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Existence and uniqueness of symmetric solutions for fractional differential equations with multi-order fractional integral conditions

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Abstract

In this paper, we investigate the existence and uniqueness of symmetric solutions for fractional differential equations with multi-order fractional integral boundary conditions, by means of standard fixed point theorems. Examples which support our theoretical results are also presented.

MSC: 34A08; 34B15

Keywords: fractional differential equations; boundary value problems; symmetric solutions; fixed point theorems; existence; uniqueness

1 Introduction

In this paper, we study the existence and uniqueness of symmetric solutions for the following boundary value problem for nonlinear fractional differential equations with multi-order fractional integral boundary conditions:

$$\begin{cases} {}^cD^\alpha x(t) = f(t, x(t)), & 1 < \alpha \leq 2, 0 < t < T, \\ x(t) = x(T-t), & \sum_{i=1}^m \lambda_i I^{\beta_i} x(\eta_i) = \sigma, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α, x is symmetric (we recall that a function $x \in C([0,T],\mathbb{R})$ is said to be *symmetric* on $[0,T]$ if $x(t) = x(T-t)$, $t \in [0,T]$), $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and symmetric with respect to t, that is, $f(t,x) = f(T-t,x)$, $\eta_i \in (0,T)$, $\sigma, \lambda_i \in \mathbb{R}$, for all $i = 1, 2, \ldots, m$ and I^{β_i} is the Riemann-Liouville fractional integral of order $\beta_i > 0$ ($i = 1, 2, \ldots, m$) such that

$$\sum_{i=1}^m \frac{\lambda_i \eta_i^{\beta_i}}{\Gamma(\beta_i + 1)} \neq 0. \tag{1.2}$$

Fractional calculus has become very useful over the last years because of its many applications in almost all applied sciences. Fractional differential equations have been of great interest and it is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various science such as physics, mechanics, chemistry, and engineering. For details, and some recent results on the subject we refer to [1–17] and references cited therein.

Recently, many authors have focused on the existence of symmetric solutions for ordinary differential equation boundary value problems; for example, see [18–21] and the references therein. To the best of the authors' knowledge there are no papers dealing with the existence of symmetric solutions for boundary value problems for fractional differential equations. The filling of this gap is the main motivation of this paper. Here we study existence and uniqueness results for symmetric solutions for boundary value problems of nonlinear fractional differential equations with multi-order fractional integral boundary conditions.

Note that the singular case can occur when the left side of (1.2) is equal to zero. For example if $m = 3$, $\eta_1 = \eta_3 = T/2$, $\eta_2 = T$, $\lambda_1 = \lambda_3 = 1$, $\lambda_2 = -1$, $\sigma = 0$, and $\beta_i = 1$, $i = 1, 2, 3$, then the fractional integral condition of (1.1) is reduced to

$$\int_0^{\frac{T}{2}} x(s)\,ds = \int_{\frac{T}{2}}^{T} x(s)\,ds, \tag{1.3}$$

which is equivalent to the symmetric condition $x(t) = x(T - t)$. Therefore, the condition (1.2) provides the other ordinary/fractional integral boundary condition which is different from the regular symmetric condition. Now, there are two different conditions which are sufficient to give the existence and uniqueness results for the problem (1.1).

The organization of this paper is as follows: In Section 2 we present some preliminary notations, definitions and lemmas that we need in the sequel. In Section 3 we present the main existence and uniqueness results for the problem (1.1). Several new existence and uniqueness results are proved by using a variety of fixed point theorems (such as Banach's contraction principle, nonlinear contractions, Krasnoselskii's fixed point theorem, and Leray-Schauder's nonlinear alternative). Examples illustrating the obtained results are presented in Section 4.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [1, 2] and present preliminary results needed in our proofs later.

Definition 2.1 For an at least n-times differentiable function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)\,ds, \quad n-1 < q < n, n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}}\,ds, \quad q > 0,$$

provided the integral exists.

Lemma 2.1 *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q u(t) = 0$ is given by*

$$u(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$ ($n = [q] + 1$).

In view of Lemma 2.1, it follows that

$$I^{q\,c}D^q u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}, \tag{2.1}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$ ($n = [q] + 1$).

For convenience we set

$$\Omega_1 = \sum_{i=1}^{m} \lambda_i \frac{\eta_i^{\beta_i+1}}{\Gamma(\beta_i + 2)}, \qquad \Omega_2 = \sum_{i=1}^{m} \lambda_i \frac{\eta_i^{\beta_i}}{\Gamma(\beta_i + 1)}. \tag{2.2}$$

Lemma 2.2 *Let $\Omega_2 \neq 0$, $1 < \alpha \leq 2$, $\beta_i > 0$, $\eta_i \in (0, T)$, for $i = 1, 2, \ldots, m$, and $y \in C([0, T], \mathbb{R})$. Then the problem*

$$D^\alpha x(t) = y(t), \quad t \in (0, T), \tag{2.3}$$

$$x(t) = x(T - t), \qquad \sum_{i=1}^{m} \lambda_i I^{\beta_i} x(\eta_i) = \sigma, \tag{2.4}$$

has a unique solution given by

$$x(t) = I^\alpha y(t) + \frac{(\Omega_1 - \Omega_2 t)}{\Omega_2 T} I^\alpha y(T) + \frac{1}{\Omega_2}\left(\sigma - \sum_{i=1}^{m} \lambda_i I^{\alpha+\beta_i} y(\eta_i)\right). \tag{2.5}$$

Proof Using Lemma 2.1, (2.3) can be expressed as an equivalent integral equation

$$x(t) = I^\alpha y(t) + c_1 t + c_2, \tag{2.6}$$

for arbitrary constants $c_1, c_2 \in \mathbb{R}$.

Taking the Riemann-Liouville fractional integral of order $p > 0$ for (2.6), we have

$$I^p x(t) = I^{\alpha+p} y(t) + c_1 \frac{t^{p+1}}{\Gamma(p+2)} + c_2 \frac{t^p}{\Gamma(p+1)}. \tag{2.7}$$

From the first condition of (2.4), it follows that

$$c_1 = -\frac{1}{T} I^\alpha y(T).$$

The second condition of (2.4) and (2.7) with $p = \beta_i$ imply that

$$\sum_{i=1}^{m} \lambda_i I^{\alpha+\beta_i} y(\eta_i) + c_1 \sum_{i=1}^{m} \lambda_i \frac{\eta_i^{\beta_i+1}}{\Gamma(\beta_i + 2)} + c_2 \sum_{i=1}^{m} \lambda_i \frac{\eta_i^{\beta_i}}{\Gamma(\beta_i + 1)} = \sigma.$$

Putting a constant c_1, we have

$$c_2 = \frac{1}{\Omega_2}\left(\sigma - \sum_{i=1}^{m} \lambda_i I^{\alpha+\beta_i} y(\eta_i) + \frac{\Omega_1}{T} I^{\alpha} y(T)\right).$$

Substituting constants c_1 and c_2 into (2.6), we obtain (2.5) as required. □

Next we outline the fixed point theorems that will be used in the proofs of our existence and uniqueness results.

Definition 2.3 Let E be a Banach space and let $F : E \to E$ be a mapping. F is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property:

$$\|Fx - Fy\| \leq \Psi\big(\|x - y\|\big), \quad \forall x, y \in E.$$

Lemma 2.3 (Boyd and Wong) [22] *Let E be a Banach space and let $F : E \to E$ be a nonlinear contraction. Then F has a unique fixed point in E.*

Lemma 2.4 (Krasnoselskii's fixed point theorem) [23] *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that* (a) $Ax + By \in M$ *whenever* $x, y \in M$; (b) *A is compact and continuous*; (c) *B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Lemma 2.5 (Nonlinear alternative for single valued maps) [24] *Let E be a Banach space, C a closed, convex subset of E, X an open subset of C and $0 \in X$. Suppose that $F : \overline{X} \to C$ is a continuous, compact* (*that is, $F(\overline{X})$ is a relatively compact subset of C*) *map. Then either*

(i) *F has a fixed point in $\overline{X}$, or*
(ii) *there is a $x \in \partial X$* (*the boundary of X in C*) *and $\lambda \in (0, 1)$ with $x = \lambda F(x)$.*

3 Main results

Let $\mathcal{C} = C([0, T], \mathbb{R})$ denotes the Banach space of all continuous functions from $[0, T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t\in[0,T]} |x(t)|$. Throughout this paper, for convenience, the expression $I^a f(s, x(s))(b)$ means

$$I^a f\big(s, x(s)\big)(b) = \frac{1}{\Gamma(a)} \int_0^b (b - s)^{a-1} f\big(s, x(s)\big)\, ds, \quad t \in [0, T],$$

where $a \in \{\alpha, \alpha + \beta_i\}$ and $b \in \{t, T, \eta_i\}$, $i = 1, 2, \ldots, m$.

As in Lemma 2.2, we define an operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(\mathcal{F}x)(t) = {} & I^{\alpha} f\big(s, x(s)\big)(t) + \left(\frac{\Omega_1 - \Omega_2 t}{\Omega_2 T}\right) I^{\alpha} f\big(s, x(s)\big)(T) \\ & + \frac{1}{\Omega_2}\left(\sigma - \sum_{i=1}^{m} \lambda_i I^{\alpha+\beta_i} f\big(s, x(s)\big)(\eta_i)\right). \end{aligned} \tag{3.1}$$

It should be noticed that the problem (1.1) has solutions if and only if the operator $\mathcal{F}$ has fixed points.

In the following subsections we prove existence, as well as existence and uniqueness results, for the boundary value problem (1.1) by using a variety of fixed point theorems.

We set

$$\Lambda = \frac{T^\alpha}{\Gamma(\alpha+1)} + \left(\frac{|\Omega_1| + |\Omega_2|T}{|\Omega_2|T}\right)\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{1}{|\Omega_2|}\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)}, \tag{3.2}$$

and

$$\Phi = \frac{|\sigma|}{|\Omega_2|}. \tag{3.3}$$

3.1 Existence and uniqueness result via Banach's fixed point theorem

The first existence and uniqueness result is based on Banach's contraction mapping principle (Banach's fixed point theorem).

Theorem 3.1 *Assume that* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is a symmetric continuous function and*

(H_1) *there exists a constant* $L > 0$ *such that* $|f(t,x) - f(t,y)| \le L|x-y|$, *for each* $t \in [0,T]$ *and* $x, y \in \mathbb{R}$.

If

$$L\Lambda < 1, \tag{3.4}$$

where Λ *is defined by* (3.2), *then the boundary value problem* (1.1) *has a unique symmetric solution on* $[0,T]$.

Proof We transform the problem (1.1) into a fixed point problem, $x = \mathcal{F}x$, where the operator $\mathcal{F}$ is defined as in (3.1). Observe that the fixed points of the operator $\mathcal{F}$ are solutions of the problem (1.1). Applying Banach's contraction mapping principle, we shall show that $\mathcal{F}$ has a unique fixed point.

We let $\sup_{t\in[0,T]} |f(t,0)| = M < \infty$ and choose

$$r \ge \frac{\Lambda M + \Phi}{1 - L\Lambda},$$

where the constant Φ is defined by (3.3).

Now, we show that $\mathcal{F}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$. For any $x \in B_r$, we have

$$\begin{aligned}
\bigl|(\mathcal{F}x)(t)\bigr| &\le \sup_{t\in[0,T]}\Biggl\{I^\alpha\bigl|f\bigl(s,x(s)\bigr)\bigr|(t) + \left(\frac{|\Omega_1| + |\Omega_2|t}{|\Omega_2|T}\right)I^\alpha\bigl|f\bigl(s,x(s)\bigr)\bigr|(T) \\
&\quad + \frac{1}{|\Omega_2|}|\sigma| + \frac{1}{|\Omega_2|}\sum_{i=1}^{m}|\lambda_i|I^{\alpha+\beta_i}\bigl|f\bigl(s,x(s)\bigr)\bigr|(\eta_i)\Biggr\} \\
&\le I^\alpha\bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr|\bigr)(T) \\
&\quad + \left(\frac{|\Omega_1| + |\Omega_2|T}{|\Omega_2|T}\right)I^\alpha\bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr|\bigr)(T)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\Omega_2|}|\sigma|+\frac{1}{|\Omega_2|}\sum_{i=1}^{m}|\lambda_i|I^{\alpha+\beta_i}\big(\big|f\big(s,x(s)\big)-f(s,0)\big|+\big|f(s,0)\big|\big)(\eta_i)\\
&\le (Lr+M)I^{\alpha}(1)(T)+(Lr+M)\left(\frac{|\Omega_1|+|\Omega_2|T}{|\Omega_2|T}\right)I^{\alpha}(1)(T)\\
&+\frac{1}{|\Omega_2|}|\sigma|+(Lr+M)\frac{1}{|\Omega_2|}\sum_{i=1}^{m}|\lambda_i|I^{\alpha+\beta_i}(1)(\eta_i)\\
&\le (Lr+M)\frac{T^{\alpha}}{\Gamma(\alpha+1)}+(Lr+M)\left(\frac{|\Omega_1|+|\Omega_2|T}{|\Omega_2|T}\right)\frac{T^{\alpha}}{\Gamma(\alpha+1)}\\
&+\frac{1}{|\Omega_2|}|\sigma|+(Lr+M)\frac{1}{|\Omega_2|}\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)}\\
&=(Lr+M)\Lambda+\Phi\le r,
\end{aligned}
$$

which implies that $\mathcal{F}B_r\subset B_r$.

Next, we let $x,y\in\mathcal{C}$. Then, for $t\in[0,T]$, we have

$$
\begin{aligned}
&\big|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\big|\\
&\le I^{\alpha}\big|f\big(s,x(s)\big)-f\big(s,y(s)\big)\big|(t)+\left(\frac{|\Omega_1|+|\Omega_2|T}{|\Omega_2|T}\right)I^{\alpha}\big|f\big(s,x(s)\big)-f\big(s,y(s)\big)\big|(T)\\
&+\frac{1}{|\Omega_2|}\sum_{i=1}^{m}|\lambda_i|I^{\alpha+\beta_i}\big|f\big(s,x(s)\big)-f\big(s,y(s)\big)\big|(\eta_i)\\
&\le L\|x-y\|\frac{T^{\alpha}}{\Gamma(\alpha+1)}+L\|x-y\|\left(\frac{|\Omega_1|+|\Omega_2|T}{|\Omega_2|T}\right)\frac{T^{\alpha}}{\Gamma(\alpha+1)}\\
&+L\|x-y\|\frac{1}{|\Omega_2|}\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)}\\
&=L\Lambda\|x-y\|,
\end{aligned}
$$

which implies that $\|\mathcal{F}x-\mathcal{F}y\|\le L\Lambda\|x-y\|$. As $L\Lambda<1$, $\mathcal{F}$ is a contraction. Therefore, we deduce, by Banach's contraction mapping principle, that $\mathcal{F}$ has a fixed point which is the unique symmetric solution of the problem (1.1). The proof is completed. □

3.2 Existence and uniqueness result via Banach's fixed point theorem and Hölder's inequality

In this subsection we give another existence and uniqueness theorem for the boundary value problem (1.1) by using Banach's fixed point theorem and Hölder's inequality.

Theorem 3.2 *Suppose that* $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ *is a symmetric continuous function satisfying the following assumption*:

(H_2) $|f(t,x)-f(t,y)|\le\delta(t)|x-y|$, *for* $t\in[0,T]$, $x,y\in\mathbb{R}$, *and* $\delta\in L^{\frac{1}{\omega}}([0,T],\mathbb{R}^+)$, $\omega\in(0,1)$.

Denote $\|\delta\| = (\int_0^T |\delta(s)|^{\frac{1}{\omega}}\,ds)^{\omega}$. *If*

$$\|\delta\|\left\{\frac{T^{\alpha-\omega}}{\Gamma(\alpha)}\left(\frac{1-\omega}{\alpha-\omega}\right)^{1-\omega} + \left(\frac{|\Omega_1|+|\Omega_2|T}{|\Omega_2|T}\right)\frac{T^{\alpha-\omega}}{\Gamma(\alpha)}\left(\frac{1-\omega}{\alpha-\omega}\right)^{1-\omega} \right.$$
$$\left. + \frac{1}{|\Omega_2|}\sum_{i=1}^{m}\frac{|\lambda_i|\eta_i^{\alpha+\beta_i-\omega}}{\Gamma(\alpha+\beta_i)}\left(\frac{1-\omega}{\alpha+\beta_i-\omega}\right)^{1-\omega}\right\} < 1,$$

then the boundary value problem (1.1) *has a unique symmetric solution on* $[0,T]$.

Proof For $x,y \in C([0,T],\mathbb{R})$ and for each $t \in [0,T]$, by Hölder's inequality, we have

$$\begin{aligned}
&\bigl|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\bigr| \\
&\quad\le I^{\alpha}\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|(t) + \left(\frac{|\Omega_1|+|\Omega_2|T}{|\Omega_2|T}\right)I^{\alpha}\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|(T) \\
&\qquad + \frac{1}{|\Omega_2|}\sum_{i=1}^{m}|\lambda_i|I^{\alpha+\beta_i}\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|(\eta_i) \\
&\quad\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\delta(s)\bigl|x(s)-y(s)\bigr|\,ds \\
&\qquad + \left(\frac{|\Omega_1|+|\Omega_2|T}{|\Omega_2|T}\right)\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\delta(s)\bigl|x(s)-y(s)\bigr|\,ds \\
&\qquad + \frac{1}{|\Omega_2|}\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\alpha+\beta_i)}\int_0^{\eta_i}(\eta_i-s)^{\alpha+\beta_i-1}\delta(s)\bigl|x(s)-y(s)\bigr|\,ds \\
&\quad\le \frac{1}{\Gamma(\alpha)}\left(\int_0^t \bigl((t-s)^{\alpha-1}\bigr)^{\frac{1}{1-\omega}}\,ds\right)^{1-\omega}\left(\int_0^t \bigl(\delta(s)\bigr)^{\frac{1}{\omega}}\,ds\right)^{\omega}\|x-y\| \\
&\qquad + \left(\frac{|\Omega_1|+|\Omega_2|T}{|\Omega_2|T}\right)\frac{1}{\Gamma(\alpha)}\left(\int_0^T \bigl((T-s)^{\alpha-1}\bigr)^{\frac{1}{1-\omega}}\,ds\right)^{1-\omega}\left(\int_0^T \bigl(\delta(s)\bigr)^{\frac{1}{\omega}}\,ds\right)^{\omega}\|x-y\| \\
&\qquad + \frac{1}{|\Omega_2|}\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\alpha+\beta_i)}\left(\int_0^{\eta_i}\bigl((\eta_i-s)^{\alpha+\beta_i-1}\bigr)^{\frac{1}{1-\omega}}\,ds\right)^{1-\omega}\left(\int_0^{\eta_i}\bigl(\delta(s)\bigr)^{\frac{1}{\omega}}\,ds\right)^{\omega}\|x-y\| \\
&\quad\le \|\delta\|\left[\frac{T^{\alpha-\omega}}{\Gamma(\alpha)}\left(\frac{1-\omega}{\alpha-\omega}\right)^{1-\omega} + \left(\frac{|\Omega_1|+|\Omega_2|T}{|\Omega_2|T}\right)\frac{T^{\alpha-\omega}}{\Gamma(\alpha)}\left(\frac{1-\omega}{\alpha-\omega}\right)^{1-\omega}\right. \\
&\qquad \left. + \frac{1}{|\Omega_2|}\sum_{i=1}^{m}\frac{|\lambda_i|\eta_i^{\alpha+\beta_i-\omega}}{\Gamma(\alpha+\beta_i)}\left(\frac{1-\omega}{\alpha+\beta_i-\omega}\right)^{1-\omega}\right]\|x-y\|.
\end{aligned}$$

It follows that $\mathcal{F}$ is a contraction mapping. Hence Banach's fixed point theorem implies that $\mathcal{F}$ has a unique fixed point, which is the unique symmetric solution of the boundary value problem (1.1). The proof is completed. □

3.3 Existence and uniqueness result via nonlinear contractions

In this subsection we establish an existence and uniqueness result for the boundary value problem (1.1) by using Boyd and Wong's fixed point theorem for nonlinear contractions (Lemma 2.3).

Theorem 3.3 *Let* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a symmetric continuous function satisfying the assumption*:

(H_3) $|f(t,x) - f(t,y)| \leq h(t)\frac{|x-y|}{H^*+|x-y|}$, $t \in [0,T]$, $x, y \geq 0$, *where* $h : [0,T] \to \mathbb{R}^+$ *is continuous and a constant* H^* *defined by*

$$H^* = I^\alpha h(T) + \left(\frac{|\Omega_1| + |\Omega_2|T}{|\Omega_2|T}\right) I^\alpha h(T) + \frac{1}{|\Omega_2|}\sum_{i=1}^m |\lambda_i| I^{\alpha+\beta_i} h(\eta_i). \tag{3.5}$$

Then the boundary value problem (1.1) *has a unique symmetric solution on* $[0,T]$.

Proof We define the operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ as in (3.1) and a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Psi(\varepsilon) = \frac{H^*\varepsilon}{H^* + \varepsilon}, \quad \forall \varepsilon \geq 0.$$

Note that the function Ψ satisfies $\Psi(0) = 0$ and $\Psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$.

For any $x, y \in \mathcal{C}$ and for each $t \in [0,T]$, we have

$$\begin{aligned}
&\bigl|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\bigr| \\
&\quad\leq I^\alpha \bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|(t) + \left(\frac{|\Omega_1| + |\Omega_2|T}{|\Omega_2|T}\right) I^\alpha \bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|(T) \\
&\qquad + \frac{1}{|\Omega_2|}\sum_{i=1}^m |\lambda_i| I^{\alpha+\beta_i} \bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|(\eta_i) \\
&\quad\leq I^\alpha \left(h(s)\frac{|x-y|}{H^* + |x-y|}\right)(T) + \left(\frac{|\Omega_1| + |\Omega_2|T}{|\Omega_2|T}\right) I^\alpha \left(h(s)\frac{|x-y|}{H^* + |x-y|}\right)(T) \\
&\qquad + \frac{1}{|\Omega_2|}\sum_{i=1}^m |\lambda_i| I^{\alpha+\beta_i} \left(h(s)\frac{|x-y|}{H^* + |x-y|}\right)(\eta_i) \\
&\quad\leq \frac{\Psi\|x-y\|}{H^*}\left(I^\alpha h(T) + \left(\frac{|\Omega_1| + |\Omega_2|T}{|\Omega_2|T}\right) I^\alpha h(T) + \frac{1}{|\Omega_2|}\sum_{i=1}^m |\lambda_i| I^{\alpha+\beta_i} h(\eta_i)\right) \\
&\quad\leq \Psi\bigl(\|x-y\|\bigr).
\end{aligned}$$

This implies that $\|\mathcal{F}x - \mathcal{F}y\| \leq \Psi(\|x-y\|)$. Therefore $\mathcal{F}$ is a nonlinear contraction. Hence, by Lemma 2.3 the operator $\mathcal{F}$ has a unique fixed point which is the unique symmetric solution of the problem (1.1). This completes the proof. □

3.4 Existence result via Krasnoselskii's fixed point theorem

The next existence theorem is based on Krasnoselskii's fixed point theorem (Lemma 2.4).

Theorem 3.4 *Let* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a symmetric continuous function satisfying* (H_1). *In addition we assume that*:

(H_4) $|f(t,x)| \leq \phi(t)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$, *and* $\phi \in C([0,T], \mathbb{R}^+)$.

Then the boundary value problem (1.1) *has at least one symmetric solution on* $[0,T]$ *provided*

$$\left(\frac{|\Omega_1|+|\Omega_2|T}{|\Omega_2|T}\right)\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{1}{|\Omega_2|}\sum_{i=1}^m|\lambda_i|\frac{\eta_i^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)}<1. \tag{3.6}$$

Proof Setting $\sup_{t\in[0,T]}|\varphi(t)|=\|\varphi\|$ and choosing

$$\rho\ge\|\varphi\|\Lambda+\Phi \tag{3.7}$$

(where Λ and Φ are defined by (3.2) and (3.3), respectively), we consider $B_\rho=\{x\in\mathcal{C}([0,T],\mathbb{R}):\|x\|\le\rho\}$. We define the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on B_ρ by

$$\begin{aligned}&\mathcal{F}_1x(t)=I^\alpha f\big(s,x(s)\big)(t),\quad t\in[0,T],\\&\mathcal{F}_2x(t)=\left(\frac{\Omega_1-\Omega_2t}{\Omega_2T}\right)I^\alpha f\big(s,x(s)\big)(T)+\frac{1}{\Omega_2}\left(\sigma-\sum_{i=1}^m\lambda_iI^{\alpha+\beta_i}f\big(s,x(s)\big)(\eta_i)\right),\quad t\in[0,T].\end{aligned}$$

For any $x,y\in B_\rho$, we have

$$\begin{aligned}&\big|\mathcal{F}_1x(t)+\mathcal{F}_2y(t)\big|\\&\quad\le\sup_{t\in[0,T]}\left\{I^\alpha\big|f\big(s,x(s)\big)\big|(t)+\left(\frac{|\Omega_1|+|\Omega_2|t}{|\Omega_2|T}\right)I^\alpha\big|f\big(s,y(s)\big)\big|(T)\right.\\&\qquad\left.+\frac{1}{|\Omega_2|}|\sigma|+\frac{1}{|\Omega_2|}\sum_{i=1}^m|\lambda_i|I^{\alpha+\beta_i}\big|f\big(s,y(s)\big)\big|(\eta_i)\right\}\\&\quad\le\|\varphi\|\left[\frac{T^\alpha}{\Gamma(\alpha+1)}+\left(\frac{|\Omega_1|+|\Omega_2|T}{|\Omega_2|T}\right)\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{1}{|\Omega_2|}\sum_{i=1}^m|\lambda_i|\frac{\eta_i^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)}\right]+\frac{|\sigma|}{|\Omega_2|}\\&\quad=\|\varphi\|\Lambda+\Phi\le\rho.\end{aligned}$$

This shows that $\mathcal{F}_1x+\mathcal{F}_2y\in B_\rho$. It is easy to see using (3.6) that $\mathcal{F}_2$ is a contraction mapping.

Continuity of f implies that the operator $\mathcal{F}_1$ is continuous. Also, $\mathcal{F}_1$ is uniformly bounded on B_ρ as

$$\|\mathcal{F}_1x\|\le\frac{T^\alpha}{\Gamma(\alpha+1)}\|\phi\|.$$

Now we prove the compactness of the operator $\mathcal{F}_1$.

We define $\sup_{(t,x)\in[0,T]\times B_\rho}|f(t,x)|=\bar{f}<\infty$, and consequently we have

$$\begin{aligned}\big|\mathcal{F}_1x(t_2)-\mathcal{F}_1x(t_1)\big|&=\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]f\big(s,x(s)\big)\,ds\right.\\&\qquad\left.+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f\big(s,x(s)\big)\,ds\right|\\&\le\frac{\bar{f}}{\Gamma(\alpha+1)}\big|t_1^\alpha-t_2^\alpha\big|,\end{aligned}$$

which is independent of x and tends to zero as $t_2 - t_2 \to 0$. Thus, $\mathcal{F}_1$ is equicontinuous. So $\mathcal{F}_1$ is relatively compact on B_ρ. Hence, by Arzelá-Ascoli's theorem, $\mathcal{F}_1$ is compact on B_ρ. Thus all the assumptions of Lemma 2.4 are satisfied. So the conclusion of Lemma 2.4 implies that the boundary value problem (1.1) has at least one symmetric solution on $[0, T]$. □

3.5 Existence result via Leray-Schauder's nonlinear alternative

By using Leray-Schauder's nonlinear alternative (Lemma 2.5) we give in this subsection our last existence theorem.

Theorem 3.5 *Assume that*:

(H_5) *there exist a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in C([0,T],\mathbb{R}^+)$ *such that*

$$|f(t,x)| \le p(t)\psi(|x|) \quad \textit{for each } (t,x) \in [0,T] \times \mathbb{R};$$

(H_6) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{\psi(M)\|p\|\Lambda + \Phi} > 1,$$

where Λ *and* Φ *are defined by* (3.2) *and* (3.3), *respectively.*

Then the boundary value problem (1.1) *has at least one symmetric solution on* $[0, T]$.

Proof Let the operator $\mathcal{F}$ be defined by (3.1). Firstly, we shall show that $\mathcal{F}$ *maps bounded sets (balls) into bounded sets in* $C([0,T],\mathbb{R})$. For a number $r > 0$, let $B_r = \{x \in C([0,T],\mathbb{R}) : \|x\| \le r\}$ be a bounded ball in $C([0,T],\mathbb{R})$. Then for $t \in [0,T]$ we have

$$\begin{aligned}
&\left|(\mathcal{F}x)(t)\right| \\
&\quad\le \sup_{t\in[0,T]}\left\{I^\alpha\left|f(s,x(s))\right|(t) + \left(\frac{|\Omega_1| + |\Omega_2|t}{|\Omega_2|T}\right)I^\alpha\left|f(s,x(s))\right|(T)\right. \\
&\qquad\left. + \frac{1}{|\Omega_2|}|\sigma| + \frac{1}{|\Omega_2|}\sum_{i=1}^m|\lambda_i|I^{\alpha+\beta_i}\left|f(s,x(s))\right|(\eta_i)\right\} \\
&\quad\le \psi(\|x\|)I^\alpha p(s)(T) + \psi(\|x\|)\left(\frac{|\Omega_1| + |\Omega_2|T}{|\Omega_2|T}\right)I^\alpha p(s)(T) \\
&\qquad + \frac{1}{|\Omega_2|}|\sigma| + \psi(\|x\|)\frac{1}{|\Omega_2|}\sum_{i=1}^m|\lambda_i|I^{\alpha+\beta_i}p(s)(\eta_i) \\
&\quad\le \psi(\|x\|)\|p\|\frac{T^\alpha}{\Gamma(\alpha+1)} + \psi(\|x\|)\|p\|\left(\frac{|\Omega_1| + |\Omega_2|T}{|\Omega_2|T}\right)\frac{T^\alpha}{\Gamma(\alpha+1)} \\
&\qquad + \frac{1}{|\Omega_2|}|\sigma| + \psi(\|x\|)\|p\|\frac{1}{|\Omega_2|}\sum_{i=1}^m|\lambda_i|\frac{\eta_i^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)} \\
&\quad\le \psi(\|x\|)\|p\|\left[\frac{T^\alpha}{\Gamma(\alpha+1)} + \left(\frac{|\Omega_1| + |\Omega_2|T}{|\Omega_2|T}\right)\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{1}{|\Omega_2|}\sum_{i=1}^m|\lambda_i|\frac{\eta_i^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)}\right] \\
&\qquad + \frac{|\sigma|}{|\Omega_2|},
\end{aligned}$$

and consequently,

$$\|\mathcal{F}x\| \leq \psi(r)\|p\|\Lambda + \Phi.$$

Next we will show that $\mathcal{F}$ *maps bounded sets into equicontinuous sets of* $C([0,T],\mathbb{R})$. Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$ and $u \in B_r$. Then we have

$$\begin{aligned}
&\left|(\mathcal{F}x)(\tau_2) - (\mathcal{F}x)(\tau_1)\right| \\
&\quad\leq \frac{1}{\Gamma(\alpha)}\left|\int_0^{\tau_1}\left[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}\right]f\left(s,x(s)\right)ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha-1}f\left(s,x(s)\right)ds\right| \\
&\qquad + \frac{1}{T}(\tau_2 - \tau_1)I^{\alpha}\left|f\left(s,x(s)\right)\right|(T) \\
&\quad\leq \frac{\psi(r)}{\Gamma(\alpha)}\left|\int_0^{\tau_1}\left[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}\right]p(s)\,ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha-1}p(s)\,ds\right| \\
&\qquad + \frac{\psi(r)}{T}(\tau_2 - \tau_1)I^{\alpha}\left|p(s)\right|(T).
\end{aligned}$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore by Arzelá-Ascoli's theorem the operator $\mathcal{F} : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ is completely continuous.

Let x be a solution. Then, for $t \in [0,T]$, and the following similar computations to the first step, we have

$$\left|x(t)\right| \leq \psi\left(\|x\|\right)\|p\|\Lambda + \Phi,$$

which leads to

$$\frac{\|x\|}{\psi(\|x\|)\|p\|\Lambda + \Phi} \leq 1.$$

In view of (H_6), there exists M such that $\|x\| \neq M$. Let us set

$$X = \left\{x \in C\left([0,T],\mathbb{R}\right) : \|x\| < M\right\}.$$

We see that the operator $\mathcal{F} : \overline{X} \to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of X, there is no $x \in \partial X$ such that $x = \nu\mathcal{F}x$ for some $\nu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 2.5), we deduce that $\mathcal{F}$ has a fixed point $x \in \overline{X}$ which is a symmetric solution of the boundary value problem (1.1). This completes the proof. □

4 Examples

In this section, we present some examples to illustrate our results.

Example 4.1 Consider the following nonlinear fractional differential equation with multi-order fractional integral conditions:

$$\begin{cases}
{}^cD^{\frac{3}{2}}x(t) = \frac{\sin((t-2)^2)}{19}\left(\frac{|x(t)|}{3+|x(t)|} + 1\right)|x(t)| + \frac{3}{4}, \quad 0 < t < 4, \\
x(t) = x(4-t), \\
\frac{1}{5}I^{\frac{1}{2}}x(\frac{1}{2}) + \frac{1}{3}I^{\sqrt{3}}x(1) + \frac{2}{3}I^{\frac{1}{2}}x(\frac{3}{2}) + \sqrt{2}I^{\frac{1}{3}}x(1) + \frac{1}{5}I^{\sqrt{2}}x(\frac{1}{2}) = 2.
\end{cases} \tag{4.1}$$

Here $\alpha = 3/2$, $T = 4$, $m = 5$, $\sigma = 2$, $\beta_1 = 1/2$, $\beta_2 = \sqrt{3}$, $\beta_3 = 1/2$, $\beta_4 = 1/3$, $\beta_5 = \sqrt{2}$, $\lambda_1 = 1/5$, $\lambda_2 = 1/3$, $\lambda_3 = 2/3$, $\lambda_4 = \sqrt{2}$, $\lambda_5 = 1/5$, $\eta_1 = 1/2$, $\eta_2 = 1$, $\eta_3 = 3/2$, $\eta_4 = 1$, $\eta_5 = 1/2$, and $f(t,x) = (\sin((t-2)^2)/19)((|x|/(3+|x|)) + 1)|x| + (3/4)$. Since $|f(t,x) - f(t,y)| \leq (4/57)|x-y|$, then (H_1) is satisfied with $L = 4/57$. We can show that $\Omega_2 = 2.934752823 \neq 0$ and

$$\begin{aligned}\Lambda &= \frac{T^\alpha}{\Gamma(\alpha+1)} + \left(\frac{|\Omega_1| + |\Omega_2|T}{|\Omega_2|T}\right)\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{1}{|\Omega_2|}\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)} \\ &= 13.74958140.\end{aligned}$$

Thus $L\Lambda = 0.9648829050 < 1$. Hence, by Theorem 3.1, the problem (4.1) has a unique symmetric solution on $[0,4]$.

Example 4.2 Consider the following nonlinear fractional differential equation with multi-order fractional integral conditions:

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) = \frac{e^{(\frac{1}{2}-t)^2}}{18}\cdot\frac{|x(t)|}{1+|x(t)|} + \frac{1}{2}, \quad 0 < t < 1, \\ x(t) = x(1-t), \\ \frac{1}{8}I^{\frac{1}{9}}x(\frac{1}{2}) + \frac{1}{34}I^{\frac{1}{5}}x(\frac{1}{3}) + \frac{1}{37}I^{\frac{1}{7}}x(\frac{1}{5}) + \frac{1}{49}I^{\frac{1}{2}}x(\frac{2}{7}) + \frac{1}{25}I^{\frac{1}{8}}x(\frac{\pi}{11}) = \frac{3}{2}. \end{cases} \tag{4.2}$$

Here $\alpha = 3/2$, $T = 1$, $m = 5$, $\sigma = 3/2$, $\beta_1 = 1/9$, $\beta_2 = 1/5$, $\beta_3 = 1/7$, $\beta_4 = 1/2$, $\beta_5 = 1/8$, $\lambda_1 = 1/8$, $\lambda_2 = 1/34$, $\lambda_3 = 1/37$, $\lambda_4 = 1/49$, $\lambda_5 = 1/25$, $\eta_1 = 1/2$, $\eta_2 = 1/3$, $\eta_3 = 1/5$, $\eta_4 = 2/7$, $\eta_5 = \pi/11$, and $f(t,x) = (e^{((1/2)-t)^2}/18)(|x|/(1+|x|)) + (1/2)$. We choose $h(t) = e^{t^2}/9$ and we obtain $\Omega_2 = 0.2195131282 \neq 0$,

$$\begin{aligned}H^* &= I^\alpha h(T) + \left(\frac{|\Omega_1| + |\Omega_2|T}{|\Omega_2|T}\right)I^\alpha h(T) + \frac{1}{|\Omega_2|}\sum_{i=1}^{m}|\lambda_i|I^{\alpha+\beta_i}h(\eta_i) \\ &= 0.2749628138.\end{aligned}$$

Clearly,

$$\left|f(t,x) - f(t,y)\right| = \frac{e^{(\frac{1}{2}-t)^2}}{18}\left(\frac{|x| - |y|}{1 + |x| + |y| + |x||y|}\right) \leq \frac{e^{t^2}}{9}\left(\frac{|x-y|}{0.2749628138 + |x-y|}\right).$$

Hence, by Theorem 3.3, the problem (4.2) has a unique symmetric solution on $[0,1]$.

Example 4.3 Consider the following nonlinear fractional differential equation with multi-order fractional integral conditions:

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) = \frac{e^{-(t-\frac{1}{2})^2}\sin((t-\frac{1}{2})^2)}{2}\cdot\frac{|x(t)|}{1+|x(t)|} + 2t(1-t), \quad 0 < t < 1, \\ x(t) = x(1-t), \\ \frac{1}{199}I^{\frac{1}{2}}x(\frac{1}{3}) + \frac{1}{256}I^{\frac{1}{3}}x(\frac{2}{3}) + \frac{1}{10}I^{\frac{1}{4}}x(\frac{1}{5}) + \frac{1}{189}I^{\frac{1}{3}}x(\frac{1}{4}) + \frac{1}{191}I^{\frac{1}{3}}x(\frac{1}{6}) = 3. \end{cases} \tag{4.3}$$

Here $\alpha = 3/2$, $T = 1$, $m = 5$, $\sigma = 3$, $\beta_1 = 1/2$, $\beta_2 = 1/3$, $\beta_3 = 1/4$, $\beta_4 = 1/3$, $\beta_5 = 1/3$, $\lambda_1 = 1/199$, $\lambda_2 = 1/256$, $\lambda_3 = 1/10$, $\lambda_4 = 1/189$, $\lambda_5 = 1/191$, $\eta_1 = 1/3$, $\eta_2 = 2/3$, $\eta_3 = 1/5$, $\eta_4 = 1/4$,

$\eta_5 = 1/6$, and $f(t,x) = (e^{-(t-(1/2))^2}\sin((t-(1/2))^2)/2)(|x|/(1+|x|)) + 2t(1-t)$. Since $|f(t,x) - f(t,y)| \leq (1/2)|x-y|$, (H_1) is satisfied with $L = 1/2$. We find that $\Omega_2 = 0.08783388964 \neq 0$,

$$\left(\frac{|\Omega_1| + |\Omega_2|T}{|\Omega_2|T}\right)\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{1}{|\Omega_2|}\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)} = 0.9472241016 < 1.$$

Clearly,

$$\big|f(t,x)\big| = \left|\frac{e^{-(t-\frac{1}{2})^2}\sin((t-\frac{1}{2})^2)}{2}\cdot\frac{|x(t)|}{1+|x(t)|} + 2t(1-t)\right| \leq \frac{e^{-(t-\frac{1}{2})^2}}{2} + 2t(1-t).$$

Hence, by Theorem 3.4, the problem (4.3) has at least one symmetric solution on $[0,1]$.

Example 4.4 Consider the following nonlinear fractional differential equation with multi-order fractional integral conditions:

$$\begin{cases} {}^cD^{\frac{4}{3}}x(t) = \frac{1}{225}(t-3)^2(\frac{x^2(t)}{1+|x(t)|} + \frac{|x(t)|+2}{3+|x(t)|}), \quad 0 < t \leq 6, \\ x(t) = x(1-t), \\ \frac{1}{2}I^{\frac{1}{2}}x(\frac{1}{2}) + \frac{1}{3}I^{\frac{1}{\sqrt{2}}}x(\frac{1}{4}) + \frac{3}{11}I^{\frac{1}{5}}x(\frac{2}{7}) + \frac{1}{7}I^{\frac{1}{3}}x(\frac{1}{5}) + \frac{1}{5}I^{\frac{1}{\sqrt{3}}}x(\frac{1}{2}) = \frac{1}{225}. \end{cases} \tag{4.4}$$

Here $\alpha = 4/3$, $T = 6$, $m = 5$, $\sigma = 1/225$, $\beta_1 = 1/2$, $\beta_2 = 1/\sqrt{2}$, $\beta_3 = 1/5$, $\beta_4 = 1/3$, $\beta_5 = 1/\sqrt{3}$, $\lambda_1 = 1/2$, $\lambda_2 = 1/3$, $\lambda_3 = 3/11$, $\lambda_4 = 1/7$, $\lambda_5 = 1/5$, $\eta_1 = 1/2$, $\eta_2 = 1/4$, $\eta_3 = 2/7$, $\eta_4 = 1/5$, $\eta_5 = 1/2$, and $f(t,x) = (1/225)(t-3)^2((x^2/(1+|x|)) + ((|x|+2)/(3+|x|)))$. Then, we get $\Omega_2 = 1.011549229 \neq 0$,

$$\begin{aligned}\Lambda &= \frac{T^\alpha}{\Gamma(\alpha+1)} + \left(\frac{|\Omega_1| + |\Omega_2|T}{|\Omega_2|T}\right)\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{1}{|\Omega_2|}\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)} \\ &= 18.87497313,\end{aligned}$$

and

$$\Phi = \frac{|\sigma|}{|\Omega_2|} = 0.004393700590.$$

Clearly,

$$\big|f(t,x)\big| = \left|\frac{1}{225}(t-3)^2\left(\frac{x^2}{1+|x|} + \frac{|x|+2}{3+|x|}\right)\right| \leq \frac{1}{225}(t-3)^2\big(|x|+1\big).$$

Choosing $p(t) = (1/225)(t-3)^2$ and $\psi(|x|) = |x| + 1$, we can show that

$$\frac{M}{\psi(M)\|p\|\Lambda + \Phi} > 1,$$

implies that $M > 3.099548142$. Hence, by Theorem 3.5, the problem (4.3) has at least one symmetric solution on $[0,6]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in this article. They read and approved the final manuscript.

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