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Blow-up for a parabolic system with nonlocal sources and nonlocal boundary conditions

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Abstract

This paper deals with blow-up properties of solutions to a nonlocal parabolic system with nonlocal boundary conditions. The global existence and finite time blow-up criteria are obtained. Moreover, for some special cases, we establish the precise blow-up rate estimates.

Keywords: global existence; finite time blow-up; nonlocal sources; nonlocal boundary conditions; blow-up rate

1 Introduction

In this article, we consider the positive solution of the following parabolic equations with nonlocal boundary conditions:

$$\begin{cases} u_t = f(u)(\Delta u + a\int_\Omega v\,dx), & x\in\Omega, t>0,\\ v_t = g(v)(\Delta v + b\int_\Omega u\,dx), & x\in\Omega, t>0,\\ u(x,t) = \int_\Omega \phi(x,y)u(y,t)\,dy, & x\in\partial\Omega, t>0,\\ v(x,t) = \int_\Omega \psi(x,y)v(y,t)\,dy, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), \qquad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N ($N\geq 1$) with smooth boundary $\partial\Omega$, $a,b>0$, while $\phi(x,y)$, $\psi(x,y)$ are nonnegative and continuous on $\partial\Omega\times\overline{\Omega}$, $u_0(x), v_0(x)\in C^{2,\theta}(\overline{\Omega})$ with $0<\theta<1$, $u_0(x), v_0(x)\geq 0$, $u_0(x)\not\equiv 0$, $v_0(x)\not\equiv 0$, and satisfy the compatibility conditions

$$u_0(x) = \int_\Omega \phi(x,y)u_0(y)\,dy, \qquad v_0(x) = \int_\Omega \psi(x,y)v_0(y)\,dy, \quad x\in\partial\Omega,$$

respectively.

There have been many articles dealing with properties of solutions to degenerate parabolic equations with homogeneous Dirichlet boundary condition (see [1–4] and references therein). For example, Deng *et al.* [5] studied the parabolic equation with nonlocal source

$$u_t = f(u)\left(\Delta u + a\int_\Omega u\,dx\right), \quad x\in\Omega, t>0, \tag{1.2}$$

which is subjected to homogeneous Dirichlet boundary condition. They proved that there exists no global positive solution if $\int^{\infty} 1/(sf(s))\,ds < \infty$ and $\int_\Omega \varphi(x)\,dx > 1/a$, where φ is the unique positive solution of the linear elliptic problem

$$-\Delta\varphi = 1, \quad x \in \Omega; \qquad \varphi(x) = 0, \quad x \in \partial\Omega.$$

In [6], Chen and Wang extended the problem (1.2) to the following system:

$$u_t = f(u)\left(\Delta u + a\int_\Omega v\,dx\right), \qquad v_t = g(v)\left(\Delta v + b\int_\Omega u\,dx\right), \quad x \in \Omega, t > 0, \tag{1.3}$$

with homogeneous Dirichlet boundary condition. Under some conditions, they proved the solution of (1.3) blows up in finite time and even blows up globally.

However, parabolic equations with both nonlocal source and nonlocal boundary condition have been studied as well. For instance, the problem of the following form:

$$\begin{cases} u_t = f(u)(\Delta u + \int_\Omega g(u)\,dx), & x \in \Omega, t > 0, \\ u(x,t) = \int_\Omega K(x,y)u^l(y,t)\,dy, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

was considered by Lin and Liu [7] for the case $l = 1$ and by Zhong and Tian [8] for the case $g(u) = u$. They established global existence and nonexistence of solutions, and they discussed the blow-up properties of solutions.

Porous medium equations with local sources or with nonlocal sources subjected to nonlocal boundary conditions were also studied (see [9–12]). They discussed the conditions of existence and blow-up. For other works on parabolic equations and systems with nonlocal boundary conditions, we refer readers to [13–23] and the references therein.

Motivated by those works above, we will study the problem (1.1) and want to understand how the functions $f(u)$, $g(v)$ and the weight functions $\phi(x,y)$, $\psi(x,y)$ in the boundary condition play substantial roles in determining the blow-up or not of the solutions.

In this article, we make some assumptions on $f(s)$, $g(s)$ as follows:

(H1) $f, g \in C([0,\infty)) \cap C^1((0,\infty))$, $f(0) = g(0) = 0$, and $f, g > 0$, $f', g' \geq 0$ in $(0,\infty)$.

(H2) Either $\liminf_{s\to\infty} \frac{f(s)}{g(s)} > 0$ or $\liminf_{s\to\infty} \frac{g(s)}{f(s)} > 0$ holds.

In view of the symmetry of the problem, we may suppose that $\liminf_{s\to\infty} \frac{f(s)}{g(s)} > 0$ in (H2) throughout this paper. For any $\eta > 0$, we can get a constant $K_0 > 0$ such that (see [5])

$$f(s) > K_0 g(s) \quad \text{for } s > \eta. \tag{1.5}$$

Let us introduce the following elliptic problem:

$$\begin{cases} -\Delta\varphi_1(x) = -\Delta\varphi_2(x) = 1, & x \in \Omega, \\ \varphi_1(x) = \int_\Omega \phi(x,y)\varphi_1(y)\,dy, & x \in \partial\Omega, \\ \varphi_2(x) = \int_\Omega \psi(x,y)\varphi_2(y)\,dy, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

where $\int_\Omega \phi(x,y)\,dy < 1$, $\int_\Omega \psi(x,y)\,dy < 1$. Then there exists a unique positive solution $(\varphi_1(x), \varphi_2(x))$ of (1.6) (see [19]).

Define

$$\mu_1 = \int_\Omega \varphi_1(x)\,dx, \qquad \mu_2 = \int_\Omega \varphi_2(x)\,dx.$$

The main results of this paper are the following theorems.

Theorem 1.1 *Suppose that* $\int_\Omega \phi(x,y)\,dy < 1$, $\int_\Omega \psi(x,y)\,dy < 1$ *for any* $x \in \partial\Omega$. *If* $ab \le \frac{1}{\mu_1\mu_2}$ *or* $\int_{s_0}^\infty \frac{ds}{sf(s)} = \infty$ *or* $\int_{s_0}^\infty \frac{ds}{sg(s)} = \infty$ *for some* $s_0 > 0$, *then the solution* (u,v) *of* (1.1) *exists globally.*

Theorem 1.2 *Suppose that* $\int_\Omega \phi(x,y)\,dy < 1$, $\int_\Omega \psi(x,y)\,dy < 1$ *for any* $x \in \partial\Omega$. *If* $ab > \frac{1}{\mu_1\mu_2}$ *and* $\int_{s_0}^\infty \frac{ds}{sg(s)} < \infty$ *for some* $s_0 > 0$, *then the solution* (u,v) *of* (1.1) *blows up in finite time.*

Theorem 1.3 *Suppose that* $\int_\Omega \phi(x,y)\,dy \ge 1$ *or* $\int_\Omega \psi(x,y)\,dy \ge 1$ *for any* $x \in \partial\Omega$. *If* $\int_{s_0}^\infty \frac{ds}{sg(s)} < \infty$ *for some* $s_0 > 0$, *then the solution* (u,v) *of* (1.1) *blows up in finite time.*

To estimate the blow-up rate, we need an additional assumption on the initial data $u_0(x)$, $v_0(x)$:

(H3) There exists a constant $\varepsilon \ge \max\{\varepsilon_1, \varepsilon_2\}$, such that

$$\Delta u_0 + a\int_\Omega v_0\,dx - \varepsilon u_0^{k_1+1-p} \ge 0, \qquad \Delta v_0 + b\int_\Omega u_0\,dx - \varepsilon v_0^{k_2+1-q} \ge 0,$$

where $0 < p, q < 1$ and ε_1, ε_2, k_1, k_2 are given in Section 5.

Theorem 1.4 *Suppose that* $\int_\Omega \phi(x,y)\,dy < 1$, $\int_\Omega \psi(x,y)\,dy < 1$ *for any* $x \in \partial\Omega$. *Let* $f(u) = u^p$, $g(v) = v^q$ $(0 < p, q < 1)$. *If* (H3) *holds and the solution* (u,v) *of* (1.1) *blows up in finite time* T^*. *Then there exist positive constants* c_i $(i = 1,2,3,4)$ *such that*

$$c_1 \le \max_{x\in\overline{\Omega}} u(x,t)\bigl(T^* - t\bigr)^{(2-q)/(p+q-pq)} \le c_2,$$

$$c_3 \le \max_{x\in\overline{\Omega}} v(x,t)\bigl(T^* - t\bigr)^{(2-p)/(p+q-pq)} \le c_4.$$

This paper is organized as follows. In Section 2, we establish the comparison principle. In Sections 3 and 4, some criteria regarding to global existence and finite time blow-up for (1.1) are given, respectively. In the last section, for some special cases, the blow-up rate estimate is established.

2 Comparison principle

We start with the definition of a subsolution and a supersolution of (1.1) and then get to the comparison principle. Set $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$ and $\overline{Q}_T = \overline{\Omega} \times [0,T)$.

Definition 2.1 A vector function $(\underline{u}(x,t), \underline{v}(x,t))$ defined on $\overline{Q}_T$, for some $T > 0$, is called a subsolution of problem (1.1), if $\underline{u}(x,t), \underline{v}(x,t) \in C(\overline{Q}_T) \cap C^{2,1}(Q_T)$ and satisfy

$$\begin{cases} \underline{u}_t \le f(\underline{u})(\Delta\underline{u} + a\int_\Omega \underline{v}\,dx), & x \in \Omega, t > 0, \\ \underline{v}_t \le g(\underline{v})(\Delta\underline{v} + b\int_\Omega \underline{u}\,dx), & x \in \Omega, t > 0, \\ (\underline{u}, \underline{v}) \le (\int_\Omega \phi(x,y)\underline{u}(y,t)\,dy, \int_\Omega \psi(x,y)\underline{v}(y,t)\,dy), & x \in \partial\Omega, t > 0, \\ (\underline{u}(x,0), \underline{v}(x,0)) \le (u_0(x), v_0(x)), & x \in \Omega. \end{cases} \tag{2.1}$$

Similarly, a vector function $(\overline{u}(x,t),\overline{v}(x,t)) \in [C(\overline{Q}_T) \cap C^{2,1}(Q_T)]^2$ is a supersolution of (1.1) if the reversed inequalities hold in (2.1). A solution of problem (1.1) is a vector function which is both a subsolution and a supersolution of (1.1).

The following comparison principle plays a crucial role in our proofs, which can be obtained by similar arguments to [20, 21], and its proof is given here for the sake of completeness.

Lemma 2.1 *Suppose that* $w(x,t), z(x,t) \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ *and satisfy*

$$\begin{cases} w_t - d_1\Delta w \geq \sum_{j=1}^n a_{1j}w_{x_j} + c_{11}w + c_{12}\int_\Omega c_{13}z(x,t)\,dx, & (x,t) \in Q_T, \\ z_t - d_2\Delta z \geq \sum_{j=1}^n a_{2j}z_{x_j} + c_{21}z + c_{22}\int_\Omega c_{23}w(x,t)\,dx, & (x,t) \in Q_T, \\ w(x,t) \geq \int_\Omega c_{31}(x,y)w(y,t)\,dy, & (x,t) \in S_T, \\ z(x,t) \geq \int_\Omega c_{32}(x,y)z(y,t)\,dy, & (x,t) \in S_T, \\ w(x,0) > 0, \qquad z(x,0) > 0, & x \in \overline{\Omega}, \end{cases} \tag{2.2}$$

where $d_i = d_i(x,t)$, $a_{ij} = a_{ij}(x,t)$ $(i = 1,2,\ j = 1,2,\dots,n)$ *and* $c_{ij} = c_{ij}(x,t)$ $(i = 1,2,\ j = 1,2,3)$ *are bounded continuous functions,* $d_i(x,t), c_{ij}(x,t) \geq 0$ $(i = 1,2,\ j = 2,3)$ *in* Q_T, $c_{3j}(x,y) \geq 0$ $(j = 1,2)$ *on* $\partial\Omega \times \Omega$ *and* $\int_\Omega c_{3j}(x,y)\,dy > 0$ *on* $\partial\Omega$. *Then* $w(x,t), z(x,t) > 0$ *on* $\overline{Q}_T$.

Proof Let $\hat{c}_i = \sup_{(x,t)\in\overline{Q}_T} |c_{i1}|$, $i = 1,2$. Set $U = e^{-\gamma t}w$, $V = e^{-\gamma t}z$ with $\gamma > \max\{\hat{c}_1, \hat{c}_2\}$. Then, for $(x,t) \in Q_T$, we have

$$\begin{aligned} U_t - d_1\Delta U &\geq \sum_{j=1}^n a_{1j}U_{x_j} + (c_{11} - \gamma)U + c_{12}\int_\Omega c_{13}V(x,t)\,dx, \\ V_t - d_2\Delta V &\geq \sum_{j=1}^n a_{2j}V_{x_j} + (c_{21} - \gamma)V + c_{22}\int_\Omega c_{23}U(x,t)\,dx. \end{aligned} \tag{2.3}$$

Also

$$\begin{aligned} &U \geq \int_\Omega c_{31}(x,y)U(y,t)\,dy, \qquad V \geq \int_\Omega c_{32}(x,y)V(y,t)\,dy, \quad (x,t) \in S_T, \\ &U(x,0) = w(x,0) > 0, \qquad V(x,0) = z(x,0) > 0, \quad x \in \overline{\Omega}. \end{aligned} \tag{2.4}$$

It suffices to show that $U,V > 0$ on $\overline{Q}_T$. Since $U(x,0), V(x,0) > 0$, there exists $\delta > 0$ such that $U,V > 0$ for $(x,t) \in \overline{\Omega} \times (0,\delta)$. Suppose for a contradiction that $\overline{t} = \sup\{t \in (0,T) : U,V > 0 \text{ on } \overline{\Omega} \times [0,t)\} < T$. Then $U,V \geq 0$ on $\overline{Q}_{\overline{t}}$, and at least one of U, V vanishes at $(\overline{x},\overline{t})$ for $\overline{x} \in \overline{\Omega}$. Without loss of generality, we assume that $U(\overline{x},\overline{t}) = 0 = \inf_{(x,t)\in\overline{Q}_{\overline{t}}} U(x,t)$. If $(\overline{x},\overline{t}) \in Q_{\overline{t}}$, by virtue of the first inequality of (2.3), we get

$$U_t - d_1\Delta U \geq \sum_{j=1}^n a_{1j}U_{x_j} + (c_{11} - \gamma)U, \quad (x,t) \in Q_{\overline{t}}.$$

Then the strong maximum principle implies that $U \equiv 0$ in $Q_{\overline{t}}$, and this is a contradiction. If $(\overline{x},\overline{t}) \in S_{\overline{t}}$, this results in the contradiction by (2.4) that

$$0 = U(\overline{x},\overline{t}) = e^{-\gamma t}w(\overline{x},\overline{t}) > \int_\Omega c_{31}(\overline{x},y)U(y,\overline{t})\,dy > 0$$

due to $\int_\Omega c_{31}(x,y)\,dy > 0$ on $\partial\Omega$. This proves $U,V > 0$, and in turn $w,z > 0$ on $\overline{Q}_T$. □

Lemma 2.2 *Suppose that $w(x,t), z(x,t) \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and satisfy*

$$\begin{cases} w_t - d_1 \Delta w \geq \sum_{j=1}^n a_{1j} w_{x_j} + c_{11} w + c_{12} \int_\Omega c_{13} z(x,t)\,dx, & (x,t) \in Q_T, \\ z_t - d_2 \Delta z \geq \sum_{j=1}^n a_{2j} z_{x_j} + c_{21} z + c_{22} \int_\Omega c_{23} w(x,t)\,dx, & (x,t) \in Q_T, \\ w(x,t) \geq \int_\Omega c_{31}(x,y) w(y,t)\,dy, & (x,t) \in S_T, \\ z(x,t) \geq \int_\Omega c_{32}(x,y) z(y,t)\,dy, & (x,t) \in S_T, \\ w(x,0) \geq 0, \quad z(x,0) \geq 0, & x \in \overline{\Omega}, \end{cases} \tag{2.5}$$

where $d_i = d_i(x,t)$, $a_{ij} = a_{ij}(x,t)$ $(i = 1,2, j = 1,2,\dots,n)$ and $c_{ij} = c_{ij}(x,t)$ $(i = 1,2, j = 1,2,3)$ are bounded continuous functions, $d_i(x,t), c_{ij}(x,t) \geq 0$ $(i = 1,2, j = 2,3)$ in Q_T, $c_{3j}(x,y) \geq 0$ $(j = 1,2)$ on $\partial\Omega \times \Omega$ and $\int_\Omega c_{3j}(x,y)\,dy > 0$ on $\partial\Omega$. Then $w(x,t), z(x,t) \geq 0$ on $\overline{Q}_T$.

Proof Set $w(x,t) = \alpha(x)U(x,t)$, $z(x,t) = \alpha(x)V(x,t)$, where $\alpha(x) \in C^2(\overline{\Omega})$ satisfies

$$\begin{aligned} &\alpha(x) > 0 \quad \text{on } \overline{\Omega}; \\ &\alpha(x) = 1, \qquad \int_\Omega \alpha(y) c_{3j}(x,y)\,dy \leq \frac{1}{2} \quad \text{on } \partial\Omega, j = 1,2. \end{aligned}$$

A direct computation yields

$$\begin{aligned} U_t - d_1 \Delta U \geq{}& \sum_{j=1}^n \left(a_{1j} + d_1 \frac{2\alpha_{x_j}}{\alpha} \right) U_{x_j} + \left(c_{11} + d_1 \frac{\Delta\alpha}{\alpha} + \sum_{j=1}^n a_{1j} \frac{\alpha_{x_j}}{\alpha} \right) U \\ &+ \frac{c_{12}}{\alpha(x)} \int_\Omega c_{13} \alpha(x) V(x,t)\,dx, \quad (x,t) \in Q_T, \\ V_t - d_2 \Delta V \geq{}& \sum_{j=1}^n \left(a_{2j} + d_2 \frac{2\alpha_{x_j}}{\alpha} \right) V_{x_j} + \left(c_{21} + d_2 \frac{\Delta\alpha}{\alpha} + \sum_{j=1}^n a_{2j} \frac{\alpha_{x_j}}{\alpha} \right) V \\ &+ \frac{c_{22}}{\alpha(x)} \int_\Omega c_{23} \alpha(x) U(x,t)\,dx, \quad (x,t) \in Q_T, \\ U \geq{}& \int_\Omega c_{31}(x,y) \alpha(y) U(y,t)\,dy, \quad (x,t) \in S_T, \\ V \geq{}& \int_\Omega c_{32}(x,y) \alpha(y) V(y,t)\,dy, \quad (x,t) \in S_T, \\ U(x,0) ={}& w(x,0)/\alpha(x) \geq 0, \qquad V(x,0) = z(x,0)/\alpha(x) \geq 0, \quad x \in \overline{\Omega}. \end{aligned} \tag{2.6}$$

Define

$$\begin{aligned} &\overline{b}_i = \sup_{(x,t)\in Q_T} \left| c_{i1} + d_i \frac{\Delta\alpha}{\alpha} + \sum_{j=1}^n a_{ij} \frac{\alpha_{x_j}}{\alpha} \right|, \\ &\overline{c}_i = \sup_{(x,t)\in Q_T} \frac{c_{i2}}{\alpha(x)}, \qquad \overline{d}_i = \sup_{(x,t)\in Q_T} c_{i3}\alpha(x), \quad i = 1,2, \end{aligned}$$

and set

$$\widetilde{U} = U + \tilde{\varepsilon} e^{\gamma t}, \qquad \widetilde{V} = V + \tilde{\varepsilon} e^{\gamma t},$$

with $\gamma > \max\{\overline{b}_1 + \overline{c}_1 \overline{d}_1 |\Omega|, \overline{b}_2 + \overline{c}_2 \overline{d}_2 |\Omega|\}$, $\tilde{\varepsilon} > 0$.

Using (2.6), we have

$$\begin{aligned}
&\widetilde{U}_t - d_1\Delta\widetilde{U} \geq \sum_{j=1}^{n}\left(a_{1j} + d_1\frac{2\alpha_{x_j}}{\alpha}\right)\widetilde{U}_{x_j} + \left(c_{11} + d_1\frac{\Delta\alpha}{\alpha} + \sum_{j=1}^{n} a_{1j}\frac{\alpha_{x_j}}{\alpha}\right)\widetilde{U} \\
&\qquad + \frac{c_{12}}{\alpha(x)}\int_\Omega c_{13}\alpha(x)\widetilde{V}(x,t)\,dx, \quad (x,t)\in Q_T, \\
&\widetilde{V}_t - d_2\Delta\widetilde{V} \geq \sum_{j=1}^{n}\left(a_{2j} + d_2\frac{2\alpha_{x_j}}{\alpha}\right)\widetilde{V}_{x_j} + \left(c_{21} + d_2\frac{\Delta\alpha}{\alpha} + \sum_{j=1}^{n} a_{2j}\frac{\alpha_{x_j}}{\alpha}\right)\widetilde{V} \\
&\qquad + \frac{c_{22}}{\alpha(x)}\int_\Omega c_{23}\alpha(x)\widetilde{U}(x,t)\,dx, \quad (x,t)\in Q_T, \\
&\widetilde{U} \geq \int_\Omega c_{31}(x,y)\alpha(y)\widetilde{U}(y,t)\,dy, \quad (x,t)\in S_T, \\
&\widetilde{V} \geq \int_\Omega c_{32}(x,y)\alpha(y)\widetilde{V}(y,t)\,dy, \quad (x,t)\in S_T, \\
&\widetilde{U}(x,0) = w(x,0)/\alpha(x) + \tilde{\varepsilon} > 0, \qquad \widetilde{V}(x,0) = z(x,0)/\alpha(x) + \tilde{\varepsilon} > 0, \quad x\in\overline{\Omega}.
\end{aligned}$$

By Lemma 2.1, we know that $\widetilde{U}, \widetilde{V} > 0$, *i.e.* $U + \tilde{\varepsilon}e^{\gamma t} > 0$, $V + \tilde{\varepsilon}e^{\gamma t} > 0$ on $\overline{Q}_T$. It follows by $\tilde{\varepsilon} \to 0^+$ that $U, V \geq 0$ and hence $w, z \geq 0$ on $\overline{Q}_T$. □

By the above lemmas, we obtain the following comparison principle of problem (1.1).

Lemma 2.3 *Let* $(\underline{u}, \underline{v})$ *and* $(\overline{u}, \overline{v})$ *be a nonnegative subsolution and supersolution of* (1.1) *on* $\overline{Q}_T$, *respectively. Then* $(\overline{u}, \overline{v}) \geq (\underline{u}, \underline{v})$ *on* $\overline{Q}_T$, *if* $\overline{u}, \overline{v} \geq \eta$ *or* $\underline{u}, \underline{v} \geq \eta$ *for some small positive constant* η *hold.*

The existence of positive classical solutions of (1.1) local in time can be obtained by using the fixed point theorem in [22], and the representation formula and the contraction mapping principle as in [7]. By the above comparison principle, we get the uniqueness of the solution to the problem. The proof is more or less standard, so is omitted here.

3 Global existence

In this section, we will give some sufficient conditions for the existence of solution and prove Theorem 1.1.

Proof of Theorem 1.1 Case 1: we assume that $ab \leq \frac{1}{\mu_1\mu_2}$ holds. Since the functions $\varphi_1(x)$, $\varphi_2(x)$ are positive and continuous, we can find two large positives constants k_1 and k_2 such that

$$\begin{aligned}
&u_0(x) \leq k_1\varphi_1(x), \qquad v_0(x) \leq k_2\varphi_2(x), \quad x\in\overline{\Omega}, \\
&a\mu_2 \leq k_1/k_2 \leq 1/(b\mu_1).
\end{aligned} \tag{3.1}$$

Set

$$w(x,t) = k_1\varphi_1(x), \qquad z(x,t) = k_2\varphi_2(x).$$

Applying (1.6) and (3.1), we get

$$\begin{aligned}
&w_t - f(w)\left(\Delta w + a\int_\Omega z\,dx\right) = -f(k_1\varphi_1)(-k_1 + ak_2\mu_2) \geq 0, \quad (x,t) \in Q_T,\\
&z_t - g(z)\left(\Delta z + b\int_\Omega w\,dx\right) = -g(k_2\varphi_2)(-k_2 + bk_1\mu_1) \geq 0, \quad (x,t) \in Q_T,\\
&w(x,t) = k_1\varphi_1(x) = \int_\Omega \phi(x,y)w(y,t)\,dy, \quad (x,t) \in S_T,\\
&z(x,t) = \int_\Omega \psi(x,y)z(y,t)\,dy, \quad (x,t) \in S_T,\\
&w(x,0) \geq u_0(x), \qquad z(x,0) \geq v_0(x), \quad x \in \overline{\Omega}.
\end{aligned}$$

The above inequalities show that (w,z) is a supersolution of (1.1), and Lemma 2.3 asserts that $(w,z) \geq (u,v)$ for $(x,t) \in \overline{Q}_T$. Therefore, the solution (u,v) of (1.1) exists globally.

Case 2: we assume that $\int_{s_0}^\infty \frac{ds}{sf(s)} = \infty$ holds. It follows from (1.5) that there exists $K_0 > 0$ such that

$$f(s) \geq K_0 g(s) \quad \text{for } s > M = \max\Big\{\max_{x\in\overline{\Omega}} u_0(x), \max_{x\in\overline{\Omega}} v_0(x)\Big\}.$$

Choose $A = \max\{a|\Omega|, b|\Omega|/K_0\}$, and consider the ordinary differential equation (ODE)

$$s'(t) = Af\big(s(t)\big)s(t), \quad t > 0; \qquad s(0) = M. \tag{3.2}$$

By the hypothesis (H1) and the theory of ODE, there exists a unique solution $s(t)$ to (3.2), and $s(t)$ is increasing. Since $\int_{s_0}^\infty \frac{ds}{sf(s)} = \infty$ for some $s_0 > 0$, $s(t)$ exists globally and $s(t) \geq s_0$. Let $w(x,t) = z(x,t) = s(t)$, and note that $\int_\Omega \phi(x,y)\,dy < 1$, $\int_\Omega \psi(x,y)\,dy < 1$ on $\partial\Omega$. Then we have

$$\begin{aligned}
&w_t - f(w)\left(\Delta w + a\int_\Omega z\,dx\right)\\
&\quad = s'(t) - f\big(s(t)\big)a|\Omega|s(t) = \big(A - a|\Omega|\big)f\big(s(t)\big)s(t) \geq 0, \quad (x,t) \in Q_T,\\
&z_t - g(z)\left(\Delta z + b\int_\Omega w\,dx\right)\\
&\quad = s'(t) - g\big(s(t)\big)b|\Omega|s(t) \geq \big(AK_0 - b|\Omega|\big)g\big(s(t)\big)s(t) \geq 0, \quad (x,t) \in Q_T,\\
&w(x,t) = s(t) > \int_\Omega \phi(x,y)w(y,t)\,dy, \quad (x,t) \in S_T,\\
&z(x,t) > \int_\Omega \psi(x,y)z(y,t)\,dy, \quad (x,t) \in S_T,\\
&w(x,0) = s(0) \geq u_0(x), \qquad z(x,0) = s(0) \geq v_0(x), \quad x \in \overline{\Omega}.
\end{aligned}$$

The above inequalities show that (w,z) is a supersolution of problem (1.1). By using Lemma 2.3, we see that the solution (u,v) of (1.1) exists globally.

Case 3: we assume that $\int_{s_0}^\infty \frac{ds}{sg(s)} = \infty$ holds. We choose two positive constants l_1, l_2 such that

$$\frac{a|\Omega|l_2}{l_1} < 1, \qquad r = \max\Big\{1\Big/\Big(l_1 \min_{x\in\overline{\Omega}} \varphi_1(x)\Big), 1/l_2\Big\}.$$

Consider the following ODE:

$$s'(t) = \frac{bl_1\mu_1}{l_2} g\big(l_2 s(t)\big)s(t), \quad t > 0; \qquad s(0) = r(M+1). \tag{3.3}$$

Here $M = \max\{\max_{x\in\overline{\Omega}} u_0(x), \max_{x\in\overline{\Omega}} v_0(x)\}$. In view of $\int_{s_0}^{\infty} \frac{ds}{sg(s)} = \infty$ for some $s_0 > 0$, we know that $s(t)$ exists globally.

Let

$$w(x,t) = l_1\varphi_1(x)s(t), \qquad z(x,t) = l_2 s(t).$$

Then by (3.3) we have

$$\begin{aligned}
&w_t - f(w)\left(\Delta w + a\int_\Omega z\,dx\right) \\
&\quad = l_1\varphi_1 s'(t) - f\big(l_1\varphi_1 s(t)\big)\big(-l_1 + al_2|\Omega|\big)s(t) \\
&\quad = l_1\varphi_1 \frac{bl_1\mu_1}{l_2} g\big(l_2 s(t)\big)s(t) + f\big(l_1\varphi_1 s(t)\big)s(t)\big(l_1 - al_2|\Omega|\big) \ge 0, \quad (x,t)\in Q_T, \\
&z_t - g(z)\left(\Delta z + b\int_\Omega w\,dx\right) = l_2 s'(t) - g\big(l_2 s(t)\big)bl_1\mu_1 s(t) = 0, \quad (x,t)\in Q_T, \\
&w(x,t) = l_1 s(t)\int_\Omega \phi(x,y)\varphi_1(y)\,dy = \int_\Omega \phi(x,y)w(y,t)\,dy, \quad (x,t)\in S_T, \\
&z(x,t) > \int_\Omega \psi(x,y)z(y,t)\,dy, \quad (x,t)\in S_T, \\
&w(x,0) = l_1\varphi_1(x)s(0) > u_0(x), \qquad z(x,0) = l_2 s(0) > v_0(x), \quad x\in\overline{\Omega}.
\end{aligned}$$

These formulas show that (w,z) is a supersolution of (1.1). Therefore, $(w,z) \ge (u,v)$. Since (w,z) exists globally, so does (u,v). This completes the proof. □

4 Blow-up results

In this section, we assume that (u,v) is a positive solution of (1.1) on $\overline{Q}_T$, where T is the maximal existence time.

Proof of Theorem 1.2 Set $\underline{K}_i = \min_{x\in\overline{\Omega}} \varphi_i(x)$, $\overline{K}_i = \max_{x\in\overline{\Omega}} \varphi_i(x)$, $i = 1,2$. In view of $ab > \frac{1}{\mu_1\mu_2}$, then there exist positive constants $l_1, l_2 > 1$ such that

$$l_1\underline{K}_1, l_2\underline{K}_2 \ge 1 \quad \text{and} \quad \frac{1}{b\mu_1} < \frac{l_1}{l_2} < a\mu_2. \tag{4.1}$$

Taking $\delta = \frac{1}{2}\min\{\min_{x\in\overline{\Omega}} u_0(x), \min_{x\in\overline{\Omega}} v_0(x)\}$ and $r = \min\{\frac{1}{l_1\overline{K}_1}, \frac{1}{l_2\overline{K}_2}\}$, it follows from (1.5) that there exists $K_0' > 0$ such that

$$f(s) \ge K_0' g(s) \quad \text{for } s > r\delta.$$

Choose $B = \min\{\frac{K_0'(al_2\mu_2 - l_1)}{l_1\overline{K}_1}, \frac{bl_1\mu_1 - l_2}{l_2\overline{K}_2}\}$, and consider the following ODE:

$$s'(t) = Bg\big(s(t)\big)s(t), \quad t > 0; \qquad s(0) = r\delta. \tag{4.2}$$

Since $\int_{s_0}^{\infty} \frac{ds}{sg(s)} < \infty$ for some $s_0 > 0$, $s(t)$ blows up in finite time.

Let

$$w(x,t) = l_1\varphi_1(x)s(t), \qquad z(x,t) = l_2\varphi_2(x)s(t).$$

Then (4.1) and (4.2) imply that

$$\begin{aligned}
&w_t - f(w)\left(\Delta w + a\int_\Omega z\,dx\right)\\
&\quad = l_1\varphi_1 s'(t) - f\bigl(l_1\varphi_1 s(t)\bigr)(-l_1 + al_2\mu_2)s(t)\\
&\quad \le l_1\overline{K}_1 Bg\bigl(s(t)\bigr)s(t) - K_0' g\bigl(l_1\varphi_1 s(t)\bigr)(-l_1 + al_2\mu_2)s(t)\\
&\quad \le \bigl(l_1\overline{K}_1 B - K_0'(al_2\mu_2 - l_1)\bigr)g\bigl(s(t)\bigr)s(t) \le 0, \quad (x,t)\in Q_T,\\
&z_t - g(z)\left(\Delta z + b\int_\Omega w\,dx\right)\\
&\quad = l_2\varphi_2 s'(t) - g\bigl(l_2\varphi_2 s(t)\bigr)(-l_2 + bl_1\mu_1)s(t)\\
&\quad \le l_2\overline{K}_2 Bg\bigl(s(t)\bigr)s(t) - g\bigl(l_2\varphi_2 s(t)\bigr)(bl_1\mu_1 - l_2)s(t)\\
&\quad \le \bigl(l_2\overline{K}_2 B - (bl_1\mu_1 - l_2)\bigr)g\bigl(s(t)\bigr)s(t) \le 0, \quad (x,t)\in Q_T,\\
&w(x,t) = \int_\Omega \phi(x,y)w(y,t)\,dy, \qquad z(x,t) = \int_\Omega \psi(x,y)z(y,t)\,dy, \quad (x,t)\in S_T,\\
&w(x,0) = l_1\varphi_1(x)s(0) < u_0(x), \qquad z(x,0) = l_2\varphi_2(x)s(0) < v_0(x), \quad x\in\overline{\Omega}.
\end{aligned}$$

The above inequalities imply that (w,z) is a subsolution of (1.1), so $(w,z)\le(u,v)$. Due to (w,z) blowing up in finite time, (u,v) blows up in finite time, and this completes the proof. □

Proof of Theorem 1.3 (i) Suppose that $\int_\Omega \phi(x,y)\,dy \ge 1$, $\int_\Omega \psi(x,y)\,dy \ge 1$ on $\partial\Omega$. By (1.5), there exists a positive constant $K_0'' > 0$ such that

$$f(s) \ge K_0'' g(s) \quad \text{for } s > m = \frac{1}{2}\min\Bigl\{\min_{x\in\overline{\Omega}} u_0(x), \min_{x\in\overline{\Omega}} v_0(x)\Bigr\}.$$

Let $C = \min\{aK_0''|\Omega|, b|\Omega|\}$, and consider the following ODE:

$$s'(t) = Cg\bigl(s(t)\bigr)s(t), \quad t>0; \qquad s(0) = m. \tag{4.3}$$

Since $\int_{s_0}^\infty \frac{ds}{sg(s)} < \infty$ for some $s_0 > 0$, the solution $s(t)$ of (4.3) blows up.

Let $w(x,t) = z(x,t) = s(t)$, then we obtain

$$\begin{aligned}
&w_t - f(w)\left(\Delta w + a\int_\Omega z\,dx\right)\\
&\quad = s'(t) - f\bigl(s(t)\bigr)a|\Omega|s(t) = Cg\bigl(s(t)\bigr)s(t) - a|\Omega|f\bigl(s(t)\bigr)s(t)\\
&\quad \le \bigl(C - aK_0''|\Omega|\bigr)g\bigl(s(t)\bigr)s(t) \le 0, \quad (x,t)\in Q_T,\\
&z_t - g(z)\left(\Delta z + b\int_\Omega w\,dx\right)
\end{aligned}$$

$$
\begin{aligned}
&= s'(t) - g\big(s(t)\big)b|\Omega|s(t) = \big(C - b|\Omega|\big)g\big(s(t)\big)s(t) \le 0, \quad (x,t) \in Q_T, \\
w(x,t) &= s(t) \le \int_\Omega \phi(x,y)w(y,t)\,dy, \quad (x,t) \in S_T, \\
z(x,t) &\le \int_\Omega \psi(x,y)z(y,t)\,dy, \quad (x,t) \in S_T, \\
w(x,0) &= s(0) \le u_0(x), \qquad z(x,0) = s(0) \le v_0(x), \quad x \in \overline{\Omega}.
\end{aligned}
$$

The above inequalities show that (w,z) is a subsolution of problem (1.1), and Lemma 2.3 shows that $(w,z) \le (u,v)$, so the solution (u,v) of (1.1) blows up in finite time.

(ii) Suppose that $\int_\Omega \phi(x,y)\,dy < 1$, $\int_\Omega \psi(x,y)\,dy \ge 1$ on $\partial\Omega$. Let $\lambda > 0$ be the first eigenvalue of the eigenvalue problem

$$
-\Delta\Phi = \lambda\Phi, \quad x \in \Omega; \qquad \Phi(x) = 0, \quad x \in \partial\Omega, \tag{4.4}
$$

and $\Phi(x)$ be the corresponding eigenfunction with $\max_{x\in\overline{\Omega}} \Phi(x) = 1$, $\Phi(x) > 0$ in Ω. Set $\underline{K} = \min_{x\in\Omega} \Phi(x) > 0$, ε_* is a small enough positive constant such that $\varepsilon_* < \min\{\frac{a|\Omega|}{\lambda}, \frac{1}{\underline{K}}\}$ and $s(t)$ is the solution of ODE as follows:

$$
s'(t) = Dg\big(\varepsilon_*\underline{K}s(t)\big)s(t), \quad t > 0; \qquad s(0) = \varepsilon_*\underline{K}m, \tag{4.5}
$$

where $D = \min\{\frac{a|\Omega|-\varepsilon_*\lambda}{\varepsilon_* K_0'''}, b\varepsilon_*\underline{K}|\Omega|\}$, m is given at the beginning of this proof, and $K_0''' > 0$ is determined by (1.5) and satisfies $f(s) > K_0'''g(s)$ for $s \ge \varepsilon_*\underline{K}m$. Since $\int_{s_0}^\infty \frac{ds}{sg(s)} < \infty$ for some $s_0 > 0$, the solution to (4.5) blows up in finite time.

Let

$$
w(x,t) = \varepsilon_*\Phi(x)s(t), \qquad z(x,t) = s(t).
$$

By (4.4), we have

$$
\begin{aligned}
&w_t - f(w)\left(\Delta w + a\int_\Omega z\,dx\right) \\
&\quad = \varepsilon_*\Phi(x)s'(t) - f\big(\varepsilon_*\Phi(x)s(t)\big)\big(-\varepsilon_*\lambda\Phi(x)s(t) + a|\Omega|s(t)\big) \\
&\quad = \varepsilon_*\Phi(x)Dg\big(\varepsilon_*\underline{K}s(t)\big)s(t) - \big(a|\Omega| - \varepsilon_*\lambda\Phi(x)\big)f\big(\varepsilon_*\Phi(x)s(t)\big)s(t) \\
&\quad \le \big[\varepsilon_* DK_0''' - \big(a|\Omega| - \varepsilon_*\lambda\big)\big]f\big(\varepsilon_*\Phi(x)s(t)\big)s(t) \le 0, \quad (x,t) \in Q_T, \\
&z_t - g(z)\left(\Delta z + b\int_\Omega w\,dx\right) \\
&\quad = s'(t) - g\big(s(t)\big)b\varepsilon_* s(t)\int_\Omega \Phi\,dx \\
&\quad \le Dg\big(\varepsilon_*\underline{K}s(t)\big)s(t) - b\varepsilon_*\underline{K}|\Omega|g\big(s(t)\big)s(t) \\
&\quad \le \big(D - b\varepsilon_*\underline{K}|\Omega|\big)g\big(s(t)\big)s(t) \le 0, \quad (x,t) \in Q_T, \\
&w(x,t) = 0 \le \int_\Omega \phi(x,y)w(y,t)\,dy, \qquad z(x,t) \le \int_\Omega \psi(x,y)z(y,t)\,dy, \quad (x,t) \in S_T, \\
&w(x,0) = \varepsilon_*\Phi(x)s(0) \le \varepsilon_*^2\underline{K}m \le u_0(x), \qquad z(x,0) = s(0) \le v_0(x), \quad x \in \overline{\Omega}.
\end{aligned}
$$

All the above inequalities show that $(w,z) = (\varepsilon_* \Phi(x)s(t), s(t))$ is a subsolution of (1.1). By Lemma 2.3, we know $(u,v) \geq (w,z)$. Since (w,z) blows up in finite time, so does (u,v).

(iii) Suppose that $\int_\Omega \phi(x,y)\,dy \geq 1$, $\int_\Omega \psi(x,y)\,dy < 1$ on $\partial\Omega$. In this case, the proof can be treated as case (ii), so we omit it here. This completes the proof of Theorem 1.3. □

5 Blow-up rate estimates

Now, we will consider the blow-up rate of the solution to (1.1) in the special case that $f(u) = u^p$, $g(v) = v^q$ $(0 < p, q < 1)$ and $\int_\Omega \phi(x,y)\,dy < 1$, $\int_\Omega \psi(x,y)\,dy < 1$ for any $x \in \partial\Omega$, *i.e.*

$$\begin{cases} u_t = u^p(\Delta u + a\int_\Omega v\,dx), & x \in \Omega, t > 0, \\ v_t = v^q(\Delta v + b\int_\Omega u\,dx), & x \in \Omega, t > 0, \\ u(x,t) = \int_\Omega \phi(x,y)u(y,t)\,dy, & x \in \partial\Omega, t > 0, \\ v(x,t) = \int_\Omega \psi(x,y)v(y,t)\,dy, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{5.1}$$

It can be seen from Theorem 1.2 that the solution (u,v) to (5.1) blows up in finite time T^*. Denote $U(t) = \max_{x\in\overline{\Omega}} u(x,t)$, $V(t) = \max_{x\in\overline{\Omega}} v(x,t)$, which are Lipschitz continuous (see [23] or [12]). From (5.1), we have $U(t)$, $V(t)$ satisfying

$$U_t \leq a|\Omega|U^p V, \qquad V_t \leq b|\Omega|UV^q, \quad \text{a.e. } t \in (0,T^*). \tag{5.2}$$

Let $\rho_1 = 2 - p$, $\rho_2 = 2 - q$, by virtue of Young's inequality, there exists $C_1 > 0$ such that

$$\left(U^{\rho_1} + V^{\rho_2}\right)_t \leq (a\rho_1 + b\rho_2)|\Omega|\left(U^{\rho_1}\right)^{1/\rho_1}\left(V^{\rho_2}\right)^{1/\rho_2} \leq C_1\left(U^{\rho_1} + V^{\rho_2}\right)^{(\rho_1+\rho_2)/\rho_1\rho_2}.$$

Integrating the above inequality over (t, T^*), we can get

$$U^{\rho_1}(t) + V^{\rho_2}(t) \geq C_2\left(T^* - t\right)^{-\rho_1\rho_2/d}, \tag{5.3}$$

where $C_2 > 0$ is a constant and $d = 1 - (1-p)(1-q)$. Let $k_1 = d/\rho_2$, $k_2 = d/\rho_1$.

Lemma 5.1 *There exists a constant* $\varepsilon > 0$, *which is defined in* (H3) *such that*

$$u_t - \varepsilon u^{k_1+1} \geq 0, \qquad v_t - \varepsilon v^{k_2+1} \geq 0, \quad (x,t) \in \Omega \times (0,T^*). \tag{5.4}$$

Proof Set $J_1(x,t) = u_t - \varepsilon u^{k_1+1}$, $J_2(x,t) = v_t - \varepsilon v^{k_2+1}$. A series of computation yields

$$\begin{aligned} &J_{1t} - u^p\Delta J_1 - 2p\varepsilon u^{k_1}J_1 - au^p\int_\Omega J_2\,dx \\ &\quad= pu^{-1}J_1^2 + \varepsilon(k_1+1)k_1u^{k_1+p-1}|\nabla u|^2 + p\varepsilon^2u^{2k_1+1} \\ &\qquad+ a\varepsilon u^p\int_\Omega v^{k_2+1}\,dx - a\varepsilon(k_1+1)u^{k_1+p}\int_\Omega v\,dx \\ &\quad\geq p\varepsilon^2u^{2k_1+1} + a\varepsilon u^p\int_\Omega v^{k_2+1}\,dx - a\varepsilon(k_1+1)u^{k_1+p}\int_\Omega v\,dx \\ &\quad= a\varepsilon u^p\left[(p\varepsilon/a)u^{2k_1+1-p} + \int_\Omega v^{k_2+1}\,dx - (k_1+1)u^{k_1}\int_\Omega v\,dx\right]. \end{aligned}$$

Since $\frac{k_1}{2k_1+1-p}+\frac{1}{k_2+1}=1$, then the Hölder inequality and Young's inequality imply

$$\begin{aligned}u^{k_1}\int_\Omega v\,dx &\le |\Omega|^{k_2/(k_2+1)}u^{k_1}\left(\int_\Omega v^{k_2+1}\,dx\right)^{1/(k_2+1)}\\&\le |\Omega|^{k_2/(k_2+1)}\left[\frac{k_1}{2k_1+1-p}\left(\sigma u^{k_1}\right)^{(2k_1+1-p)/k_1}+\frac{1}{k_2+1}\sigma^{-(k_2+1)}\int_\Omega v^{k_2+1}\,dx\right],\end{aligned}$$

where $\sigma=(\frac{k_1+1}{k_2+1})^{1/(k_2+1)}|\Omega|^{k_2/(k_2+1)^2}$. Taking $\varepsilon_1=\frac{ak_2}{p}(\frac{k_1+1}{k_2+1})^{1+(1/k_2)}|\Omega|$, then

$$J_{1t}-u^p\Delta J_1-2p\varepsilon u^{k_1}J_1-au^p\int_\Omega J_2\,dx\ge p\varepsilon(\varepsilon-\varepsilon_1)u^{2k_1+1}\ge 0.$$

Similarly, we can determine a number $\varepsilon_2=\frac{bk_1}{q}(\frac{k_2+1}{k_1+1})^{1+(1/k_1)}|\Omega|$ satisfying

$$J_{2t}-v^q\Delta J_2-2q\varepsilon v^{k_2}J_2-bv^q\int_\Omega J_1\,dx\ge q\varepsilon(\varepsilon-\varepsilon_2)v^{2k_2+1}\ge 0.$$

For $(x,t)\in\partial\Omega\times(0,T^*)$, we have

$$\begin{aligned}J_1(x,t)&=u_t-\varepsilon u^{k_1+1}\\&=\int_\Omega\phi(x,y)u_t(y,t)\,dy-\varepsilon\left(\int_\Omega\phi(x,y)u(y,t)\,dy\right)^{k_1+1}\\&=\int_\Omega\phi(x,y)J_1(y,t)\,dy+\varepsilon\left[\int_\Omega\phi(x,y)u^{k_1+1}(y,t)\,dy-\left(\int_\Omega\phi(x,y)u(y,t)\,dy\right)^{k_1+1}\right].\end{aligned}$$

Using the Hölder inequality and noting that $\int_\Omega\phi(x,y)\,dy<1$, we have

$$\begin{aligned}&\int_\Omega\phi(x,y)u^{k_1+1}(y,t)\,dy-\left(\int_\Omega\phi(x,y)u(y,t)\,dy\right)^{k_1+1}\\&\quad\ge\int_\Omega\phi(x,y)u^{k_1+1}(y,t)\,dy\left[1-\left(\int_\Omega\phi(x,y)\,dy\right)^{k_1}\right]\ge 0.\end{aligned}$$

Hence

$$J_1(x,t)\ge\int_\Omega\phi(x,y)J_1(y,t)\,dy.$$

By a similar argument, we have

$$J_2(x,t)\ge\int_\Omega\psi(x,y)J_2(y,t)\,dy.$$

On the other hand, (H3) implies that $J_1(x,0)\ge 0$, $J_2(x,0)\ge 0$, $x\in\Omega$. By Lemma 2.2, we have $J_1\ge 0$, $J_2\ge 0$. This completes the proof. □

It follows from (5.4) that

$$U_t-\varepsilon U^{k_1+1}\ge 0,\qquad V_t-\varepsilon V^{k_2+1}\ge 0,\quad (x,t)\in\Omega\times\left(0,T^*\right).\tag{5.5}$$

Integrating (5.5) from (t, T^*), we conclude that

$$U(t) \le (\varepsilon k_1)^{-1/k_1}\left(T^* - t\right)^{-1/k_1}, \qquad V(t) \le (\varepsilon k_2)^{-1/k_2}\left(T^* - t\right)^{-1/k_2}. \tag{5.6}$$

Combining (5.5) with (5.2), we can obtain

$$\left(\frac{\varepsilon}{a|\Omega|}\right)^{\rho_2} U^{\rho_1} \le V^{\rho_2}, \qquad \left(\frac{\varepsilon}{b|\Omega|}\right)^{\rho_1} V^{\rho_2} \le U^{\rho_1}. \tag{5.7}$$

From (5.3) and (5.7), we conclude that

$$\begin{aligned} U(t) &\ge C_2^{1/\rho_1}\left[1 + \left(\frac{\varepsilon}{b|\Omega|}\right)^{-\rho_1}\right]^{-1/\rho_1}\left(T^* - t\right)^{-1/k_1}, \\ V(t) &\ge C_2^{1/\rho_2}\left[1 + \left(\frac{\varepsilon}{a|\Omega|}\right)^{-\rho_2}\right]^{-1/\rho_2}\left(T^* - t\right)^{-1/k_2}. \end{aligned} \tag{5.8}$$

By (5.6) and (5.8), we can obtain Theorem 1.4 immediately.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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