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Existence of two positive solutions for a class of second order impulsive singular integro-differential equations on the half line

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Abstract

In this paper, the author discusses the existence of two positive solutions for an infinite boundary value problem of second order impulsive singular integro-differential equations on the half line by means of the fixed point theorem of cone expansion and compression with norm type.

MSC: 45J05; 47H10

Keywords: impulsive singular integro-differential equation; infinite boundary value problem; fixed point theorem of cone expansion and compression with norm type

1 Introduction

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years (see [1–3]). Many problems have been investigated for impulsive differential equations, impulsive functional differential equations and impulsive differential inclusions. These problems include existence of solutions, stability theory, geometric properties, applications, *etc.* There is a vast literature on existence of solutions: by using upper and lower solutions together with the monotone iterative technique to obtain the extremal solutions [4–8]; by using fixed point theorems to obtain the existence of solution and multiple solutions [9–14]; by using the Leray-Schauder degree theory or fixed point index theory to obtain multiple solutions [15–19]; by using the variational method to obtain the existence of solution and existence of infinite many solutions [20–25]. In recent article [14], the author discussed the existence of two positive solutions for an infinite boundary value problem of first order impulsive singular integro-differential equations on the half line by means of the fixed point theorem of cone expansion and compression with norm type, which was established by the author in [26] (see also [27–30]). Now, in this article, we shall discuss such problem for a class of second order equations. The discussion for second order equations is more complicated than the first order case. We must introduce a new Banach space and a new cone in it to control both the unknown function and its derivative so that we can still use the fixed point theorem of cone expansion and compression with norm type.

Consider the infinite boundary value problem (IBVP) for second order impulsive singular integro-differential equation of mixed type on the half line:

$$\begin{cases} u''(t) = f(t, u(t), u'(t), (Tu)(t), (Su)(t)), & \forall t \in R'_{++}, \\ \Delta u|_{t=t_k} = I_k(u'(t_k^-)) \quad (k = 1, 2, 3, \ldots), \\ \Delta u'|_{t=t_k} = \bar{I}_k(u'(t_k^-)) \quad (k = 1, 2, 3, \ldots), \\ u(0) = 0, \qquad u'(\infty) = \beta u'(0), \end{cases} \tag{1}$$

where R denotes the set of all real numbers, $R_+ = \{x \in R : x \geq 0\}$, $R_{++} = \{x \in R : x > 0\}$, $0 < t_1 < \cdots < t_k < \cdots$, $t_k \to \infty$, $R'_{++} = R_{++} \backslash \{t_1, \ldots, t_k, \ldots\}$, $f \in C[R_{++} \times R_{++} \times R_{++} \times R_+ \times R_+, R_+]$, $I_k, \bar{I}_k \in C[R_{++}, R_+]$ $(k = 1, 2, 3, \ldots)$, $\beta > 1$, $u'(\infty) = \lim_{t\to\infty} u'(t)$ and

$$(Tu)(t) = \int_0^t K(t, s)u(s)\,ds, \qquad (Su)(t) = \int_0^\infty H(t, s)u(s)\,ds, \tag{2}$$

$K \in C[D, R_+]$, $D = \{(t, s) \in R_+ \times R_+ : t \geq s\}$, $H \in C[R_+ \times R_+, R_+]$. $\Delta u|_{t=t_k}$ and $\Delta u'|_{t=t_k}$ denote the jumps of $u(t)$ and $u'(t)$ at $t = t_k$, respectively, *i.e.*

$$\Delta u|_{t=t_k} = u\bigl(t_k^+\bigr) - u\bigl(t_k^-\bigr), \qquad \Delta u'|_{t=t_k} = u'\bigl(t_k^+\bigr) - u'\bigl(t_k^-\bigr),$$

where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively, and $u'(t_k^+)$ and $u'(t_k^-)$ represent the right and left limits of $u'(t)$ at $t = t_k$, respectively. In what follows, we always assume that

$$\lim_{t\to 0^+} f(t, u, v, w, z) = \infty, \quad \forall u, v \in R_{++}, w, z \in R_+, \tag{3}$$

$$\lim_{u\to 0^+} f(t, u, v, w, z) = \infty, \quad \forall t, v \in R_{++}, w, z \in R_+ \tag{4}$$

and

$$\lim_{v\to 0^+} f(t, u, v, w, z) = \infty, \quad \forall t, u \in R_{++}, w, z \in R_+, \tag{5}$$

i.e. $f(t, u, v, w, z)$ is singular at $t = 0, u = 0$ and $v = 0$. We also assume that

$$\lim_{v\to 0^+} I_k(v) = \infty \quad (k = 1, 2, 3, \ldots) \tag{6}$$

and

$$\lim_{v\to 0^+} \bar{I}_k(v) = \infty \quad (k = 1, 2, 3, \ldots), \tag{7}$$

i.e. $I_k(v)$ and $\bar{I}_k(v)$ $(k = 1, 2, 3, \ldots)$ are singular at $v = 0$. Let $PC[R_+, R] = \{u : u$ is a real function on R_+ such that $u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, 3, \ldots\}$ and $PC^1[R_+, R] = \{u \in PC[R_+, R] : u'(t)$ is continuous at $t \neq t_k$, and $u'(t_k^+)$ and $u'(t_k^-)$ exist for $k = 1, 2, 3, \ldots\}$. Let $u \in PC^1[R_+, R]$. For $0 < h < t_k - t_{k-1}$, by the mean value theorem, there exists $t_k - h < \xi_k < t_k$ such that

$$u(t_k) - u(t_k - h) = u'(\xi_k)h,$$

hence the left derivative of $u(t)$ at $t = t_k$, which is denoted by $u'_-(t_k)$, exists, and

$$u'_-(t_k) = \lim_{h\to 0^+} \frac{u(t_k) - u(t_k - h)}{h} = u'\bigl(t_k^-\bigr).$$

In what follows, it is understood that $u'(t_k) = u'_-(t_k)$. So, for $u \in PC^1[R_+, R]$, we have $u' \in PC[R_+, R]$.

A function $u \in PC^1[R_+, R] \cap C^2[R'_{++}, R]$ is called a positive solution of IBVP (1) if $u(t) > 0$ for $t \in R_{++}$ and $u(t)$ satisfies (1). Now, we need to introduce a new space $DPC^1[R_+, R]$ and a new cone Q in it. Let

$$DPC^1[R_+, R] = \left\{ u \in PC^1[R_+, R] : \sup_{t\in R_{++}} \frac{|u(t)|}{t} < \infty, \sup_{t\in R_+} \bigl|u'(t)\bigr| < \infty \right\}.$$

It is easy to see that $DPC^1[R_+, R]$ is a Banach space with the norm

$$\|u\|_D = \max\bigl\{ \|u\|_S, \bigl\|u'\bigr\|_B \bigr\},$$

where

$$\|u\|_S = \sup_{t\in R_{++}} \frac{|u(t)|}{t}, \qquad \bigl\|u'\bigr\|_B = \sup_{t\in R_+} \bigl|u'(t)\bigr|.$$

Let $W = \{u \in DPC^1[R_+, R] : u(t) \geq 0, u'(t) \geq 0, \forall t \in R_+\}$ and

$$Q = \left\{ u \in DPC^1[R_+, R] : \inf_{t\in R_{++}} \frac{u(t)}{t} \geq \beta^{-1}\|u\|_S, \inf_{t\in R_+} u'(t) \geq \beta^{-1}\bigl\|u'\bigr\|_B \right\}.$$

Obviously, W and Q are two cones in the space $DPC^1[R_+, R]$ and $Q \subset W$ (for details on cone theory, see [28]). Let $Q_+ = \{u \in Q : \|u\|_D > 0\}$ and $Q_{pq} = \{u \in Q : p \leq \|u\|_D \leq q\}$ for $q > p > 0$.

2 Several lemmas

Remark 1 (a) For $u \in DPC^1[R_+, R]$, we have $u(0) = 0$. This is clear since $u(0) \neq 0$ implies

$$\sup_{t\in R_{++}} \frac{|u(t)|}{t} = \infty.$$

(b) For $u \in Q_+$, we have $u(t) > 0$ for $t \in R_{++}$ and $u'(t) > 0$ for $t \in R_+$.

Lemma 1 *For $u \in Q$, we have*

$$\|u\|_S \geq \beta^{-1}\bigl\|u'\bigr\|_B, \qquad \bigl\|u'\bigr\|_B \geq \beta^{-1}\|u\|_S, \tag{8}$$

$$\beta^{-1}\|u\|_D \leq \|u\|_S \leq \|u\|_D, \qquad \beta^{-1}\|u\|_D \leq \bigl\|u'\bigr\|_B \leq \|u\|_D \tag{9}$$

and

$$\beta^{-2}\|u\|_D \leq \frac{u(t)}{t} \leq \|u\|_D, \quad \forall t \in R_{++}; \qquad \beta^{-2}\|u\|_D \leq u'(t) \leq \|u\|_D, \quad \forall t \in R_+. \tag{10}$$

Proof Since (8) implies (9) and (8) and (9) imply (10), we need only to show (8).

For fixed $0 < t < t_1$, observing $u(0) = 0$ and by the mean value theorem, there exists $0 < \xi < t$ such that

$$\frac{u(t)}{t} = \frac{u(t) - u(0)}{t} = u'(\xi).$$

So,

$$\|u\|_S = \sup_{s\in R_{++}} \frac{u(s)}{s} \ge \frac{u(t)}{t} = u'(\xi) \ge \inf_{s\in R_+} u'(s) \ge \beta^{-1} \|u'\|_B.$$

On the other hand, for any $0 < t < t_1$, we have

$$\frac{u(t)}{t} \ge \beta^{-1}\|u\|_S,$$

so,

$$u'(0) = \lim_{t\to 0^+} \frac{u(t) - u(0)}{t} = \lim_{t\to 0^+} \frac{u(t)}{t} \ge \beta^{-1}\|u\|_S,$$

hence,

$$\|u'\|_B = \sup_{s\in R_+} u'(s) \ge u'(0) \ge \beta^{-1}\|u\|_S. \qquad \square$$

Let us list some conditions.

(H_1) $\sup_{t\in J} \int_0^t K(t,s)s\,ds < \infty$, $\sup_{t\in J} \int_0^\infty H(t,s)s\,ds < \infty$ and

$$\lim_{t'\to t} \int_0^\infty \left|H(t',s) - H(t,s)\right| s\,ds = 0, \quad \forall t \in R_+.$$

In this case, let

$$k^* = \sup_{t\in R_+} \int_0^t K(t,s)s\,ds, \qquad h^* = \sup_{t\in R_+} \int_0^\infty H(t,s)s\,ds.$$

(H_2) There exist $a, b \in C[R_{++}, R_+]$, $g \in C[R_{++}, R_+]$ and $G \in C[R_{++} \times R_+ \times R_+, R_+]$ such that

$$f(t,u,v,w,z) \le a(t)g(u) + b(t)G(v,w,z), \quad \forall t,u,v \in R_{++}, w,z \in R_+$$

and

$$a_r^* = \int_0^\infty a(t)g_r(t)\,dt < \infty$$

for any $r > 0$, where

$$g_r(t) = \max\{g(u) : \beta^{-2}rt \le u \le rt\}$$

and

$$b^* = \int_0^\infty b(t)\,dt < \infty.$$

(H$_3$) $I_k(v) \leq t_k \bar{I}_k(v)$, $\forall v \in R_{++}$ $(k = 1, 2, 3, \ldots)$, and there exist $\gamma_k \in R_+$ $(k = 1, 2, 3, \ldots)$ and $F \in C[R_{++}, R_+]$ such that

$$\bar{I}_k(v) \leq \gamma_k F(v), \quad \forall v \in R_{++} \ (k = 1, 2, 3, \ldots)$$

and

$$\bar{\gamma} = \sum_{k=1}^{\infty} t_k \gamma_k < \infty,$$

and, consequently,

$$\gamma^* = \sum_{k=1}^{\infty} \gamma_k \leq t_1^{-1} \bar{\gamma} < \infty.$$

It is clear: if condition (H$_3$) is satisfied, then (6) implies (7).

(H$_4$) There exists $c \in C[R_{++}, R_{++}]$ such that

$$\frac{f(t, u, v, w, z)}{c(t)v} \to \infty \quad \text{as } v \to \infty$$

uniformly for $t, u \in R_{++}$, $w, z \in R_+$, and

$$c^* = \int_0^\infty c(t)\,dt < \infty.$$

(H$_5$) There exists $d \in C[R_{++}, R_{++}]$ such that

$$\left[d(t)\right]^{-1} f(t, u, v, w, z) \to \infty \quad \text{as } v \to 0^+$$

uniformly for $t, u \in R_{++}$, $w, z \in R_+$, and

$$d^* = \int_0^\infty d(t)\,dt < \infty.$$

Remark 2 It is clear: if condition (H$_1$) is satisfied, then the operators T and S defined by (2) are bounded linear operators from $DPC^1[R_+, R]$ into $BC[R_+, R]$ (the Banach space of all bounded continuous functions $u(t)$ on R_+ with the norm $\|u\|_B = \sup_{t \in R_+} |u(t)|$) and $\|T\| \leq k^*$, $\|S\| \leq h^*$; moreover, we have $T(DPC^1[R_+, R_+]) \subset BC[R_+, R_+]$ ($BC[R_+, R_+] = \{u \in BC[R_+, R] : u(t) \geq 0, \forall t \in R_+\}$) and $S(DPC^1[R_+, R_+]) \subset BC[R_+, R_+]$.

Remark 3 Condition (H$_4$) means that the function $f(t, u, v, w, z)$ is superlinear with respect to v.

Remark 4 Condition (H_5) means that the function $f(t,u,v,w,z)$ is singular at $v=0$ and it is stronger than (5).

Remark 5 In what follows, we need the following two formulas (see [6], Lemma 1):

(a) If $u \in PC[R_+, R] \cap C^1[R'_{++}, R]$, then

$$u(t) = u(0) + \int_0^t u'(s)\,ds + \sum_{0<t_k<t} \bigl[u(t_k^+) - u(t_k^-)\bigr], \quad \forall t \in R_+. \tag{11}$$

(b) If $u \in PC^1[R_+, R] \cap C^2[R'_{++}, R]$, then

$$\begin{aligned} u(t) &= u(0) + tu'(0) + \int_0^t (t-s)u''(s)\,ds \\ &\quad + \sum_{0<t_k<t} \bigl\{\bigl[u(t_k^+) - u(t_k^-)\bigr] + (t-t_k)\bigl[u'(t_k^+) - u'(t_k^-)\bigr]\bigr\}, \quad \forall t \in R_+. \end{aligned} \tag{12}$$

We shall reduce IBVP (1) to an impulsive integral equation. To this end, we first consider operator A defined by

$$\begin{aligned} (Au)(t) &= \frac{t}{\beta-1}\left\{\int_0^\infty f\bigl(s,u(s),u'(s),(Tu)(s),(Su)(s)\bigr)\,ds + \sum_{k=1}^\infty \bar{I}_k\bigl(u'(t_k^-)\bigr)\right\} \\ &\quad + \int_0^t (t-s) f\bigl(s,u(s),u'(s),(Tu)(s),(Su)(s)\bigr)\,ds \\ &\quad + \sum_{0<t_k<t} \bigl\{I_k\bigl(u'(t_k^-)\bigr) + (t-t_k)\bar{I}_k\bigl(u'(t_k^-)\bigr)\bigr\}, \quad \forall t \in R_+. \end{aligned} \tag{13}$$

In what follows, we write $J_1 = [0,t_1]$, $J_k = (t_{k-1}, t_k]$ $(k = 2,3,4,\ldots)$.

Lemma 2 *If conditions* (H_1)-(H_3) *are satisfied, then operator A defined by* (13) *is a continuous operator from Q_+ into Q; moreover, for any $q > p > 0$, $A(Q_{pq})$ is relatively compact.*

Proof Let $u \in Q_+$ and $\|u\|_B = r$. Then $r > 0$ and, by (10) and Remark 1(a),

$$\beta^{-2} rt \le u(t) \le rt, \qquad \beta^{-2} r \le u'(t) \le r, \quad \forall t \in R_+. \tag{14}$$

By conditions (H_1), (H_2) and (14), we have (for k^*, h^*, $a(t)$, $g(u)$, $b(t)$, $G(v,w,z)$, $g_r(t)$ and a_r^*, b^*, see conditions (H_1) and (H_2))

$$f\bigl(t,u(t),u'(t),(Tu)(t),(Su)(t)\bigr) \le a(t)g_r(t) + G_r b(t), \quad \forall t \in R_{++}, \tag{15}$$

where

$$G_r = \max\bigl\{g(v,w,z) : \beta^{-2} r \le v \le r, 0 \le w \le k^* r, 0 \le z \le h^* r\bigr\},$$

which implies the convergence of the infinite integral

$$\int_0^\infty f\bigl(t,u(t),u'(t),(Tu)(t),(Su)(t)\bigr)\,dt \tag{16}$$

and

$$\int_0^\infty f\bigl(t,u(t),u'(t),(Tu)(t),(Su)(t)\bigr)\,dt \le a_r^* + G_r b^*. \tag{17}$$

On the other hand, by condition (H_3) and (14), we have

$$\bar{I}_k\bigl(u'\bigl(t_k^-\bigr)\bigr) \le N_r\gamma_k \quad (k=1,2,3,\ldots), \tag{18}$$

where

$$N_r = \max\bigl\{F(v) : \beta^{-2}r \le v \le r\bigr\},$$

which implies the convergence of the infinite series

$$\sum_{k=1}^\infty \bar{I}_k\bigl(u'\bigl(t_k^-\bigr)\bigr) \tag{19}$$

and

$$\sum_{k=1}^\infty \bar{I}_k\bigl(u'\bigl(t_k^-\bigr)\bigr) \le N_r\gamma^*. \tag{20}$$

In addition, from (13) we get

$$\begin{aligned}\frac{(Au)(t)}{t} \ge{}& \frac{1}{\beta-1}\biggl\{\int_0^\infty f\bigl(s,u(s),u'(s),(Tu)(s),(Su)(s)\bigr)\,ds \\ &+ \sum_{k=1}^\infty \bar{I}_k\bigl(u'\bigl(t_k^-\bigr)\bigr)\biggr\}, \quad \forall t\in R_{++}.\end{aligned} \tag{21}$$

Moreover, by condition (H_3), we have

$$I_k(v) \le t_k\bar{I}_k(v), \quad \forall v\in R_{++}\ (k=1,2,3,\ldots),$$

so, (13) gives

$$\begin{aligned}\frac{(Au)(t)}{t} \le{}& \frac{1}{\beta-1}\biggl\{\int_0^\infty f\bigl(s,u(s),u'(s),(Tu)(s),(Su)(s)\bigr)\,ds + \sum_{k=1}^\infty \bar{I}_k\bigl(u'\bigl(t_k^-\bigr)\bigr)\biggr\} \\ &+ \int_0^\infty f\bigl(s,u(s),u'(s),(Tu)(s),(Su)(s)\bigr)\,ds + \sum_{k=1}^\infty \bar{I}_k\bigl(u'\bigl(t_k^-\bigr)\bigr) \\ ={}& \frac{\beta}{\beta-1}\biggl\{\int_0^\infty f\bigl(s,u(s),u'(s),(Tu)(s),(Su)(s)\bigr)\,ds \\ &+ \sum_{k=1}^\infty \bar{I}_k\bigl(u'\bigl(t_k^-\bigr)\bigr)\biggr\}, \quad \forall t\in R_{++}.\end{aligned} \tag{22}$$

On the other hand, by (13), we have

$$(Au)'(t) = \frac{1}{\beta-1}\left\{\int_0^\infty f\bigl(s,u(s),u'(s),(Tu)(s),(Su)(s)\bigr)\,ds + \sum_{k=1}^\infty \bar{I}_k\bigl(u'\bigl(t_k^-\bigr)\bigr)\right\} + \int_0^t f\bigl(s,u(s),u'(s),(Tu)(s),(Su)(s)\bigr)\,ds + \sum_{0<t_k<t} \bar{I}_k\bigl(u'\bigl(t_k^-\bigr)\bigr), \quad \forall t \in R_+, \tag{23}$$

so,

$$(Au)'(t) \ge \frac{1}{\beta-1}\left\{\int_0^\infty f\bigl(s,u(s),u'(s),(Tu)(s),(Su)(s)\bigr)\,ds + \sum_{k=1}^\infty \bar{I}_k\bigl(u'\bigl(t_k^-\bigr)\bigr)\right\}, \quad \forall t \in R_+ \tag{24}$$

and

$$(Au)'(t) \le \frac{\beta}{\beta-1}\left\{\int_0^\infty f\bigl(s,u(s),u'(s),(Tu)(s),(Su)(s)\bigr)\,ds + \sum_{k=1}^\infty \bar{I}_k\bigl(u'\bigl(t_k^-\bigr)\bigr)\right\}, \quad \forall t \in R_+. \tag{25}$$

It follows from (13), (21)-(25) that $Au \in Q$, *i.e.* $Au \in DPC^1[R_+, R]$ and

$$\inf_{t\in R_{++}} \frac{(Au)(t)}{t} \ge \beta^{-1}\|Au\|_S,$$
$$\inf_{t\in R_+} (Au)'(t) \ge \beta^{-1}\bigl\|(Au)'\bigr\|_B,$$

and, by (17), (20), (22) and (25),

$$\|Au\|_S \le \frac{\beta}{\beta-1}\bigl(a_r^* + G_r b^* + N_r\gamma^*\bigr), \tag{26}$$

$$\bigl\|(Au)'\bigr\|_B \le \frac{\beta}{\beta-1}\bigl(a_r^* + G_r b^* + N_r\gamma^*\bigr). \tag{27}$$

Thus, we have proved that A maps Q_+ into Q.

Now, we are going to show that A is continuous. Let $u_n, \bar{u} \in Q_+$, $\|u_n - \bar{u}\|_D \to 0$ ($n \to \infty$). Write $\|\bar{u}\|_D = 2\bar{r}$ ($\bar{r} > 0$) and we may assume that

$$\bar{r} \le \|u_n\|_D \le 3\bar{r} \quad (n = 1, 2, 3, \ldots).$$

So, (9) and (10) imply

$$\beta^{-2}\bar{r} \le \frac{u_n(t)}{t} \le 3\bar{r}, \qquad \beta^{-2}\bar{r} \le \frac{\bar{u}(t)}{t} \le 3\bar{r}, \quad \forall t \in R_{++}\ (n = 1, 2, 3, \ldots) \tag{28}$$

and

$$\beta^{-2}\bar{r} \le u_n'(t) \le 3\bar{r}, \qquad \beta^{-2}\bar{r} \le \bar{u}'(t) \le 3\bar{r}, \quad \forall t \in R_+\ (n = 1, 2, 3, \ldots). \tag{29}$$

By (13), we have

$$\begin{aligned}
&\frac{|(Au_n)(t)-(A\bar{u})(t)|}{t}\\
&\quad\le \frac{1}{\beta-1}\biggl\{\int_0^\infty \bigl|f\bigl(s,u_n(s),u'(s),(Tu_n)(s),(Su_n)(s)\bigr)\\
&\qquad - f\bigl(s,\bar{u}(s),\bar{u}'(s),(T\bar{u})(s),(S\bar{u})(s)\bigr)\bigr|\,ds + \sum_{k=1}^{\infty}\bigl|\bar{I}_k\bigl(u_n'(t_k^-)\bigr)-\bar{I}_k\bigl(\bar{u}'(t_k^-)\bigr)\bigr|\biggr\}\\
&\qquad + \int_0^t \bigl|f\bigl(s,u_n(s),u_n'(s),(Tu_n)(s),(Su_n)(s)\bigr) - f\bigl(s,\bar{u}(s),\bar{u}'(s),(T\bar{u})(s),(T\bar{u})(s)\bigr)\bigr|\,ds\\
&\qquad + \frac{1}{t}\sum_{0<t_k<t}\bigl|I_k\bigl(u_n'(t_k^-)\bigr)-I_k\bigl(\bar{u}'(t_k^-)\bigr)\bigr| + \sum_{0<t_k<t}\bigl|\bar{I}_k\bigl(u_n'(t_k^-)\bigr)-\bar{I}_k\bigl(\bar{u}(t_k^-)\bigr)\bigr|,\\
&\qquad \forall t\in R_{++}\ (n=1,2,3,\ldots). \qquad (30)
\end{aligned}$$

When $0<t\le t_1$, we have

$$\sum_{0<t_k<t}\bigl|I_k\bigl(u_n'(t_k^-)\bigr)-I_k\bigl(\bar{u}'(t_k^-)\bigr)\bigr| = 0,$$

so,

$$\begin{aligned}
&\sup_{t\in R_{++}}\frac{1}{t}\sum_{0<t_k<t}\bigl|I_k\bigl(u_n'(t_k^-)\bigr)-I_k\bigl(\bar{u}'(t_k^-)\bigr)\bigr|\\
&\quad = \sup_{t_1<t<\infty}\frac{1}{t}\sum_{0<t_k<t}\bigl|I_k\bigl(u_n'(t_k^-)\bigr)-I_k\bigl(\bar{u}'(t_k^-)\bigr)\bigr|\\
&\quad \le \frac{1}{t_1}\sum_{k=1}^{\infty}\bigl|I_k\bigl(u_n'(t_k^-)\bigr)-I_k\bigl(\bar{u}'(t_k^-)\bigr)\bigr|. \qquad (31)
\end{aligned}$$

It follows from (30) and (31) that

$$\begin{aligned}
\|Au_n - A\bar{u}\|_S &= \sup_{t\in R_{++}}\frac{|(Au_n)(t)-(A\bar{u})(t)|}{t}\\
&\le \frac{1}{t_1}\sum_{k=1}^{\infty}\bigl|I_k\bigl(u_n'(t_k^-)\bigr)-I_k\bigl(\bar{u}'(t_k^-)\bigr)\bigr|\\
&\quad + \frac{\beta}{\beta-1}\biggl\{\int_0^\infty \bigl|f\bigl(s,u_n(s),u_n'(s),(Tu_n)(s),(Su_n)(s)\bigr)\\
&\quad - f\bigl(s,\bar{u}(s),\bar{u}'(s),(T\bar{u})(s),(S\bar{u})(s)\bigr)\bigr|\,ds\\
&\quad + \sum_{k=1}^{\infty}\bigl|\bar{I}_k\bigl(u_n'(t_k^-)\bigr)-\bar{I}_k\bigl(\bar{u}'(t_k^-)\bigr)\bigr|\biggr\} \quad (n=1,2,3,\ldots). \qquad (32)
\end{aligned}$$

It is clear that

$$\begin{aligned}
&f\bigl(t,u_n(t),u_n'(t),(Tu_n)(t),(Su_n)(t)\bigr)\\
&\quad \to f\bigl(t,\bar{u}(t),\bar{u}'(t),(T\bar{u})(t),(S\bar{u})(t)\bigr) \quad \text{as } n\to\infty, \forall t\in R_{++} \qquad (33)
\end{aligned}$$

and, similar to (15) and observing (28), we have

$$\begin{aligned}&\left|f\left(t,u_n(t),u_n'(t),(Tu_n)(t),(Su_n)(t)\right)-f\left(t,\bar{u}(t),\bar{u}'(t),(T\bar{u})(t),(S\bar{u})(t)\right)\right|\\&\quad\leq 2\left[a(t)\bar{g}(t)+\bar{G}b(t)\right]=\sigma(t),\quad \forall t\in R_{++}\ (n=1,2,3,\ldots),\end{aligned}\tag{34}$$

where

$$\begin{aligned}&\bar{g}(t)=\max\left\{g(u):\beta^{-2}\bar{r}t\leq u\leq 3\bar{r}t\right\},\\&\bar{G}(t)=\max\left\{g(v,w,z):\beta^{-2}\bar{r}\leq v\leq 3\bar{r},0\leq w\leq 3k^*\bar{r},0\leq z\leq 3h^*\bar{r}\right\}.\end{aligned}$$

It is easy to see that condition (H_2) implies

$$a_{pq}^*=\int_0^\infty a(t)g_{pq}(t)\,dt<\infty\tag{35}$$

for any $q>p>0$, where

$$g_{pq}(t)=\max\left\{g(u):\beta^{-2}pt\leq u\leq qt\right\}.\tag{36}$$

So,

$$\int_0^\infty a(t)\bar{g}(t)\,dt<\infty,$$

and therefore,

$$\int_0^\infty \sigma(t)\,dt<\infty.\tag{37}$$

It follows from (33), (34), (37) and the dominated convergence theorem that

$$\begin{aligned}&\lim_{n\to\infty}\int_0^\infty\left|f\left(t,u_n(t),u_n'(t),(Tu_n)(t),(Su_n)(t)\right)-f\left(t,\bar{u}(t),\bar{u}'(t),(T\bar{u})(t),(S\bar{u})(t)\right)\right|dt\\&\quad=0.\end{aligned}\tag{38}$$

On the other hand, similar to (18) and observing (29), we have

$$\bar{I}_k\left(u_n'\left(t_k^-\right)\right)\leq\bar{N}_r\gamma_k,\qquad \bar{I}_k\left(\bar{u}'\left(t_k^-\right)\right)\leq\bar{N}_r\gamma_k\quad (k,n=1,2,3,\ldots),\tag{39}$$

where

$$\bar{N}_r=\max\left\{F(v):\beta^{-2}\bar{r}\leq v\leq 3\bar{r}\right\}.$$

For any given $\epsilon>0$, by (39) and condition (H_3), we can choose a positive integer k_0 such that

$$\sum_{k=k_0+1}^\infty t_k\bar{I}_k\left(u_n'\left(t_k^-\right)\right)<\epsilon\quad (n=1,2,3,\ldots)$$

and

$$\sum_{k=k_0+1}^{\infty} t_k \bar{I}_k\big(\bar{u}'\big(t_k^-\big)\big) < \epsilon,$$

so,

$$\sum_{k=k_0+1}^{\infty} I_k\big(u_n'\big(t_k^-\big)\big) < \epsilon \quad (n = 1, 2, 3, \ldots), \tag{40}$$

$$\sum_{k=k_0+1}^{\infty} I_k\big(\bar{u}'\big(t_k^-\big)\big) < \epsilon, \tag{41}$$

$$\sum_{k=k_0+1}^{\infty} \bar{I}_k\big(u_n'\big(t_k^-\big)\big) \le \frac{1}{t_1} \sum_{k=k_0+1}^{\infty} t_k \bar{I}_k\big(u_n'\big(t_k^-\big)\big) < t_1^{-1}\epsilon \tag{42}$$

and

$$\sum_{k=k_0+1}^{\infty} \bar{I}_k\big(\bar{u}'\big(t_k^-\big)\big) \le \frac{1}{t_1} \sum_{k=k_0+1}^{\infty} t_k \bar{I}_k\big(\bar{u}'\big(t_k^-\big)\big) < t_1^{-1}\epsilon. \tag{43}$$

It is clear that

$$I_k\big(u_n'\big(t_k^-\big)\big) \to I_k\big(\bar{u}'\big(t_k^-\big)\big) \quad \text{as } n \to \infty \ (k = 1, 2, 3, \ldots)$$

and

$$\bar{I}_k\big(u_n'\big(t_k^-\big)\big) \to \bar{I}_k\big(\bar{u}'\big(t_k^-\big)\big) \quad \text{as } n \to \infty \ (k = 1, 2, 3, \ldots),$$

so, we can choose a positive integer n_0 such that

$$\sum_{k=1}^{k_0} \big| I_k\big(u_n'\big(t_k^-\big)\big) - I_k\big(\bar{u}'\big(t_k^-\big)\big)\big| < \epsilon, \quad \forall n > n_0 \tag{44}$$

and

$$\sum_{k=1}^{k_0} \big| \bar{I}_k\big(u_n'\big(t_k^-\big)\big) - \bar{I}_k\big(\bar{u}'\big(t_k^-\big)\big)\big| < \epsilon, \quad \forall n > n_0. \tag{45}$$

From (40)-(45), we get

$$\sum_{k=1}^{\infty} \big| I_k\big(u_n'\big(t_k^-\big)\big) - I_k\big(\bar{u}'\big(t_k^-\big)\big)\big| < 3\epsilon, \quad \forall n > n_0$$

and

$$\sum_{k=1}^{\infty} \big| \bar{I}_k\big(u_n'\big(t_k^-\big)\big) - \bar{I}_k\big(\bar{u}'\big(t_k^-\big)\big)\big| < \big(1 + 2t_1^{-1}\big)\epsilon, \quad \forall n > n_0,$$

hence

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\left|I_k\left(u_n'\left(t_k^-\right)\right)-I_k\left(\bar{u}'\left(t_k^-\right)\right)\right|=0 \tag{46}$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\left|\bar{I}_k\left(u_n'\left(t_k^-\right)\right)-\bar{I}_k\left(\bar{u}'\left(t_k^-\right)\right)\right|=0. \tag{47}$$

It follows from (32), (38), (46) and (47) that

$$\lim_{n\to\infty}\|Au_n-A\bar{u}\|_S=0. \tag{48}$$

On the other hand, from (23) it is easy to get

$$\begin{aligned}\left\|(Au_n)'-(A\bar{u})'\right\|_B\le\frac{\beta}{\beta-1}\Biggl\{&\int_0^{\infty}\bigl|f\bigl(s,u_n(s),u_n'(s),(Tu_n)(s),(Su_n)(s)\bigr)\\&-f\bigl(s,\bar{u}(s),\bar{u}'(s),(T\bar{u})(s),(S\bar{u})(s)\bigr)\bigr|\,ds\\&+\sum_{k=1}^{\infty}\left|\bar{I}_k\left(u_n'\left(t_k^-\right)\right)-\bar{I}_k\left(\bar{u}'\left(t_k^-\right)\right)\right|\Biggr\}.\end{aligned} \tag{49}$$

So, (49), (38) and (47) imply

$$\lim_{n\to\infty}\left\|(Au_n)'-(A\bar{u})'\right\|_B=0. \tag{50}$$

It follows from (48) and (50) that $\|Au_n-A\bar{u}\|_D\to 0$ as $n\to\infty$, and the continuity of A is proved.

Finally, we prove that $A(Q_{pq})$ is relatively compact, where $q>p>0$ are arbitrarily given. Let $\bar{u}_n\in Q_{pq}$ $(n=1,2,3,\ldots)$. Then, by (10),

$$\beta^{-2}pt\le\bar{u}_n(t)\le qt,\qquad\beta^{-2}p\le\bar{u}_n'(t)\le q,\quad\forall t\in R_+\ (n=1,2,3,\ldots). \tag{51}$$

Similar to (15), (18), (26) and observing (51), we have

$$\begin{aligned}&f\bigl(t,\bar{u}_n(t),\bar{u}_n'(t),(T\bar{u}_n)(t),(S\bar{u}_n)(t)\bigr)\\&\quad\le a(t)g_{pq}(t)+G_{pq}b(t),\quad\forall t\in R_{++}\ (n=1,2,3,\ldots),\end{aligned} \tag{52}$$

$$\bar{I}_k\bigl(\bar{u}_n'\bigl(t_k^-\bigr)\bigr)\le N_{pq}\gamma_k\quad(k,n=1,2,3,\ldots) \tag{53}$$

and

$$\|A\bar{u}_n\|_S\le\frac{\beta}{\beta-1}\bigl(a_{pq}^*+G_{pq}b^*+N_{pq}\gamma^*\bigr)\quad(n=1,2,3,\ldots), \tag{54}$$

where $g_{pq}(t)$ and a_{pq}^* are defined by (36) and (35), respectively, and

$$\begin{aligned}G_{pq}&=\max\bigl\{g(v,w,z):\beta^{-2}p\le v\le q,0\le w\le k^*q,0\le z\le h^*q\bigr\},\\N_{pq}&=\max\bigl\{F(v):\beta^{-2}p\le v\le q\bigr\}.\end{aligned}$$

From (54) we see that functions $\{(A\bar{u}_n)(t)\}$ $(n = 1, 2, 3, \ldots)$ are uniformly bounded on $[0, r]$ for any $r > 0$. On the other hand, by (13) and (52)-(54), we have

$$\begin{aligned}
0 &\le (A\bar{u}_n)\bigl(t'\bigr) - (A\bar{u}_n)(t) \\
&= \frac{t' - t}{\beta - 1}\left\{\int_0^\infty f\bigl(s, \bar{u}_n(s), \bar{u}_n'(s), (T\bar{u}_n)(s), (S\bar{u}_n)(s)\bigr)\,ds + \sum_{k=1}^\infty \bar{I}_k\bigl(\bar{u}_n'\bigl(t_k^-\bigr)\bigr)\right\} \\
&\quad + \bigl(t' - t\bigr)\int_0^t f\bigl(s, \bar{u}_n(s), \bar{u}_n'(s), (T\bar{u}_n)(s), (S\bar{u}_n)(s)\bigr)\,ds \\
&\quad + \int_t^{t'} \bigl(t' - s\bigr) f\bigl(s, \bar{u}_n(s), \bar{u}_n'(s), (T\bar{u}_n)(s), (S\bar{u}_n)(s)\bigr)\,ds \\
&\quad + \bigl(t' - t\bigr)\sum_{0<t_k<t} \bar{I}_k\bigl(\bar{u}_n'\bigl(t_k^-\bigr)\bigr) \\
&\le \frac{t' - t}{\beta - 1}\bigl(a_{pq}^* + G_{pq}b^* + N_{pq}\gamma^*\bigr) + \bigl(t' - t\bigr)\bigl(a_{pq}^* + G_{pq}b^*\bigr) \\
&\quad + (t_k - t_{k-1})\int_t^{t'} \bigl[a(s)g_{pq}(s) + G_{pq}b(s)\bigr]\,ds + \bigl(t' - t\bigr)N_{pq}\gamma^*, \\
&\quad \forall t, t' \in J_k, t' > t\ (k, n = 1, 2, 3, \ldots),
\end{aligned}$$

which implies that functions $\{w_n(t)\}$ $(n = 1, 2, 3, \ldots)$ defined by (for any fixed k)

$$w_n(t) = \begin{cases} (A\bar{u}_n)(t), & \forall t \in J_k = (t_{k-1}, t_k], \\ (A\bar{u}_n)(t_{k-1}^+), & \forall t = t_{k-1} \end{cases} \qquad (n = 1, 2, 3, \ldots)$$

($(A\bar{u}_n)(t_{k-1}^+)$ denotes the right limit of $(A\bar{u}_n)(t)$ at $t = t_{k-1}$) are equicontinuous on $\bar{J}_k = [t_{k-1}, t_k]$. Consequently, by the Ascoli-Arzela theorem, $\{w_n(t)\}$ has a subsequence which is convergent uniformly on $\bar{J}_k$. So, functions $\{A\bar{u}_n(t)\}$ $(n = 1, 2, 3, \ldots)$ have a subsequence which is convergent uniformly on J_k. Now, by the diagonal method, we can choose a subsequence $\{(A\bar{u}_{n_i})(t)\}$ $(i = 1, 2, 3, \ldots)$ of $\{(A\bar{u}_n)(t)\}$ $(n = 1, 2, 3, \ldots)$ such that $\{(A\bar{u}_{n_i})(t)\}$ $(i = 1, 2, 3, \ldots)$ is convergent uniformly on each J_k $(k = 1, 2, 3, \ldots)$. Let

$$\lim_{i\to\infty} (A\bar{u}_{n_i})(t) = \bar{w}(t), \quad \forall t \in R_+. \tag{55}$$

Similarly, we can discuss $\{(A\bar{u}_n)'(t)\}$ $(n = 1, 2, 3, \ldots)$. Similar to (27) and by (23), we have

$$\bigl\|(A\bar{u}_n)'\bigr\|_B \le \frac{\beta}{\beta - 1}\bigl(a_{pq}^* + G_{pq}b^* + N_{pq}\gamma^*\bigr) \quad (n = 1, 2, 3, \ldots) \tag{56}$$

and

$$\begin{aligned}
0 \le (A\bar{u}_n)'\bigl(t'\bigr) - (A\bar{u}_n)'(t) &= \int_t^{t'} f\bigl(s, \bar{u}_n(s), \bar{u}_n'(s), (T\bar{u}_n)(s), (S\bar{u}_n)(s)\bigr)\,ds \\
&\le \int_t^{t'} \bigl[a(s)g_{pq}(s) + G_{pq}b(s)\bigr]\,ds, \quad \forall t, t' \in J_k, t' > t\ (n = 1, 2, 3, \ldots),
\end{aligned}$$

and by a similar method, we can prove that $\{(A\bar{u}_{n_i})'(t)\}$ $(n = 1, 2, 3, \ldots)$ has a subsequence which is convergent uniformly on each J_k $(k - 1, 2, 3, \ldots)$. For the sake of simplicity of notation, we may assume that $\{(A\bar{u}_{n_i})'(t)\}$ $(i = 1, 2, 3, \ldots)$ itself converges uniformly on each J_k $(k = 1, 2, 3, \ldots)$. Let

$$\lim_{i\to\infty} (A\bar{u}_{n_i})'(t) = y(t). \tag{57}$$

By (55), (57) and the uniformity of convergence, we have

$$\bar{w}'(t) = y(t), \quad \forall t \in R_+, \tag{58}$$

and so, $\bar{w} \in PC^1[R_+, R]$. From (54) and (56), we get

$$\|\bar{w}\|_S \le \frac{\beta}{\beta - 1}\big(a^*_{pq} + G_{pq}b^* + N_{pq}\gamma^*\big)$$

and

$$\big\|\bar{w}'\big\|_B \le \frac{\beta}{\beta - 1}\big(a^*_{pq} + G_{pq}b^* + N_{pq}\gamma^*\big).$$

Consequently, $\bar{w} \in DPC^1[R_+, R]$ and

$$\|\bar{w}\|_D \le \frac{\beta}{\beta - 1}\big(a^*_{pq} + G_{pq}b^* + N_{pq}\gamma^*\big).$$

Let $\epsilon > 0$ be arbitrarily given. Choose a sufficiently large positive number η such that

$$\int_\eta^\infty a(t)g_{pq}(t)\,dt + G_{pq}\int_\eta^\infty b(t)\,dt + N_{pq}\sum_{t_k\ge\eta}\gamma_k < \epsilon. \tag{59}$$

For any $\eta < t < \infty$, we have, by (23), (52) and (53),

$$\begin{aligned}
0 &\le (A\bar{u}_{n_i})'(t) - (A\bar{u}_{n_i})'(\eta) \\
&= \int_\eta^t f\big(s, \bar{u}_{n_i}(s), \bar{u}'_{n_i}(s), (T\bar{u}_{n_i})(s), (S\bar{u}_{n_i})(s)\big)\,ds + \sum_{\eta\le t_k<t} \bar{I}_k\big(\bar{u}'_{n_i}\big(t_k^-\big)\big) \\
&\le \int_\eta^t a(s)g_{pq}(s)\,ds + G_{pq}\int_\eta^t b(s)\,ds + N_{pq}\sum_{\eta\le t_k<t}\gamma_k \quad (i = 1, 2, 3, \ldots),
\end{aligned}$$

which implies by virtue of (59) that

$$0 \le (A\bar{u}_{n_i})'(t) - (A\bar{u}_{n_i})'(\eta) < \epsilon, \quad \forall t > \eta\ (i = 1, 2, 3, \ldots). \tag{60}$$

Letting $i \to \infty$ in (60) and observing (57) and (58), we get

$$0 \le \bar{w}'(t) - \bar{w}'(\eta) \le \epsilon, \quad \forall t > \eta. \tag{61}$$

On the other hand, since $\{(A\bar{u}_{n_i})'(t)\}$ converges uniformly to $\bar{w}'(t)$ on $[0, \eta]$ as $i \to \infty$, there exists a positive integer i_0 such that

$$\big|(A\bar{u}_{n_i})'(t) - \bar{w}'(t)\big| < \epsilon, \quad \forall t \in [0, \eta], i > i_0. \tag{62}$$

It follows from (60)-(62) that

$$\begin{aligned}\left|(A\bar{u}_{n_i})'(t)-\bar{w}'(t)\right| &\le \left|(A\bar{u}_{n_i})'(t)-(A\bar{u}_{n_i})'(\eta)\right|+\left|(A\bar{u}_{n_i})'(\eta)-\bar{w}'(\eta)\right| \\ &\quad+\left|\bar{w}'(\eta)-\bar{w}'(t)\right|<3\epsilon, \quad \forall t>\eta, i>i_0. \end{aligned} \tag{63}$$

By (62) and (63), we have

$$\left\|(A\bar{u}_{n_i})'-\bar{w}'\right\|_B\le 3\epsilon, \quad \forall i>i_0,$$

hence

$$\lim_{i\to\infty}\left\|(A\bar{u}_{n_i})'-\bar{w}'\right\|_B=0. \tag{64}$$

It is clear that (13) implies

$$(A\bar{u}_{n_i})\big(t_k^+\big)-(A\bar{u}_{n_i})\big(t_k^-\big)=I_k\big(\bar{u}'_{n_i}\big(t_k^-\big)\big) \quad (k,i=1,2,3,\ldots). \tag{65}$$

By virtue of the uniformity of convergence of $\{(A\bar{u}_{n_i})(t)\}$, we see that

$$\lim_{i\to\infty}(A\bar{u}_{n_i})\big(t_k^-\big)=\bar{w}\big(t_k^-\big), \qquad \lim_{i\to\infty}(A\bar{u}_{n_i})\big(t_k^+\big)=\bar{w}\big(t_k^+\big) \quad (k=1,2,3,\ldots),$$

so, (65) implies that

$$\lim_{i\to\infty}I_k\big(\bar{u}'_{n_i}\big(t_k^-\big)\big) \quad (k=1,2,3,\ldots)$$

exist and

$$\bar{w}\big(t_k^+\big)-\bar{w}\big(t_k^-\big)=\lim_{i\to\infty}I_k\big(\bar{u}'_{n_i}\big(t_k^-\big)\big) \quad (k=1,2,3,\ldots).$$

Let

$$\lim_{i\to\infty}I_k\big(\bar{u}'_{n_i}\big(t_k^-\big)\big)=\alpha_k \quad (k=1,2,3,\ldots).$$

Then $\alpha_k\ge 0$ $(k=1,2,3,\ldots)$ and

$$\bar{w}\big(t_k^+\big)-\bar{w}\big(t_k^-\big)=\alpha_k \quad (k=1,2,3,\ldots). \tag{66}$$

By (53) and condition (H_3), we have

$$I_k\big(\bar{u}'_{n_i}\big(t_k^-\big)\big)\le N_{pq}t_k\gamma_k \quad (k,i=1,2,3,\ldots), \tag{67}$$

so,

$$\alpha_k\le N_{pq}t_k\gamma_k \quad (k=1,2,3,\ldots). \tag{68}$$

For any given $\epsilon > 0$, choose a sufficiently large positive integer k_0 such that

$$N_{pq} \sum_{k=k_0+1}^{\infty} t_k \gamma_k < \epsilon, \tag{69}$$

and then, choose another sufficiently large integer i_1 such that

$$\left| I_k\left(\bar{u}'_{n_i}\left(t_k^-\right)\right) - \alpha_k \right| < \frac{\epsilon}{k_0}, \quad \forall i > i_1 \ (k = 1, 2, \ldots, k_0). \tag{70}$$

It follows from (67)-(70) that

$$\begin{aligned} \sum_{k=1}^{\infty} \left| I_k\left(\bar{u}'_{n_i}\left(t_k^-\right)\right) - \alpha_k \right| &\leq \sum_{k=1}^{k_0} \left| I_k\left(\bar{u}'_{n_i}\left(t_k^-\right)\right) - \alpha_k \right| \\ &\quad + \sum_{k=k_0+1}^{\infty} I_k\left(\bar{u}'_{n_i}\left(t_k^-\right)\right) + \sum_{k=k_0+1}^{\infty} \alpha_k < 3\epsilon, \quad \forall i > i_1, \end{aligned}$$

hence

$$\lim_{i\to\infty} \sum_{k=1}^{\infty} \left| I_k\left(\bar{u}'_{n_i}\left(t_k^-\right)\right) - \alpha_k \right| = 0. \tag{71}$$

By formula (11) and (65), (66), we have

$$(A\bar{u}_{n_i})(t) = \int_0^t (A\bar{u}_{n_i})'(s)\, ds + \sum_{0<t_k<t} I_k\left(\bar{u}'_{n_i}\left(t_k^-\right)\right), \quad \forall t \in R_+ \ (i = 1, 2, 3, \ldots)$$

and

$$\bar{w}(t) = \int_0^t \bar{w}'(s)\, ds + \sum_{0<t_k<t} \alpha_k, \quad \forall t \in R_+,$$

which imply

$$\begin{aligned} \left| (A\bar{u}_{n_i})(t) - \bar{w}(t) \right| &\leq t \left\| (A\bar{u}_{n_i})' - \bar{w}' \right\|_B \\ &\quad + \sum_{0<t_k<t} \left| I_k\left(\bar{u}'_{n_i}\left(t_k^-\right)\right) - \alpha_k \right|, \quad \forall t \in R_+ \ (i = 1, 2, 3, \ldots). \end{aligned} \tag{72}$$

Since

$$\sum_{0<t_k<t} \left| I_k\left(\bar{u}'_{n_i}\left(t_k^-\right)\right) - \alpha_k \right| = 0, \quad \forall 0 < t \leq t_1,$$

(72) implies

$$\|A\bar{u}_{n_i} - \bar{w}\|_S \leq \left\| (A\bar{u}_{n_i})' - \bar{w}' \right\|_B + t_1^{-1} \sum_{k=1}^{\infty} \left| I_k\left(\bar{u}'_{n_i}\left(t_k^-\right)\right) - \alpha_k \right| \quad (i = 1, 2, 3, \ldots). \tag{73}$$

By (64), (71) and (73), we have

$$\lim_{i\to\infty} \|A\bar{u}_{n_i} - \bar{w}\|_S = 0. \tag{74}$$

It follows from (64) and (74) that $\|A\bar{u}_{n_i} - \bar{w}\|_D \to 0$ as $i \to \infty$, and the relative compactness of $A(Q_{pq})$ is proved. □

Lemma 3 *Let conditions* (H_1)-(H_3) *be satisfied. Then* $u \in Q_+ \cap C^2[R'_{++}, R]$ *is a positive solution of IBVP* (1) *if and only if* $u \in Q_+$ *is a solution of the following impulsive integral equation*:

$$\begin{aligned} u(t) = {}& \frac{t}{\beta - 1}\left\{\int_0^\infty f\big(s, u(s), u'(s), (Tu)(s), (Su)(s)\big)\,ds + \sum_{k=1}^\infty \bar{I}_k\big(u'\big(t_k^-\big)\big)\right\} \\ &+ \int_0^t (t-s) f\big(s, u(s), u'(s), (Tu)(s), (Su)(s)\big)\,ds \\ &+ \sum_{0<t_k<t}\big\{I_k\big(u'\big(t_k^-\big)\big) + (t - t_k)\bar{I}_k\big(u'\big(t_k^-\big)\big)\big\}, \quad \forall t \in R_+, \end{aligned} \tag{75}$$

i.e. u is a fixed point of operator A defined by (13).

Proof If $u \in Q_+ \cap C^2[R'_{++}, R]$ is a positive solution of IBVP (1), then, by (1) and formula (12), we have

$$\begin{aligned} u(t) = {}& tu'(0) + \int_0^t (t-s) f\big(s, u(s), u'(s), (Tu)(s), (Su)(s)\big)\,ds \\ &+ \sum_{0<t_k<t}\big\{I_k\big(u'\big(t_k^-\big)\big) + (t - t_k)\bar{I}_k\big(u'\big(t_k^-\big)\big)\big\}, \quad \forall t \in R_+. \end{aligned} \tag{76}$$

Differentiation of (76) gives

$$u'(t) = u'(0) + \int_0^t f\big(s, u(s), u'(s), (Tu)(s), (Su)(s)\big)\,ds + \sum_{0<t_k<t} \bar{I}_k\big(u'\big(t_k^-\big)\big), \quad \forall t \in R_+. \tag{77}$$

Under conditions (H_1)-(H_3), we have shown in the proof of Lemma 2 that the infinite integral (15) and the infinite series (19) are convergent. So, by taking limits as $t \to \infty$ in both sides of (77) and using the relation $u'(\infty) = \beta u'(0)$, we get

$$u'(0) = \frac{1}{\beta - 1}\left\{\int_0^\infty f\big(s, u(s), u'(s), (Tu)(s), (Su)(s)\big)\,ds + \sum_{k=1}^\infty \bar{I}_k\big(u'\big(t_k^-\big)\big)\right\}. \tag{78}$$

Now, substituting (78) into (76), we see that $u(t)$ satisfies equation (75).

Conversely, if $u \in Q_+$ is a solution of equation (75), then direct differentiation of (75) twice gives

$$\begin{aligned} u'(t) = {}& \frac{1}{\beta - 1}\left\{\int_0^\infty f\big(s, u(s), u'(s), (Tu)(s), (Su)(s)\big)\,ds + \sum_{k=1}^\infty \bar{I}_k\big(u'\big(t_k^-\big)\big)\right\} \\ &+ \int_0^t f\big(s, u(s), u'(s), (Tu)(s), (Su)(s)\big)\,ds + \sum_{0<t_k<t} \bar{I}_k\big(u'\big(t_k^-\big)\big), \quad \forall t \in R_+ \end{aligned} \tag{79}$$

and

$$u''(t) = f\big(t, u(t), u'(t), (Tu)(t), (Su)(t)\big), \quad \forall t \in R'_{++}.$$

So, $u \in C^2[R'_{++}, R]$ and

$$\Delta u|_{t=t_k} = I_k\big(u'\big(t_k^-\big)\big), \qquad \Delta u'|_{t=t_k} = \bar{I}_k\big(u'\big(t_k^-\big)\big) \quad (k = 1, 2, 3, \ldots).$$

Moreover, taking limits as $t \to \infty$ in (79), we see that $u'(\infty)$ exists and

$$u'(\infty) = \frac{\beta}{\beta - 1}\left\{\int_0^\infty f\big(s, u(s), u'(s), (Tu)(s), (Su)(s)\big)\, ds + \sum_{k=1}^{\infty} \bar{I}_k\big(u'\big(t_k^-\big)\big)\right\} = \beta u'(0).$$

Hence, $u(t)$ is a positive solution of IBVP (1). □

Lemma 4 (Fixed point theorem of cone expansion and compression with norm type, see Corollary 1 in [26] or Theorem 3 in [27] or Theorem 2.3.4 in [28], see also [29, 30]) *Let P be a cone in a real Banach space E and Ω_1, Ω_2 be two bounded open sets in E such that $\theta \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, where θ denotes the zero element of E and $\bar{\Omega}_i$ denotes the closure of Ω_i* ($i = 1, 2$). *Let the operator $A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ be completely continuous* (*i.e. continuous and compact*). *Suppose that one of the following two conditions is satisfied*:

(a) $\|Ax\| \leq \|x\|$, $\forall x \in P \cap \partial\Omega_1$; $\|Ax\| \geq \|x\|$, $\forall x \in P \cap \partial\Omega_2$, *where $\partial\Omega_i$ denotes the boundary of Ω_i* ($i = 1, 2$).

(b) $\|Ax\| \geq \|x\|$, $\forall x \in P \cap \partial\Omega_1$; $\|Ax\| \leq \|x\|$, $\forall x \in P \cap \partial\Omega_2$.

Then A has at least one fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

3 Main theorem

Theorem *Let conditions* (H_1)-(H_5) *be satisfied. Assume that there exists $r > 0$ such that*

$$\frac{\beta}{\beta - 1}\big(a_r^* + G_r b^* + N_r \gamma^*\big) < r, \tag{80}$$

where a_r^*, b^* and γ^* are defined in conditions (H_1) *and* (H_2), *and, G_r and N_r are defined by two equalities below* (15) *and* (18), *respectively. Then IBVP* (1) *has at least two positive solutions $u^*, u^{**} \in Q_+ \cap C^2[R'_{++}, R]$ such that*

$$0 < \inf_{t \in R_{++}} \frac{u^*(t)}{t} \leq \sup_{t \in R_{++}} \frac{u^*(t)}{t} < r,$$

$$0 < \inf_{t \in R_+} \big(u^*\big)'(t) \leq \sup_{t \in R_+} \big(u^*\big)'(t) < r,$$

$$\beta^{-2} r < \inf_{t \in R_{++}} \frac{u^{**}(t)}{t} \leq \sup_{t \in R_{++}} \frac{u^{**}(t)}{t} < \infty$$

and

$$\beta^{-2} r < \inf_{t \in R_+} \big(u^{**}\big)'(t) \leq \sup_{t \in R_+} \big(u^{**}\big)'(t) < \infty.$$

Proof By Lemma 2 and Lemma 3, operator A defined by (13) is continuous from Q_+ into Q, and we need to prove that A has two fixed points u^* and u^{**} in Q_+ such that $0 < \|u^*\|_D < r < \|u^{**}\|_D$.

By condition (H_4), there exists $r_1 > 0$ such that

$$f(t,u,v,w,z) \geq \frac{\beta^2(\beta-1)}{c^*}c(t)v, \quad \forall t,u \in R_{++}, v \geq r_1, w,z \in R_+. \tag{81}$$

Choose

$$r_2 > \max\{\beta^2 r_1, r\}. \tag{82}$$

For $u \in Q$, $\|u\|_D = r_2$, we have, by (10) and (82),

$$u'(t) \geq \beta^{-2} r_2 > r_1, \quad \forall t \in R_+,$$

so, (23) and (81) imply

$$\begin{aligned}(Au)'(t) &\geq \frac{1}{\beta-1}\int_0^\infty f\bigl(s,u(s),u'(s),(Tu)(s),(Su)(s)\bigr)\,ds \\ &\geq \frac{\beta^2}{c^*}\int_0^\infty c(s)u'(s)\,ds \geq \frac{r_2}{c^*}\int_0^\infty c(s)\,ds = r_2, \quad \forall t \in R_+,\end{aligned}$$

and consequently,

$$\left\|(Au)'\right\|_B \geq r_2,$$

hence

$$\|Au\|_D \geq \|u\|_D, \quad \forall u \in Q, \|u\|_D = r_2. \tag{83}$$

By condition (H_5), there exists $r_3 > 0$ such that

$$f(t,u,v,w,z) \geq \frac{(\beta-1)r}{d^*}d(t), \quad \forall t,u \in R_{++}, 0 < v < r_3, w,z \in R_+. \tag{84}$$

Choose

$$0 < r_4 < \min\{r_3, r\}. \tag{85}$$

For $u \in Q$, $\|u\|_D = r_4$, we have, by (10) and (85),

$$r_3 > r_4 \geq u'(t) \geq \beta^{-2} r_4 > 0,$$

so, we get, by (23) and (84),

$$\begin{aligned}(Au)'(t) &\geq \frac{1}{\beta-1}\int_0^\infty f\bigl(s,u(s),u'(s),(Tu)(s),(Su)(s)\bigr)\,ds \\ &\geq \frac{r}{d^*}\int_0^\infty d(s)\,ds = r > r_4, \quad \forall t \in R_+,\end{aligned}$$

hence

$$\|(Au)'\|_B > r_4,$$

and consequently,

$$\|Au\|_D > \|u\|_D, \quad \forall u \in Q, \|u\|_D = r_4. \tag{86}$$

On the other hand, for $u \in Q$, $\|u\|_D = r$, (26) and (27) imply

$$\|Au\|_D \leq \frac{\beta}{\beta - 1}\big(a_r^* + G_r b^* + N_r \gamma^*\big). \tag{87}$$

Thus, from (80) and (87), we get

$$\|Au\|_D < \|u\|_D, \quad \forall u \in Q, \|u\|_D = r. \tag{88}$$

By (82) and (85) we know $0 < r_4 < r < r_2$, and, by Lemma 2, operator A is completely continuous from $Q_{r_4 r_2}$ into Q. Hence, (83), (86), (88) and Lemma 4 imply that A has two fixed points $u^*, u^{**} \in Q_{r_4 r_2}$ such that $r_4 < \|u^*\|_D < r < \|u^{**}\|_D \leq r_2$. The proof is complete. □

Example Consider the infinite boundary value problem for second order impulsive singular integro-differential equation of mixed type on the half line:

$$\begin{cases} u''(t) = \frac{e^{-2t}}{45t^{\frac{1}{3}}}\big(\frac{1}{[u(t)]^{\frac{1}{3}}} + \frac{1}{u'(t)} + [u'(t)]^2\big) + \frac{e^{-3t}}{40t^{\frac{1}{3}}}\{(\int_0^t e^{-(t+2)s} u(s)\,ds)^2 \\ \qquad + (\int_0^\infty \frac{u(s)\,ds}{(1+t+s)^3})^3\}, \quad \forall 0 < t < \infty, t \neq k\ (k = 1, 2, 3, \ldots), \\ \Delta u|_{t=k} = 3^{-k-4} \frac{1}{u'(k^-) + \sqrt{u'(k^-)}} \quad (k = 1, 2, 3, \ldots), \\ \Delta u'|_{t=k} = k^{-1} 3^{-k-4} \frac{1}{\sqrt{u'(k^-)}} \quad (k = 1, 2, 3, \ldots), \\ u(0) = 0, \qquad u'(\infty) = 2u'(0). \end{cases} \tag{89}$$

Conclusion IBVP (89) has at least two positive solutions $u^*, u^{**} \in PC^1[R_+, R] \cap C^2[R'_{++}, R]$ such that

$$0 < \inf_{0<t<\infty} \frac{u^*(t)}{t} \leq \sup_{0<t<\infty} \frac{u^*(t)}{t} < 1,$$

$$0 < \inf_{0\leq t<\infty} \big(u^*\big)'(t) \leq \sup_{0\leq t<\infty} \big(u^*\big)'(t) < 1,$$

$$\frac{1}{4} < \inf_{0<t<\infty} \frac{u^{**}(t)}{t} \leq \sup_{0<t<\infty} \frac{u^{**}(t)}{t} < \infty$$

and

$$\frac{1}{4} < \inf_{0\leq t<\infty} \big(u^{**}\big)'(t) \leq \sup_{0\leq t<\infty} \big(u^{**}\big)'(t) < \infty.$$

Proof System (89) is an IBVP of form (1). In this situation, $t_k = k$ $(k = 1, 2, 3, \ldots)$, $K(t,s) = e^{-(t+2)s}$, $H(t,s) = (1 + t + s)^{-3}$, $\beta = 2$, and

$$f(t,u,v,w,z) = \frac{e^{-2t}}{45t^{\frac{1}{3}}}\left(\frac{1}{u^{\frac{1}{3}}} + \frac{1}{v} + v^2\right) + \frac{e^{-3t}}{40t^{\frac{1}{3}}}\big(w^2 + z^3\big), \quad \forall t, u, v \in R_{++}, w, z \in R_+,$$

$$I_k(v) = 3^{-k-4}\frac{1}{v+\sqrt{v}}, \quad \forall v \in R_{++}\ (k = 1,2,3,\ldots),$$
$$\bar{I}_k(v) = k^{-1}3^{-k-4}\frac{1}{\sqrt{v}}, \quad \forall v \in R_{++}\ (k = 1,2,3,\ldots).$$

It is clear that (3)-(7) are satisfied, so, (89) is a singular problem. It is easy to see that condition (H_1) is satisfied and $k^* \le 1$, $h^* \le 1$. We have

$$f(t,u,v,w,z) \le \frac{e^{-2t}}{45t^{\frac{1}{3}}}\frac{1}{u^{\frac{1}{3}}} + \frac{e^{-2t}}{t^{\frac{1}{3}}}\left\{\frac{1}{45}\left(\frac{1}{v}+v^2\right) + \frac{1}{40}\left(w^2+z^3\right)\right\},$$

so, condition (H_2) is satisfied for

$$a(t) = \frac{e^{-2t}}{45t^{\frac{1}{3}}}, \qquad g(u) = \frac{1}{u^{\frac{1}{3}}}, \qquad b(t) = \frac{e^{-2t}}{t^{\frac{1}{3}}}$$

and

$$g(v,w,z) = \frac{1}{45}\left(\frac{1}{v}+v^2\right) + \frac{1}{40}\left(w^2+z^3\right)$$

with

$$\begin{aligned} g_r(t) &= \max\left\{u^{-\frac{1}{3}} : \frac{rt}{4} \le u \le rt\right\} = \left(\frac{4}{r}\right)^{\frac{1}{3}} t^{-\frac{1}{3}}, \\ a_r^* &= \int_0^\infty a(t)g_r(t)\,dt = \frac{1}{45}\left(\frac{4}{r}\right)^{\frac{1}{3}}\int_0^\infty \frac{e^{-2t}}{t^{\frac{2}{3}}}\,dt < \infty \end{aligned} \tag{90}$$

and

$$b^* = \int_0^\infty \frac{e^{-2t}}{t^{\frac{1}{3}}}\,dt < \infty. \tag{91}$$

It is obvious that condition (H_3) is satisfied for $\gamma_k = k^{-1}3^{-k-4}$ ($\gamma^* = \frac{1}{162}$) and $F(v) = v^{-\frac{1}{2}}$. From

$$f(t,u,v,w,z) \ge \frac{e^{-2t}}{45t^{\frac{1}{3}}}v^2, \quad \forall t,u,v \in R_{++}, w,z \in R_+$$

and

$$f(t,u,v,w,z) \ge \frac{e^{-2t}}{45t^{\frac{1}{3}}}\frac{1}{v}, \quad \forall t,u,v \in R_{++}, w,z \in R_+,$$

we see that conditions (H_4) and (H_5) are satisfied for

$$c(t) = \frac{e^{-2t}}{t^{\frac{1}{3}}} \quad \left(c^* = b^*, \text{see (91)}\right)$$

and

$$d(t) = \frac{e^{-2t}}{t^{\frac{1}{3}}} \quad \left(d^* = b^*\right),$$

respectively. Finally, we check that inequality (80) is satisfied for $r = 1$, *i.e.*

$$2\left(a_1^* + G_1 b^* + N_1 \gamma^*\right) < 1. \tag{92}$$

By (90) and (91), we have

$$\begin{aligned} a_1^* &= \frac{4^{\frac{1}{3}}}{45} \int_0^\infty \frac{e^{-2t}}{t^{\frac{2}{3}}}\, dt < \frac{4^{\frac{1}{3}}}{45} \left(\int_0^1 \frac{dt}{t^{\frac{2}{3}}} + \int_1^\infty e^{-2t}\, dt \right) \\ &= \frac{4^{\frac{1}{3}}}{45} \left(3 + \frac{1}{2} e^{-2} \right) < \frac{1}{45} \left(\frac{8}{5} \right) \left(3 + \frac{1}{14} \right) = \frac{172}{1{,}575} \end{aligned}$$

and

$$b^* < \int_0^1 \frac{dt}{t^{\frac{1}{3}}} + \int_1^\infty e^{-2t}\, dt = \frac{3}{2} + \frac{1}{2} e^{-2} < \frac{11}{7}.$$

Moreover, it is easy to get

$$G_1 < \frac{29}{180}, \qquad N_1 = 2.$$

Hence

$$2\left(a_1^* + G_1 b^* + N_1 \gamma^*\right) < 2\left(\frac{172}{1{,}575} + \frac{319}{1{,}260} + \frac{1}{81} \right) = \frac{21{,}247}{28{,}350} < 1.$$

Consequently, (92) holds, and our conclusion follows from the theorem. □

Competing interests
The author declares that they have no competing interests.

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