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Nontrivial solutions to boundary value problem with general singular differential operator

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Abstract
In this paper, a class of boundary value problems with general singular differential operator is investigated. The nonlinear term in the boundary value problem is sign-changing and may be unbounded from below. By means of the topological degree of a completely continuous field, the existence of nontrivial solutions is obtained. Finally, an example is given to illustrate the application of our main result.

MSC: 34B16

Keywords: singularities; eigenvalue; topological degree; nontrivial solutions

1 Introduction

In [1], Han and Wu considered the singular boundary value problem

$$\begin{aligned} &u'' + h(t)f(u) = 0, \quad t \in (0,1), \\ &u(0) = u(1) = 0 \end{aligned} \tag{1.1}$$

under the following assumptions:

(H1) There exist three constants $b > 0$, $c > 0$ and $\alpha \in (0,1)$ such that

$$f(u) \geq -b - c|u|^{\alpha}, \quad u \in \mathbb{R}.$$

(H2) $h \in C((0,1),[0,+\infty))$, $h(t) \not\equiv 0$ in $(0,1)$ and $\int_0^1 t(1-t)h(t)\,dt < +\infty$.

(H3) $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Denote by $\tilde{\lambda}_1$ the first eigenvalue of the eigenvalue problem

$$u'' + \lambda h(t)u = 0, \qquad u(0) = u(1) = 0. \tag{1.2}$$

By computing the Leray-Schauder degree, they established the following result.

Theorem A *Assume that* (H1)-(H3) *hold. If*

$$\liminf_{u\to+\infty} \frac{f(u)}{u} > \tilde{\lambda}_1 \quad \textit{and} \quad \limsup_{u\to 0} \left|\frac{f(u)}{u}\right| < \tilde{\lambda}_1,$$

then the singular boundary value problem (1.1) *has at least one nontrivial solution.*

This paper is mainly focused on the existence of nontrivial solutions to singular boundary value problem with general differential operator such as

$$\begin{aligned} &u'' + a(t)u' + b(t)u + h(t)f(t,u) = 0, \quad t \in (0,1), \\ &u(0) = u(1) = 0. \end{aligned} \tag{1.3}$$

We will make the following assumptions:

(A1) $a \in C(0,1) \cap L^1(0,1)$; $b \in C((0,1),(-\infty,0))$ and $\int_0^1 s(1-s)|b(s)|\,\mathrm{d}s < +\infty$.

(A2) $h:(0,1) \to [0,\infty)$ is continuous and does not vanish identically on any subinterval of $(0,1)$; furthermore, $\int_0^1 s(1-s)h(s)\,\mathrm{d}s < +\infty$.

(A3) $f:[0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous and there exist two nonnegative functions $c,d \in C[0,1]$, $d \not\equiv 0$ and $B \in C(\mathbb{R},[0,+\infty))$ with

$$\lim_{|u|\to+\infty} \frac{B(u)}{|u|} = 0 \tag{1.4}$$

such that

$$f(t,u) \geq -c(t) - d(t)B(u), \quad t \in [0,1], u \in \mathbb{R}. \tag{1.5}$$

(A4) $\liminf_{u\to+\infty} \frac{f(t,u)}{u} > \lambda_1$ uniformly on $t \in [0,1]$.

(A5) $\limsup_{u\to 0} |\frac{f(t,u)}{u}| < \lambda_1$ uniformly on $t \in [0,1]$.

Here λ_1 is the first eigenvalue of the following linear eigenvalue problem corresponding to the boundary value problem (1.3):

$$\begin{aligned} &u'' + a(t)u' + b(t)u + \lambda h(t)u = 0, \quad t \in (0,1), \\ &u(0) = u(1) = 0. \end{aligned} \tag{1.6}$$

We state our main theorem as follows.

Theorem 1.1 *Let* (A1)-(A5) *hold, then the singular boundary value problem* (1.3) *has at least one nontrivial solution.*

Remark 1.1 (A1) and (A2) show that the functions a, b and h are all allowed to be singular at $t = 0, 1$, which means that the differential operator in the equation of problem (1.3) is more general than the classic Sturm-Liouville differential operator where $a, b \in C[0,1]$ are required. It is worth mentioning that the solvability of singular boundary value problems with general differential operators were also investigated in [2], where the results on the existence of solutions do not involve the eigenvalue of the corresponding linear eigenvalue problem, therefore the conditions laid on the functions a, b in [2] are more general than the conditions in this paper.

Remark 1.2 The existence of the first eigenvalue λ_1 and the properties of its corresponding positive eigenfunction φ_1 in (1.6) (see Lemma 2.4 in Section 2) play a very important role in the proof of our main theorem. The presence of $a(t)u' + b(t)u$, however, brings difficulties when it comes to proving that $\varphi_1 \in C^1[0,1]$. We find a way to overcome this problem. For the special case of (1.6), *i.e.*, the eigenvalue problem (1.2), the complete results on eigenvalues and eigenfunctions were obtained by Asakawa [3].

Remark 1.3 Choose $B(u) = |u|^\alpha$, $\alpha \in (0,1)$ and the functions c, d to be constants on $[0,1]$. Then (A3) can be reduced to (H1). Therefore, the main theorem in this paper generalizes Theorem A in [1].

We note that the results on positive solutions or nontrivial solutions for other kinds of boundary value problems have been considered in many publications such as [3–10] and the references therein. The existence of nontrivial solutions involving the relation between the principal eigenvalue and the growth of the nonlinearity has been investigated in [11].

The rest of the paper is arranged as follows. In Section 2, we state some notations and prove some preliminary results. In Section 3, we prove our main result. In Section 4, an example is given to illustrate the application of our main result.

2 Notations and preliminary results

In this section, we recall some notations, abstract theorems and auxiliary results, which are important for proving our main result.

We denote by $C[0,1]$ the Banach space with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Let

$$P = \{u \in C[0,1] \mid u(t) \geq 0, t \in [0,1]\}$$

be a positive cone in $C[0,1]$. Denote by $B_r = \{u \in C[0,1] \mid \|u\| < r\}$ $(r > 0)$ the open ball of radius r. We denote by $AC[0,1]$ the space of all absolute continuous functions on $[0,1]$. Let

$$AC_{\mathrm{loc}}[0,1) = \{u \mid u|_{[0,d]} \in AC[0,d] \text{ for every compact interval } [0,d] \subseteq [0,1)\},$$
$$AC_{\mathrm{loc}}(0,1] = \{u \mid u|_{[d,1]} \in AC[d,1] \text{ for every compact interval } [d,1] \subseteq (0,1]\}.$$

Lemma 2.1 [4] *Suppose that* (A1) *holds. Then*

(i) *the initial value problem*

$$\begin{aligned} &u'' + a(t)u' + b(t)u = 0, \quad t \in (0,1), \\ &u(0) = 0, \qquad u'(0) = 1 \end{aligned} \tag{2.1}$$

has a unique solution $\alpha \in AC[0,1] \cap C^1[0,1)$ *and* $\alpha' \in AC_{\mathrm{loc}}[0,1)$;

(ii) *the initial value problem*

$$\begin{aligned} &u'' + a(t)u' + b(t)u = 0, \quad t \in (0,1), \\ &u(1) = 0, \qquad u'(1) = -1 \end{aligned} \tag{2.2}$$

has a unique solution $\beta \in AC[0,1] \cap C^1(0,1]$ *and* $\beta' \in AC_{\mathrm{loc}}(0,1]$;

(iii) α *is nondecreasing on* $[0,1]$, β *is nonincreasing on* $[0,1]$.

We remark here that from the proof of Lemma 2.2 and Lemma 2.3 in [4], the existence and uniqueness of α and β for the initial value problem (2.1) and (2.2) have no relation with the sign of the functions a and b. The sign condition laid on b in (A1) is meant to obtain the monotonicity of α and β.

Let

$$G(t,s) = \frac{1}{\rho}\begin{cases}\alpha(s)\beta(t), & 0 \le s \le t \le 1,\\ \alpha(t)\beta(s), & 0 \le t \le s \le 1,\end{cases} \tag{2.3}$$

where $\rho = \alpha'(\frac{1}{2})\beta(\frac{1}{2}) - \alpha(\frac{1}{2})\beta'(\frac{1}{2})$ is a positive constant.

We give the following remark. Although its proof is trivial, the consequences of the result are of major importance.

Remark 2.1 It follows from the above lemma that there exist positive constants c_1, c_2, ρ_1 and ρ_2 such that

$$c_1 t \le \alpha(t) \le c_2 t, \qquad \rho_1(1-t) \le \beta(t) \le \rho_2(1-t), \quad t \in [0,1].$$

Furthermore, for $t, s \in [0,1]$,

$$(*) \quad \rho G(t,s) \le \rho G(s,s) = \alpha(s)\beta(s) \le c_2\rho_2 s(1-s).$$

From Lemma 2.6 in [4], the singular boundary value problem (1.3) can be converted into the equivalent integral equation

$$u(t) = \int_0^1 G(t,s)q(s)h(s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1],$$

where $q(s) = \exp(\int_{\frac{1}{2}}^{s} a(\tau)\,\mathrm{d}\tau)$. Define $\tilde{h}(t) = q(t)h(t)$ for ease of notation. Combining (A2) and the definition of q, we see that $\tilde{h}$ satisfies the following:

($\tilde{\mathrm{A}}2$) $\tilde{h} : (0,1) \to [0,\infty)$ is continuous and does not vanish identically on any subinterval of $(0,1)$; furthermore, $0 < \int_0^1 s(1-s)\tilde{h}(s)\,\mathrm{d}s < +\infty$.

For any $u \in C[0,1]$, let

$$(Au)(t) = \int_0^1 G(t,s)\tilde{h}(s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s, \tag{2.4}$$

$$(Tu)(t) = \int_0^1 G(t,s)\tilde{h}(s)u(s)\,\mathrm{d}s. \tag{2.5}$$

From Lemma 2.9 in [4], then $A : C[0,1] \to C[0,1]$ and $T : C[0,1] \to C[0,1]$ are completely continuous operators, respectively.

Lemma 2.2 *Suppose that* (A1) *and* (A2) *hold. Let* $\sigma \in (0,1)$, $r \in C(0,1)$ *be continuous and* $\int_0^1 s(1-s)|r(s)| < +\infty$. *If* ω *is a solution of*

$$\begin{aligned} &u'' + a(t)u' + b(t)u + r(t)u = 0, \quad t \in (0,1),\\ &u(\sigma) = u'(\sigma) = 0 \end{aligned} \tag{2.6}$$

such that $\omega \in C[0,1] \cap C^1(0,1)$ *and* $\omega' \in AC_{\mathrm{loc}}(0,1)$, *then* $\omega \equiv 0$.

Proof Let ω be a solution of initial value problem (2.6). Multiplying both sides of the equation of (2.6) by q and integrating it from σ to t, we get

$$\omega'(t) = -\frac{1}{q(t)}\int_\sigma^t q(s)\big(b(s)+r(s)\big)\omega(s)\,\mathrm{d}s, \quad t\in(0,1).$$

Integrating the above equation from σ to t again, we have

$$\omega(t) = -\int_\sigma^t \frac{1}{q(\tau)}\int_\sigma^\tau q(s)\big(b(s)+r(s)\big)\omega(s)\,\mathrm{d}s\,\mathrm{d}\tau, \quad t\in(0,1). \tag{2.7}$$

Let

$$\kappa_1 = \min_{t\in[0,1]} q(t), \qquad \kappa_2 = \max_{t\in[0,1]} q(t).$$

If $t\in[\sigma,1)$, then it follows from (2.7) that

$$\begin{aligned}\big|\omega(t)\big| &\le \frac{\kappa_2}{\kappa_1}\int_\sigma^t\int_\sigma^\tau \big|b(s)+r(s)\big|\big|\omega(s)\big|\,\mathrm{d}s\,\mathrm{d}\tau\\ &= \frac{\kappa_2}{\kappa_1}\int_\sigma^t (t-s)\big|b(s)+r(s)\big|\big|\omega(s)\big|\,\mathrm{d}s\\ &\le \frac{\kappa_2}{\kappa_1}\int_\sigma^t (1-s)\big|b(s)+r(s)\big|\big|\omega(s)\big|\,\mathrm{d}s.\end{aligned}$$

Since $\int_\sigma^1(1-s)|b(s)+r(s)|\,\mathrm{d}s<+\infty$, we get $\omega(t)\equiv 0$ for $t\in[\sigma,1]$ by Gronwall's inequality. Using the same argument, we can easily get $\omega(t)\equiv 0$ for $t\in[0,\sigma]$. This completes the proof. □

We now turn to the eigenvalue problem (1.6). Before stating the results on the eigenvalue and eigenfunction, we give the following lemma, which is a direct result of the Krein-Rutman theorem.

Lemma 2.3 [12, 13] *Suppose that $T: C[0,1]\to C[0,1]$ is a completely continuous linear operator and $T(P)\subset P$. If there exist $\phi\in C[0,1]\backslash(-P)$ and a constant $c>0$ such that $cT\phi\ge\phi$, then the spectral radius $r(T)\ne 0$ and T has a positive eigenfunction corresponding to its first eigenvalue $\lambda_1=(r(T))^{-1}$.*

We are now in a position to give the results on eigenvalue problem (1.6).

Lemma 2.4 *Suppose that* (A1) *and* (A2) *are satisfied. Then T has a principle eigenvalue $\lambda_1=(r(T))^{-1}$ and a positive eigenfunction $\varphi_1\in P$ corresponding to λ_1. Furthermore,*

(i) *$\varphi_1'(0)>0$, $\varphi_1'(1)<0$ and there exist positive constants ν_1, ν_2 such that $\nu_1 t(1-t)\le\varphi_1(t)\le\nu_2 t(1-t)$, $t\in[0,1]$;*

(ii) *there exist $\delta_1,\delta_2>0$ such that $\delta_1 G(t,s)\le\varphi_1(s)\le\delta_2 s(1-s)$, $t,s\in[0,1]$.*

Proof It is obvious that there is $t_1\in(0,1)$ such that $G(t_1,t_1)\tilde{h}(t_1)>0$. Thus there exists $[a_1,b_1]\in(0,1)$ such that $t_1\in(a_1,b_1)$ and $G(t,s)\tilde{h}(s)>0$ for all $t,s\in[a_1,b_1]$. Take $\psi\in C[0,1]$

such that $\psi(t) \geq 0$, $t \in [0,1]$, $\psi(t_1) > 0$ and $\psi(t) = 0$, $t \notin [a_1, b_1]$. Then, for $t \in [a_1, b_1]$,

$$(T\psi)(t) = \int_0^1 G(t,s)\tilde{h}(s)\psi(s)\,\mathrm{d}s \geq \int_{a_1}^{b_1} G(t,s)\tilde{h}(s)\psi(s)\,\mathrm{d}s > 0. \tag{2.8}$$

So there exists a constant $c > 0$ such that $c(T\psi)(t) \geq \psi(t)$, $t \in [0,1]$. From Lemma 2.3, we know that the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction φ_1 corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.

If (A1) and (A2) hold, then the conclusions of Lemma 2.1 are still valid for the initial value problems

$$u'' + a(t)u' + b(t)u + \lambda_1 h(t)u = 0, \qquad u(0) = 0, \qquad u'(0) = 1$$

and

$$u'' + a(t)u' + b(t)u + \lambda_1 h(t)u = 0, \qquad u(1) = 0, \qquad u'(1) = -1.$$

We denote the unique solutions of the above two initial problems by ξ and ζ, respectively. Then Lemma 2.1 shows that $\xi \in AC[0,1] \cap C^1[0,1)$, $\xi' \in AC_{\mathrm{loc}}[0,1)$ and $\zeta \in AC[0,1] \cap C^1(0,1]$, $\zeta' \in AC_{\mathrm{loc}}(0,1]$.

(i) From the definition of eigenvalue, we know that φ_1 satisfies

$$\begin{aligned} &\varphi_1'' + a(t)\varphi_1' + b(t)\varphi_1 + \lambda_1 h(t)\varphi_1 = 0, \quad t \in (0,1), \\ &\varphi_1(0) = \varphi_1(1) = 0. \end{aligned} \tag{2.9}$$

Since $\varphi_1(0) = \varphi_1(1) = 0$, there exists $\tau \in (0,1)$ such that $\varphi_1(\tau) = \|\varphi_1\| > 0$ and $\varphi_1'(\tau) = 0$. Let

$$\chi(t) = q(t)\varphi_1'(t)\xi(t) - q(t)\varphi_1(t)\xi'(t), \quad t \in (0,1). \tag{2.10}$$

Then $\chi \in C^1(0,1)$ and for $t \in (0,1)$, it is easy to compute that

$$\chi'(t) = \bigl(q(t)\varphi_1'(t)\bigr)'\xi(t) - \bigl(q(t)\xi'(t)\bigr)'\varphi_1(t) = 0.$$

For $0 < t < \tau$, integrating χ' from t to τ and letting $t \to 0$, we have

$$0 = \lim_{t\to 0}\int_t^\tau \chi'(s)\,\mathrm{d}s = \chi(\tau) - \lim_{t\to 0}\chi(t). \tag{2.11}$$

By Lemma 2.3 in [4] and the fact that $\xi \in C^1[0,1)$, $\xi(0) = 0$, we have

$$\lim_{t\to 0} q(t)\varphi_1'(t)\xi(t) = q(0)\lim_{t\to 0}\varphi_1'(t)\xi(t) = 0. \tag{2.12}$$

Combining (2.10), (2.11), (2.12) and

$$\lim_{t\to 0} q(t)\varphi_1(t)\xi'(t) = q(0)\varphi_1(0)\xi'(0) = 0, \qquad \chi(\tau) = -q(\tau)\varphi_1(\tau)\xi'(\tau),$$

we get

$$q(\tau)\varphi_1(\tau)\xi'(\tau) = 0.$$

Since $q(\tau) \neq 0$ and $\varphi_1(\tau) \neq 0$, we have $\xi'(\tau) = 0$. Then $\xi(\tau) \neq 0$ due to Lemma 2.2. Let us define the function ϕ by

$$\phi(t) = \varphi_1(t) - \frac{\varphi_1(\tau)}{\xi(\tau)}\xi(t), \quad t \in [0,1],$$

it is easy to check that ϕ is the solution of the following problem:

$$\phi'' + a(t)\phi' + b(t)\phi + \lambda_1 h(t)\phi = 0, \quad t \in (0,1),$$
$$\phi'(\tau) = 0, \qquad \phi(\tau) = 0.$$

Then Lemma 2.2 yields $\phi \equiv 0$, that is, $\varphi_1 = [\varphi_1(\tau)/\xi(\tau)]\xi$. In particular, $\varphi_1'(0) = \varphi_1(\tau)/\xi(\tau) \neq 0$; furthermore, $\varphi_1'(0) > 0$ by positivity of φ_1 on $(0,1)$. In the same manner, we can see that $\varphi_1 = [\varphi_1(\tau)/\zeta(\tau)]\zeta$ and $\varphi_1'(1) = -\varphi_1(\tau)/\zeta(\tau) < 0$.

Define

$$\Theta(t) = \begin{cases} \varphi_1'(0), & t = 0, \\ \frac{\varphi_1(t)}{t(1-t)}, & 0 < t < 1, \\ -\varphi_1'(1), & t = 1. \end{cases}$$

Then $\Theta(\cdot)$ is continuous and $\Theta(t) > 0$ on $[0,1]$, so there exist positive constants ν_1 and ν_2 such that $\nu_1 \leq \Theta(t) \leq \nu_2$ for $t \in [0,1]$, *i.e.*,

$$(**) \quad \nu_1 t(1-t) \leq \varphi_1(t) \leq \nu_2 t(1-t), \quad t \in [0,1].$$

(ii) From Lemma 2.1(iii), $(*)$ and $(**)$, for $t, s \in [0,1]$, we have

$$\rho G(t,s) \leq \rho G(s,s) = \alpha(s)\beta(s) \leq c_2\rho_2 s(1-s) \leq \frac{c_2\rho_2}{\nu_1}\nu_1 s(1-s) \leq \frac{c_2\rho_2}{\nu_1}\varphi_1(s).$$

Define

$$\delta_1 := \frac{\rho\nu_1}{c_2\rho_2}, \qquad \delta_2 := \nu_2,$$

then

$$\delta_1 G(t,s) \leq \varphi_1(s) \leq \delta_2 s(1-s), \quad t,s \in [0,1].$$

This completes the proof. □

Lemma 2.5 *Suppose that* (A1) *and* (A2) *are satisfied. Let*

$$P_1 = \left\{ u \in P \,\middle|\, \int_0^1 \varphi_1(t)\tilde{h}(t)u(t)\,\mathrm{d}t \geq \lambda_1^{-1}\delta_1\|u\| \right\},$$

where φ_1 *and* δ_1 *are defined by Lemma* 2.4. *Then* P_1 *is a cone in* $C[0,1]$ *and* $T(P) \subset P_1$.

Proof It follows from ($\tilde{\mathrm{A}}2$) and ($**$) that

$$\int_0^1 \varphi_1(t)\tilde{h}(t)\left|u(t)\right| \mathrm{d}t \le \delta_2 \int_0^1 t(1-t)\tilde{h}(t)\left|u(t)\right| \mathrm{d}t < +\infty$$

for any $u \in C[0,1]$, then P_1 is well defined. Furthermore, for any $u \in P$, from Lemma 2.4, we have

$$\begin{aligned}
\int_0^1 \varphi_1(t)\tilde{h}(t)(Tu)(t)\,\mathrm{d}t &= \int_0^1 \left[\varphi_1(t)\tilde{h}(t)\int_0^1 G(t,s)\tilde{h}(s)u(s)\,\mathrm{d}s\right]\mathrm{d}t \\
&= \int_0^1\int_0^1 \varphi_1(t)\tilde{h}(t)G(t,s)\tilde{h}(s)u(s)\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^1\int_0^1 \varphi_1(t)\tilde{h}(t)G(t,s)\tilde{h}(s)u(s)\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^1 \left[\tilde{h}(s)u(s)\int_0^1 G(t,s)\tilde{h}(t)\varphi_1(t)\,\mathrm{d}t\right]\mathrm{d}s \\
&= \lambda_1^{-1}\int_0^1 \tilde{h}(s)\varphi_1(s)u(s)\,\mathrm{d}s \\
&\ge \lambda_1^{-1}\delta_1\int_0^1 G(t,s)\tilde{h}(s)u(s)\,\mathrm{d}s = \lambda_1^{-1}\delta_1(Tu)(t),
\end{aligned}$$

then $\int_0^1 \varphi_1(t)\tilde{h}(t)(Tu)(t)\,\mathrm{d}t \ge \lambda_1^{-1}\delta_1\|Tu\|$, *i.e.*, $T(P) \subset P_1$.

This completes the proof. □

3 Proof of the main result

Proof of Theorem 1.1 From (A4), for $\varepsilon > 0$, there exists $L > 0$ such that

$$f(t,u) \ge \lambda_1(1+\varepsilon)u, \quad t \in [0,1], u \ge L. \tag{3.1}$$

Combining (A3) and (3.1), we can see that there exists a nonnegative function $c_1 \in C[0,1]$ such that

$$f(t,u) \ge \lambda_1(1+\varepsilon)u - c_1(t) - d(t)B(u), \quad t \in [0,1], u \in \mathbb{R}. \tag{3.2}$$

In the following we shall prove

$$u - Au \ne \mu\varphi_1, \quad \forall u \in C[0,1], \|u\| = R, \mu \ge 0, \tag{3.3}$$

provided that R is large enough.

In fact, if (3.3) is not true, then there exist $u_0 \in C[0,1]$, $\|u_0\| = R$ and $\mu_0 \ge 0$ such that

$$u_0 - Au_0 = \mu_0\varphi_1. \tag{3.4}$$

For $t \in [0,1]$ and for any $u \in C[0,1]$, we set

$$\eta_0(t) = \int_0^1 G(t,s)\tilde{h}(s)u_0(s)\,\mathrm{d}s, \qquad \eta_1(t) = \int_0^1 G(t,s)\tilde{h}(s)c_1(s)\,\mathrm{d}s, \tag{3.5}$$

$$\eta_2(t) = \int_0^1 G(t,s)\tilde{h}(s)d(s)B\bigl(u_0(s)\bigr)\,\mathrm{d}s, \tag{3.6}$$

$$g(u) = \int_0^1 \varphi_1(t)\tilde{h}(t)u(t)\,\mathrm{d}t. \tag{3.7}$$

Then from ($\tilde{\mathrm{A}}2$) and $(**)$, functional g is well defined. Thus we have

$$\begin{aligned} &u_0(t) + \eta_1(t) + \eta_2(t) \\ &\quad = (Au_0)(t) + \mu_0\varphi_1(t) + \eta_1(t) + \eta_2(t) \\ &\quad = \int_0^1 G(t,s)\tilde{h}(s)\bigl[f\bigl(s,u_0(s)\bigr) + c_1(s) + d(s)B\bigl(u_0(s)\bigr)\bigr]\,\mathrm{d}s + \mu_0\varphi_1(t). \end{aligned}$$

Recall that

$$\varphi_1(t) = \lambda_1^{-1}\int_0^1 G(t,s)\tilde{h}(s)\varphi_1(s)\,\mathrm{d}s.$$

It follows from Lemma 2.5 and (3.2) that $u_0 + \eta_1 + \eta_2 \in P_1$.

From Lemma 2.4 and (3.5)-(3.7), we have

$$\begin{aligned} g(\eta_0) &= \int_0^1 \varphi_1(t)\tilde{h}(t)\eta_0(t)\,\mathrm{d}t = \int_0^1 \left[\varphi_1(t)\tilde{h}(t)\int_0^1 G(t,s)\tilde{h}(s)u_0(s)\,\mathrm{d}s\right]\mathrm{d}t \\ &= \int_0^1 \left[\tilde{h}(s)u_0(s)\int_0^1 G(t,s)\varphi_1(t)\tilde{h}(t)\,\mathrm{d}t\right]\mathrm{d}s \\ &= \lambda_1^{-1}\int_0^1 \varphi_1(s)\tilde{h}(s)u_0(s)\,\mathrm{d}s = \lambda_1^{-1}g(u_0). \end{aligned} \tag{3.8}$$

By the same manner, we get

$$\begin{aligned} g(\eta_1) &= \lambda_1^{-1}\int_0^1 \varphi_1(s)\tilde{h}(s)c_1(s)\,\mathrm{d}s, \\ g(\eta_2) &= \lambda_1^{-1}\int_0^1 \varphi_1(s)\tilde{h}(s)d(s)B\bigl(u_0(s)\bigr)\,\mathrm{d}s. \end{aligned} \tag{3.9}$$

By (3.2), (3.7) and (3.8), we have

$$\begin{aligned} g(Au_0) &= \int_0^1 \left[\varphi_1(t)\tilde{h}(t)\int_0^1 G(t,s)\tilde{h}(s)f\bigl(s,u_0(s)\bigr)\,\mathrm{d}s\right]\mathrm{d}t \\ &\geq \lambda_1(1+\varepsilon)g(\eta_0) - g(\eta_1) - g(\eta_2) \\ &= (1+\varepsilon)g(u_0) - g(\eta_1) - g(\eta_2). \end{aligned}$$

Then, from the above inequality, (3.8), (3.9) and Lemma 2.5, we have

$$\begin{aligned} &g(Au_0) - g(u_0) \\ &\quad \geq \varepsilon g(u_0) - g(\eta_1) - g(\eta_2) \\ &\quad = \varepsilon g(u_0 + \eta_1 + \eta_2) - (1+\varepsilon)g(\eta_1) - (1+\varepsilon)g(\eta_2) \end{aligned}$$

$$\begin{aligned}
&= \varepsilon \int_0^1 \varphi_1(s)\tilde{h}(s)\big[u_0(s) + \eta_1(s) + \eta_2(s)\big]\,\mathrm{d}s - \lambda_1^{-1}(1+\varepsilon)\int_0^1 \varphi_1(s)\tilde{h}(s)c_1(s)\,\mathrm{d}s \\
&\quad - \lambda_1^{-1}(1+\varepsilon)\int_0^1 \varphi_1(s)\tilde{h}(s)d(s)B\big(u_0(s)\big)\,\mathrm{d}s \\
&\geq \varepsilon\lambda_1^{-1}\delta_1\|u_0 + \eta_1 + \eta_2\| - \lambda_1^{-1}(1+\varepsilon)\int_0^1 \varphi_1(s)\tilde{h}(s)c_1(s)\,\mathrm{d}s \\
&\quad - \lambda_1^{-1}(1+\varepsilon)\int_0^1 \varphi_1(s)\tilde{h}(s)d(s)B\big(u_0(s)\big)\,\mathrm{d}s \\
&\geq \varepsilon\lambda_1^{-1}\delta_1\big(\|u_0\| - \|\eta_1\| - \|\eta_2\|\big) - \lambda_1^{-1}(1+\varepsilon)\int_0^1 \varphi_1(s)\tilde{h}(s)c_1(s)\,\mathrm{d}s \\
&\quad - \lambda_1^{-1}(1+\varepsilon)\int_0^1 \varphi_1(s)\tilde{h}(s)d(s)B\big(u_0(s)\big)\,\mathrm{d}s.
\end{aligned} \tag{3.10}$$

We divide the proof into two cases as follows.

Case 1. $B(\cdot)$ is bounded on $\mathbb{R}$.

In this case, there exists $M_2 > 0$, for any $u \in \mathbb{R}$, $B(u) \leq M_2$. Thus,

$$\begin{aligned}
\|\eta_2\| &= \max_{t\in[0,1]}\int_0^1 G(t,s)\tilde{h}(s)d(s)B\big(u_0(s)\big)\,\mathrm{d}s \\
&\leq M_2\int_0^1 G(s,s)\tilde{h}(s)d(s)\,\mathrm{d}s.
\end{aligned} \tag{3.11}$$

Therefore, from (3.10), (3.11) and the expression of η_1, we get

$$\begin{aligned}
g(Au_0) - g(u_0) &\geq \varepsilon\lambda_1^{-1}\delta_1\big(\|u_0\| - \|\eta_1\| - \|\eta_2\|\big) - \lambda_1^{-1}(1+\varepsilon)\int_0^1 \varphi_1(s)\tilde{h}(s)c_1(s)\,\mathrm{d}s \\
&\quad - \lambda_1^{-1}(1+\varepsilon)\int_0^1 \varphi_1(s)\tilde{h}(s)d(s)B\big(u_0(s)\big)\,\mathrm{d}s \\
&\geq \varepsilon\lambda_1^{-1}\delta_1\left(\|u_0\| - \int_0^1 G(s,s)\tilde{h}(s)c_1(s)\,\mathrm{d}s - M_2\int_0^1 G(s,s)\tilde{h}(s)d(s)\,\mathrm{d}s\right) \\
&\quad - \lambda_1^{-1}(1+\varepsilon)\int_0^1 \varphi_1(s)\tilde{h}(s)c_1(s)\,\mathrm{d}s - \lambda_1^{-1}(1+\varepsilon)M_2\int_0^1 \varphi_1(s)\tilde{h}(s)d(s)\,\mathrm{d}s \\
&> 0
\end{aligned}$$

provided that we take

$$R > A + B + C + D := R_1,$$

where

$$A = \int_0^1 G(s,s)\tilde{h}(s)c_1(s)\,\mathrm{d}s, \qquad B = M_2\int_0^1 G(s,s)\tilde{h}(s)d(s)\,\mathrm{d}s,$$
$$C = \frac{(1+\varepsilon)}{\varepsilon\delta_1}\int_0^1 \varphi_1(s)\tilde{h}(s)c_1(s)\,\mathrm{d}s, \qquad D = \frac{(1+\varepsilon)}{\varepsilon\delta_1}M_2\int_0^1 \varphi_1(s)\tilde{h}(s)d(s)\,\mathrm{d}s.$$

Case 2. $B(\cdot)$ is unbounded on $\mathbb{R}$.

From (1.4), for $\rho > 0$, there exists a positive constant M_1 such that

$$B(u) < \rho|u|, \quad |u| \geq M_1. \tag{3.12}$$

In this case, there exists a positive constant $M_3 > M_1$ such that

$$B(u) \leq B(M_3), \quad |u| \leq M_3. \tag{3.13}$$

Since $\|u_0\| = R$, then for $s \in [0,1]$, $|u_0(s)| \leq R < M_3$. By (3.12) and (3.13), we have

$$\begin{aligned} \|\eta_2\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)\tilde{h}(s)d(s)B\big(u_0(s)\big)\,\mathrm{d}s \leq \int_0^1 G(s,s)\tilde{h}(s)d(s)B(M_3)\,\mathrm{d}s \\ &\leq \int_0^1 G(s,s)\tilde{h}(s)d(s)\rho M_3\,\mathrm{d}s = \rho M_3 \int_0^1 G(s,s)\tilde{h}(s)d(s)\,\mathrm{d}s. \end{aligned} \tag{3.14}$$

Therefore, from (3.10), (3.13) and (3.14), we have

$$\begin{aligned} g(Au_0) - g(u_0) &\geq \varepsilon\lambda_1^{-1}\delta_1\big(\|u_0\| - \|\eta_1\| - \|\eta_2\|\big) - \lambda_1^{-1}(1+\varepsilon)\int_0^1 \varphi_1(s)\tilde{h}(s)c_1(s)\,\mathrm{d}s \\ &\quad - \lambda_1^{-1}(1+\varepsilon)\int_0^1 \varphi_1(s)\tilde{h}(s)d(s)B\big(u_0(s)\big)\,\mathrm{d}s \\ &\geq \varepsilon\lambda_1^{-1}\delta_1\left(\|u_0\| - \int_0^1 G(s,s)\tilde{h}(s)c_1(s)\,\mathrm{d}s - \rho M_3\int_0^1 G(s,s)\tilde{h}(s)d(s)\,\mathrm{d}s\right) \\ &\quad - \lambda_1^{-1}(1+\varepsilon)\int_0^1 \varphi_1(s)\tilde{h}(s)c_1(s)\,\mathrm{d}s - \lambda_1^{-1}(1+\varepsilon)\rho M_3\int_0^1 \varphi_1(s)\tilde{h}(s)d(s)\,\mathrm{d}s \\ &> 0 \end{aligned}$$

provided that we take

$$R > A + E + C + F := R_2,$$

where

$$E = \rho M_3 \int_0^1 G(s,s)\tilde{h}(s)d(s)\,\mathrm{d}s, \qquad F = \frac{(1+\varepsilon)}{\varepsilon\delta_1}\rho M_3 \int_0^1 \varphi_1(s)\tilde{h}(s)d(s)\,\mathrm{d}s.$$

Thus, no matter which case happens, if we choose $R > \max\{R_1, R_2\}$, we have

$$g(Au_0) - g(u_0) > 0. \tag{3.15}$$

On the other hand, from (3.4), ($\tilde{\mathrm{A}}2$) and the fact that $\varphi_1(t) > 0$ for $t \in (0,1)$ and $\mu_0 \geq 0$, we have

$$g(u_0) - g(Au_0) = g(\mu_0\varphi_1) = \mu_0 g(\varphi_1) \geq 0,$$

which contradicts with (3.15). Therefore, (3.3) is true and we have

$$\deg(I - A, B_R, \theta) = 0. \tag{3.16}$$

From (A5), for $0 < \varepsilon_1 < \lambda_1$, there exists $0 < r < R$ such that

$$\left|f(t,u)\right| \le (\lambda_1 - \varepsilon_1)|u|, \quad t \in [0,1], |u| < r. \tag{3.17}$$

In the following we will prove that

$$u \neq \mu Au, \quad u \in \partial B_r, \mu \in [0,1]. \tag{3.18}$$

If (3.18) is not true, then there exist $u_1 \in \partial B_r$ and $\mu_1 \in (0,1]$ such that $u_1 = \mu_1 Au_1$, and then by (3.17) we have

$$\begin{aligned}
g\left(\left|u_1(t)\right|\right) &= g\left(\mu_1\left|(Au_1)(t)\right|\right) = \mu_1 \int_0^1 \left[\varphi_1(t)\tilde{h}(t)\left|\int_0^1 G(t,s)\tilde{h}(s)f\left(s,u_1(s)\right)\,\mathrm{d}s\right|\right]\mathrm{d}t \\
&\le \mu_1(\lambda_1 - \varepsilon_1)\int_0^1 \left[\varphi_1(t)\tilde{h}(t)\int_0^1 G(t,s)\tilde{h}(s)\left|u_1(s)\right|\,\mathrm{d}s\right]\mathrm{d}t \\
&= \mu_1(\lambda_1 - \varepsilon_1)\int_0^1 \left[\tilde{h}(s)\left|u_1(s)\right|\int_0^1 G(t,s)\varphi_1(t)\tilde{h}(t)\,\mathrm{d}t\right]\mathrm{d}s \\
&= \mu_1\left(1 - \lambda_1^{-1}\varepsilon_1\right)\int_0^1 \varphi_1(s)\tilde{h}(s)\left|u_1(s)\right|\,\mathrm{d}s \\
&= \mu_1\left(1 - \lambda_1^{-1}\varepsilon_1\right)g\left(\left|u_1(t)\right|\right),
\end{aligned}$$

which implies $g(|u_1(t)|) \le 0$.

On the other hand, from $\|u_1\| = r > 0$ and $\varphi_1(t) > 0$ for $t \in (0,1)$, we have

$$g\left(\left|u_1(t)\right|\right) = \int_0^1 \varphi_1(t)\tilde{h}(t)\left|u_1(t)\right|\,\mathrm{d}t > 0, \quad t \in [0,1],$$

which is a contradiction. Thus (3.18) holds. According to the invariance property of the Leray-Schauder degree, we have

$$\deg(I - A, B_r, \theta) = 1. \tag{3.19}$$

By (3.16), (3.19) and the additivity of the Leray-Schauder degree, we obtain

$$\deg(I - A, B_R \backslash \bar{B}_r, \theta) = \deg(I - A, B_R, \theta) - \deg(I - A, B_r, \theta) = -1.$$

Therefore, A has at least one fixed point on $B_R \backslash \bar{B}_r$, *i.e.*, problem (1.3) has at least one non-trivial solution. This completes the proof. □

4 Example

In this section, an example is given to illustrate the application of our main result (Theorem 1.1). Consider the following boundary value problem:

$$\begin{aligned}
&u'' - \frac{u}{t(1-t)} + \frac{1}{t(1-t)}f(t,u) = 0, \quad t \in (0,1), \\
&u(0) = u(1) = 0,
\end{aligned} \tag{4.1}$$

where

$$a \equiv 0, \qquad b(t) = -\frac{1}{t(1-t)}, \qquad h(t) = \frac{1}{t(1-t)},$$

$$f(t,u) = \begin{cases} -\sqrt{|u|}, & -\infty < u \le -1, \\ -u^2, & -1 < u < 0, \\ u^2, & 0 \le u < \infty. \end{cases}$$

Obviously, conditions (A1)-(A3) of Theorem 1.1 are satisfied. For the linear eigenvalue problem corresponding to (4.1), we compute that

$$\lambda_1 = 3, \qquad \varphi_1(t) = t(1-t).$$

Moreover,

$$\liminf_{u\to+\infty} \frac{f(t,u)}{u} = +\infty > \lambda_1 = 3 > 0 = \limsup_{u\to 0} \left| \frac{f(t,u)}{u} \right|,$$

which implies that (A4) and (A5) are satisfied. So, all of conditions in Theorem 1.1 are fulfilled. Therefore, the boundary value problem (4.1) has at least one nontrivial solution according to Theorem 1.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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