}\|y\|_H^2 + \frac{\gamma}{\gamma_2}\|Dy\|_H^2 \le \varepsilon\|Dy\|_H^2 + C\left\|B^*\overline{\omega}\right\|_{V^*}^2 + C. \tag{2.35}$$

Integrating the above inequality with respect to t yields

$$\|y\|_H^2 \le \|y_0\|_H^2 + C\left\|B^*\overline{\omega}\right\|_{L^2(0,T;V^*)}^2 + C. \tag{2.36}$$

By (2.36), (2.2), and (2.30), we deduce that

$$\begin{aligned}
\|y_t\|_{V^*}^2 \le{}& \left\|B^*\overline{\omega}\right\|_{V^*}^2 + \frac{\gamma}{\gamma_2}\|y\|_V^2 + \frac{\gamma}{\gamma_2}\|Du\| \\
&+ \left\|D\left(a(u)D^2u + \frac{a'(u)}{2}|Du|^2\right)\right\| + \left\|Df(u)\right\| \\
\le{}& \left\|B^*\overline{\omega}\right\|_{V^*}^2 + C\|y\|_V^2 + C \\
\le{}& \|y_0\|_H^2 + C\left\|B^*\overline{\omega}\right\|_{L^2(0,T;V^*)}^2 + C.
\end{aligned} \tag{2.37}$$

From (2.36) and (2.37), we have

$$\begin{aligned}
\|y\|_{W(0,T;V)} &= \|y\|_{L^2(0,T;V)} + \|y_t\|_{L^2(0,T;V^*)} \\
&\le C_3\left(\|y_0\|_H^2 + \|\overline{\omega}\|_{L^2(0,T;Q_0)}^2\right) + C_4.
\end{aligned}$$

The proof is completed. □

3 Optimal problem

In this section, we will study the distributed optimal control and the existence of the optimal solution is obtained based on Lions' theory.

We study the following problem when $\overline{\omega} \in L^2(0,T;Q_0)$,

$$\begin{cases}
\min \mathcal{J}(y,\overline{\omega}) = \frac{1}{2}\|Cy-z\|_S^2 + \frac{\delta}{2}\|\overline{\omega}\|_{L^2(0,T;Q_0)}^2, \\
\quad \text{s.t. } y_t - \frac{\gamma}{\gamma_2}D^2y + \frac{\gamma}{\gamma_2}D^2u + D^2((a(u) - \frac{\gamma k}{\gamma_2})D^2u + \frac{a'(u)}{2}|Du|^2) - D^2f(u) = B^*\overline{\omega}, \\
y(x,0) = y_0 = u_0 - kD^2u(x,0) + \gamma_2 D^4u, \\
u(x,t) = D^2u(x,t) = D^4u(x,t) = 0,
\end{cases}$$

where $y = u - kD^2u + \gamma_2 D^4u$.

As we know that there exists a weak solution y to (2.2), due to $u = (1 - k\partial_x^2 + \gamma_2\partial_x^2)^{-1}y$, we know that there exists a weak solution u to (2.1). Let there be given an observation operator $C \in L(W(0,T;V),S)$, in which S is a real Hilbert space and C is continuous.

We choose a performance index of tracking type

$$\mathcal{J}(y,\overline{\omega}) = \frac{1}{2}\|Cy-z\|_S^2 + \frac{\delta}{2}\|\overline{\omega}\|_{L^2(0,T;Q_0)}^2, \tag{3.1}$$

where $z \in S$ is a desired state and $\delta > 0$ is fixed.

The optimal control problem as regards the further generalized sixth order Cahn-Hilliard equation is

$$\min \mathcal{J}(y,\overline{\omega}), \tag{3.2}$$

where $(y,\overline{\omega})$ satisfies the problem (2.2).

Let $X = W(0,T;V) \times L^2(0,T;Q_0)$ and $Y = L^2(0,T;V) \times H$.

We define an operator $e = e(e_1,e_2) : X \to Y$ by

$$e(y,\overline{\omega}) = e\big(e_1(y,\overline{\omega}), e_2(y,\overline{\omega})\big),$$

where

$$\begin{cases} e_1(y,\overline{\omega}) = (D^2)^{-1}(y_t - \frac{\gamma}{\gamma_2} D^2 y + \frac{\gamma}{\gamma_2} D^2 u \\ \qquad\qquad + D^2((a(u) - \frac{\gamma k}{\gamma_2}) D^2 u + \frac{a'(u)}{2}|Du|^2) - D^2 f(u) - B^*\overline{\omega}), \\ e_2 = y(x,0) - y_0, \end{cases}$$

and D^2 is an operator from $H^1(0,1)$ to $H^{-1}(0,1)$.

Then (3.2) is rewritten as

$$\min \mathcal{J}(y,\overline{\omega}) \quad \text{subject to } e = e(y,\overline{\omega}) = 0.$$

Now, we have the following theorem.

Theorem 3.1 *There exists an optimal control solution to the problem.*

Proof Let $(y,\overline{\omega}) \in X$ satisfy the equation $e = e(y,\overline{\omega}) = 0$. In view of (3.1), we have

$$\mathcal{J}(y,\overline{\omega}) \geq \frac{\delta}{2}\|\overline{\omega}\|_{L^2(0,T;Q_0)}.$$

From Theorem 2.2, we have

$$\|y\|_{W(0,T;V)} \to \infty \quad \text{yields} \quad \|\overline{\omega}\|_{L^2(0,T;Q_0)} \to \infty.$$

Hence

$$\mathcal{J}(y,\overline{\omega}) \to +\infty, \quad \text{when } \|y,\overline{\omega}\|_X \to \infty. \tag{3.3}$$

As the norm is weakly lowered semi-continuous [14], we find that $\mathcal{J}$ is weakly lowered semi-continuous.

Since $\mathcal{J}(y,\overline{\omega}) \geq 0$ for all $(y,\overline{\omega}) \in X$ holds, there exists

$$\eta = \inf\big\{\mathcal{J}(y,\overline{\omega}) | (y,\overline{\omega}) \in X \text{ such that } e(y,\overline{\omega}) = 0\big\},$$

which means that there exists a minimizing sequence $\{(y_n,\overline{\omega^n})\}_{n\in\mathbb{N}}$ in X such that

$$\eta = \lim_{n\to\infty} \mathcal{J}\big(y_n,\overline{\omega}^n\big) \quad \text{and} \quad e = e\big(y_n,\overline{\omega}^n\big) = 0, \quad \forall n \in \mathbb{N}.$$

From (3.3), there exists an element $(y^*, \overline{\omega}^*) \in X$ such that

$$y_n \rightharpoonup y^*, \quad y \in W(0,T;V), \tag{3.4}$$

$$\overline{\omega}^n \rightharpoonup \overline{\omega}^*, \quad \overline{\omega} \in L^2(0,T;Q_0), \tag{3.5}$$

when $n \to \infty$.

From (3.4), we have

$$\lim_{n\to\infty} \int_0^T \big(y_n(t) - y^*(t), \phi(t)\big)_{V^*,V}\, dt = 0, \quad \forall \phi \in L^2(0,T;V).$$

Since $W(0,T;V)$ is compactly embedded into $L^2(0,T;L^\infty)$ and continuously embedded into $C(0,T;H)$, we derive that $y_n \to y^*$ strongly in $L^2(0,T;L^\infty)$ and $y_n \to y^*$ strongly in $C(0,T;H)$, as $n \to \infty$. Then we also derive that $u_n \to u^*$, $Du_n \to Du^*$, $D^2u_n \to D^2u^*$, $D^3u_n \to D^3u^*$, $D^4u_n \to D^4u^*$ strongly in $C(0,T;H)$, as $n \to \infty$.

As the sequence $\{y_n\}_{n\in\mathbb{N}}$ converges weakly, $\|y_n\|_{W(0,T;V)}$ is bounded. Also, we see that $\|y_n\|_{L^2(0,T;L^\infty)}$ is bounded based on the embedding theorem.

Since $y_n \to y^*$ strongly in $L^2(0,T;L^\infty)$, we derive that $\|y^*\|_{L^2(0,T;L^\infty)}$, $\|u^*\|_{L^2(0,T;L^\infty)}$, $\|D^2u^*\|_{L^2(0,T;L^\infty)}$ and $\|D^4u^*\|_{L^2(0,T;L^\infty)}$ are bounded.

Notice that

$$\begin{aligned}
&\left|\int_0^T \int_0^1 \big(D^2 f(u_n) - D^2 f(u^*)\big)\psi \, dx\, dt\right| \\
&\quad = \left|\int_0^T \int_0^1 D\big(f(u_n) - f(u^*)\big) D\psi \, dx\, dt\right| \\
&\quad \le C\left|\int_0^T \int_0^1 \big((u_n)^4 Du_n + (u_n)^2 Du_n + Du_n \right. \\
&\qquad \left. - (u^*)^4 Du^* - (u^*)^2 Du^* - Du^*\big) D\psi \, dx\, dt\right| \\
&\quad \le \int_0^T \|u_n\|_{L^\infty}^4 \|Du_n - Du^*\|_H \|D\psi\|_H \, dt \\
&\qquad + \int_0^T \big\|(u_n)^4 - (u^*)^4\big\|_H \|Du^*\|_{L^\infty} \|D\psi\|_H \, dt \\
&\qquad + \int_0^T \big\|(u_n)^2\big\|_{L^\infty} \|Du_n - Du^*\|_H \|D\psi\|_H \, dt \\
&\qquad + \int_0^T \big\|(u_n)^2 - (u^*)^2\big\|_H \|Du^*\|_{L^\infty} \|D\psi\|_H \, dt \\
&\qquad + \int_0^T \|Du_n - Du^*\|_H \|D\psi\|_H \, dt \\
&\quad \le \|u_n\|_{C(0,T;L^\infty)}^4 \|Du_n - Du^*\|_{L^2(0,T;H)} \|D\psi\|_{L^2(0,T;H)} \\
&\qquad + \big(\|u_n\|_{C(0,T;L^\infty)}^3 + \|u^*\|_{C(0,T;L^\infty)}^3\big) \|Du^*\|_{C(0,T;L^\infty)} \|u_n - u^*\|_{L^2(0,T;H)} \\
&\qquad \times \|D\psi\|_{L^2(0,T;H)} + \|u_n\|_{C(0,T;L^\infty)}^2 \|Du_n - Du^*\|_{L^2(0,T;H)} \|D\psi\|_{L^2(0,T;H)} \\
&\qquad + \big(\|u_n\|_{C(0,T;L^\infty)} + \|u^*\|_{C(0,T;L^\infty)}\big) \|u_n - u^*\|_{L^2(0,T;H)} \|Du^*\|_{C(0,T;L^\infty)}
\end{aligned}$$

$$\begin{aligned}&\times \|D\psi\|_{L^2(0,T;H)} + \big\|Du_n - Du^*\big\|_{C(0,T;H)}\|D\psi\|_{L^2(0,T;H)}\\ &\to 0, \quad \forall \psi \in L^2(0,T;V).\end{aligned} \tag{3.6}$$

As we know

$$\begin{aligned}&\left|\int_0^T\int_0^1\left(D^2\left(a(u_n)D^2u_n + \frac{a'(u_n)}{2}|Du_n|^2\right)\right.\right.\\ &\qquad\left.\left. - D^2\left(a\big(u^*\big)D^2u^* + \frac{a'(u^*)}{2}\big|Du^*\big|^2\right)\right)\psi\,dx\,dt\right|\\ &= \left|\int_0^T\int_0^1 D^2\big(a(u_n)D^2u_n - a\big(u^*\big)D^2u^*\big)\psi\,dx\,dt\right.\\ &\qquad\left. + \int_0^T\int_0^1 D^2\left(\frac{a'(u_n)}{2}|Du_n|^2 - \frac{a'(u^*)}{2}\big|Du^*\big|^2\right)\psi\,dx\,dt\right|\\ &= |I_1 + I_2|.\end{aligned}$$

Note that

$$\begin{aligned}|I_1| &= \left|\int_0^T\int_0^1 D^2\left(\left(a(u_n) - \frac{\gamma k}{\gamma_2}\right)D^2u_n - \left(a\big(u^*\big) - \frac{\gamma k}{\gamma_2}\right)D^2u^*\right)\psi\,dx\,dt\right|\\ &\le \left|\int_0^T\int_0^1 D\left(\left(a_2(u_n)^2 + a_0 - \frac{\gamma k}{\gamma_2}\right)D^2u_n\right.\right.\\ &\qquad\left.\left. - \left(a_2\big(u^*\big)^2 + a_0 - \frac{\gamma k}{\gamma_2}\right)D^2u^*\right)D\psi\,dx\,dt\right|\\ &= \left|\int_0^T\int_0^1 D\big(a_2(u_n)^2D^2u_n - a_2\big(u^*\big)^2D^2u^*\big)D\psi\,dx\,dt\right.\\ &\qquad\left. + \int_0^T\int_0^1 D\left(\left(a_0 - \frac{\gamma k}{\gamma_2}\right)D^2u_n - \left(a_0 - \frac{\gamma k}{\gamma_2}\right)D^2u^*\right)D\psi\,dx\,dt\right|\\ &= \big|I_1^1 + I_1^2\big|.\end{aligned}$$

For I_1^1, we have

$$\begin{aligned}\big|I_1^1\big| &= \left|\int_0^T\int_0^1\big(2a_2u_nDu_nD^2u_n - 2a_2u^*Du^*D^2u^* + a_2(u_n)^2D^3u_n\right.\\ &\qquad\left. - a_2\big(u^*\big)^2D^3u^*\big)D\psi\,dx\,dt\right|\\ &\le 2|a_2|\int_0^T\|u_n\|_{L^\infty}\|Du_n\|_{L^\infty}\big\|D^2u_n - D^2u^*\big\|_H\|D\psi\|_H\,dt\\ &\quad + 2|a_2|\int_0^T\big\|u_n - u^*\big\|_H\|Du_n\|_{L^\infty}\big\|D^2u^*\big\|_{L^\infty}\|D\psi\|_H\,dt\\ &\quad + 2|a_2|\int_0^T\big\|u^*\big\|_{L^\infty}\big\|Du_n - Du^*\big\|_H\big\|D^2u^*\big\|_{L^\infty}\|D\psi\|_H\,dt\\ &\quad + |a_2|\int_0^T\big(\|u_n\|_{L^\infty} + \big\|u^*\big\|_{L^\infty}\big)\big\|D^3u_n\big\|_{L^\infty}\big\|u_n - u^*\big\|_H\|D\psi\|_H\,dt\end{aligned}$$

$$
\begin{aligned}
&+ |a_2| \int_0^T \|u^*\|_{L^\infty}^2 \|D^3 u_n - D^3 u^*\|_H \|D\psi\|_H \, dt \\
\le\ & 2|a_2| \|u_n\|_{C(0,T;L^\infty)} \|Du_n\|_{C(0,T;L^\infty)} \|D^2 u_n - D^2 u^*\|_{L(0,T;H)} \|D\psi\|_{L^2(0,T;H)} \\
&+ 2|a_2| \|u_n - u^*\|_{L^2(0,T;H)} \|Du_n\|_{C(0,T;L^\infty)} \|D^2 u^*\|_{C(0,T;L^\infty)} \|D\psi\|_{L^2(0,T;H)} \\
&+ 2|a_2| \|u^*\|_{C(0,T;L^\infty)} \|Du_n - Du^*\|_{L^2(0,T;H)} \|D^2 u^*\|_{C(0,T;L^\infty)} \|D\psi\|_{L^2(0,T;H)} \\
&+ |a_2| \|u_n\|_{C(0,T;L^\infty)} \|D^3 u_n\|_{L^2(0,T;L^\infty)} \|u_n - u^*\|_{C(0,T;H)} \|D\psi\|_{L^2(0,T;H)} \\
&+ \|u^*\|_{C(0,T;L^\infty)} \|D^3 u_n\|_{L^2(0,T;L^\infty)} \|u_n - u^*\|_{C(0,T;H)} \|D\psi\|_{L^2(0,T;H)} \\
&+ |a_2| \|u^*\|_{C(0,T;L^\infty)}^2 \|D^3 u_n - D^3 u^*\|_{L^2(0,T;H)} \|D\psi\|_{L^2(0,T;H)} \\
\to\ & 0, \quad \forall \psi \in L^2(0,T;V).
\end{aligned}
$$

Also we have

$$
\begin{aligned}
I_1^2 &= \int_0^T \int_0^1 D\left(\left(a_0 - \frac{\gamma k}{\gamma_2}\right) D^2 u_n - \left(a_0 - \frac{\gamma k}{\gamma_2}\right) D^2 u^*\right) D\psi \, dx \, dt \\
&\le \int_0^T \left(a_0 - \frac{\gamma k}{\gamma_2}\right) \left\| \left(D^3 u_n - D^3 u^*\right) \right\|_H \|D\psi\|_H \, dt \\
&\le \left(a_0 - \frac{\gamma k}{\gamma_2}\right) \left\| \left(D^3 u_n - D^3 u^*\right) \right\|_{C(0,T;H)} \|D\psi\|_{L^2(0,T;H)} \\
&\to 0, \quad \forall \psi \in L^2(0,T;V).
\end{aligned}
$$

Further, similar to (3.6), we have

$$
I_2 \to 0, \quad \forall \psi \in L^2(0,T;V).
$$

From (3.5), we have

$$
\left| \int_0^T \int_0^1 \left(B^* \overline{\varpi}^n - B^* \overline{\varpi}^*\right) \psi \, dx \, dt \right| \to 0, \quad \forall \psi \in L^2(0,T;V).
$$

In view of the above discussion, we can conclude that

$$
e_1\left(y^*, \overline{\varpi}^*\right) = 0, \quad \forall n \in \mathbb{N}.
$$

Since $y^* \in W(0,T;V)$, we have $y^*(0) \in H$. From $y_n \rightharpoonup y^*$ in $W(0,T;V)$, we can infer that $y_n(0) \rightharpoonup y^*(0)$. Thus we obtain

$$
\left(y_n(0) - y^*(0), \psi\right) \to 0, \quad \forall \psi \in H,
$$

which means that $e_2(y^*, \overline{\varpi}^*) = 0$, $\forall n \in \mathbb{N}$.

Hence, we can derive that $e(y^*, \overline{\varpi}^*) = 0$, $\forall n \in \mathbb{N}$.

In conclusion, there exists an optimal solution $(y^*, \overline{\varpi}^*)$ to the problem. We can infer that there exists an optimal solution $(y^*, \overline{\varpi}^*)$ to the viscous generalized Cahn-Hilliard equation due to $u = (1 - k\partial_x^2 + \gamma_2 \partial_x^4)^{-1} y$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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