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New method for the existence and uniqueness of solution of nonlinear parabolic equation

Li Wei[1], Ravi P Agarwal[2,3] and Patricia JY Wong[4*]

*Correspondence: ejywong@ntu.edu.sg
[4]School of Electrical and Electronic Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore, 639798, Singapore
Full list of author information is available at the end of the article

Abstract
There are two contributions in this paper. The first is that the abstract result for the existence of the unique solution of certain nonlinear parabolic equation is obtained by using the properties of *H-monotone operators*, consequently, the proof is simplified compared to the corresponding discussions in the literature. The second is that the connections between resolvent of *H*-monotone operators and solutions of nonlinear parabolic equations are shown, and this strengthens the importance of *H*-monotone operators, which have already attracted the attention of mathematicians because of the connections with practical problems.

MSC: 47H05; 47H09

Keywords: *H*-monotone operator; resolvent; subdifferential; parabolic equation

1 Introduction and preliminaries

1.1 Introduction

Nonlinear boundary value problems involving the generalized p-Laplacian operator arise from many physical phenomena, such as reaction-diffusion problems, petroleum extraction, flow through porous media and non-Newtonian fluids, just to name a few. Thus, the study of such problems and their generalizations have attracted numerous attention in recent years. In particular, we would mention the books of Lieberman [1, 2] where in [1] the theory of linear and quasilinear parabolic second-order partial differential equations is elaborated, with emphasis on the Cauchy-Dirichlet problem and the oblique derivative problem in bounded space-time domains; while in [2] a detailed qualitative analysis of second-order elliptic boundary value problems that involve oblique derivatives is presented. A sample of other research work that contributes to the literature of parabolic and elliptic problems includes [3–14] listed chronologically as well as the references cited therein. For time-periodic case which is the concern of this paper, we refer the reader to [15–17].

In 2008, Wei and Agarwal [18] studied the following nonlinear elliptic boundary value problem involving the generalized p-Laplacian:

$$\begin{cases} -\operatorname{div}[(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u]+\varepsilon|u|^{q-2}u+g(x,u(x))=f(x), & \text{a.e. in } \Omega, \\ -\langle \vartheta,(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle \in \beta_x(u(x)), & \text{a.e. on } \Gamma, \end{cases} \tag{1.1}$$

where $0 \le C(x) \in L^p(\Omega)$, ε is a non-negative constant and ϑ denotes the exterior normal derivative of Γ. It is shown that (1.1) has solutions in $L^s(\Omega)$ under some conditions, where $\frac{2N}{N+1} < p \le s < +\infty$, $1 \le q < +\infty$ if $p \ge N$, and $1 \le q \le \frac{Np}{N-p}$ if $p < N$, for $N \ge 1$. We observe that the proof, which uses Theorem 1.1 (stated in Section 1.2) as the main tool, is very complicated, since one needs to check that conditions (1.10) and (1.11) and the compactness of $A + C_1$ are satisfied.

In 2010, Wei *et al.* [19] extended the work on elliptic equation to the following nonlinear parabolic equation involving the generalized p-Laplacian with mixed boundary conditions:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}[(C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u] + \varepsilon |u|^{p-2} u = f(x,t), & (x,t) \in \Omega \times (0,T), \\ -\langle \vartheta, (C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \rangle \in \beta(u) - h(x,t), & (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u(x,T), & \text{a.e. } x \in \Omega. \end{cases} \tag{1.2}$$

Some new technique has been used to tackle the existence of solutions of (1.2); specifically, the problem is divided into the following two auxiliary equations: (i) a parabolic equation with Dirichlet boundary conditions (1.3), and (ii) a parabolic equation with Neumann boundary value conditions (1.4):

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}[(C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u] + \varepsilon |u|^{p-2} u = f(x,t), & (x,t) \in \Omega \times (0,T), \\ \gamma u = w, & (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u(x,T), & \text{a.e. } x \in \Omega, \end{cases} \tag{1.3}$$

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}[(C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u] + \varepsilon |u|^{p-2} u = f(x,t), & (x,t) \in \Omega \times (0,T), \\ -\langle \vartheta, (C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \rangle \in \beta(u) - h(x,t), & (x,t) \in \Gamma \times (0,T). \end{cases} \tag{1.4}$$

By using Theorems 1.2 and 1.3 (stated in Section 1.2), it is shown that (1.3) has a unique solution. By employing Theorem 1.4, it is proved that (1.4) has a unique solution in $L^p(0,T;W^{1,p}(\Omega))$, which implies that (1.2) has a unique solution in $L^p(0,T;W^{1,p}(\Omega))$, where $2 \le p < +\infty$. However, we observe that the inequality (1.12) is not easy to check during the discussion.

Motivated by the work of Kawohl *et al.* [5, 9, 10, 13], Serrin *et al.* [3, 4, 12, 14] as well as Wei *et al.* [18, 19], in this paper we shall consider the following parabolic problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}[\alpha(|\nabla u|^p)|\nabla u|^{p-2} \nabla u] + \lambda_1 |u|^{r_1-2} u \\ \quad + \lambda_2 |u|^{r_2-2} u + g(x,u,\frac{\partial u}{\partial t}, \varepsilon \nabla u) = f(x,t), & (x,t) \in \Omega \times (0,T), \\ -\langle \vartheta, \alpha(|\nabla u|^p)|\nabla u|^{p-2} \nabla u \rangle \in \beta_x(u(x,t)), & (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u(x,T), & x \in \Omega, \end{cases} \tag{1.5}$$

where $\alpha : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ is a continuous nonlinear mapping such that $pt\alpha'(t) + (p-1)\alpha(t) > 0$, $\alpha(t) \le k_1$, for $t \ge 0$, $\lim_{t \to +\infty} \alpha(t) = k_2 > 0$, here k_1 and k_2 are positive constants.

Let $\varphi : \Gamma \times \mathbb{R} \to \mathbb{R}$ be a given function such that, for each $x \in \Gamma$, $\varphi_x = \varphi(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is a proper, convex and lower-semicontinuous function with $\varphi_x(0) = 0$. Let β_x be the subdifferential of φ_x, *i.e.*, $\beta_x \equiv \partial\varphi_x$. Suppose that $0 \in \beta_x(0)$ and for each $t \in \mathbb{R}$, the function $x \in \Gamma \to (I + \lambda\beta_x)^{-1}(t) \in \mathbb{R}$ is measurable for $\lambda > 0$. More details of (1.5) will be presented in Section 2.

There are some major differences between parabolic problems (1.2) and (1.5): (i) The main part $-\operatorname{div}[\alpha(|\nabla u|^p)|\nabla u|^{p-2}\nabla u]$ in (1.5) includes the main part $-\operatorname{div}[(C(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u]$ in (1.2); (ii) the term $g(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u)$ is considered in (1.5) but not in (1.2); (iii) $\beta_x(u(x,t))$ in (1.5) is different from $\beta(u)-h(x,t)$ in (1.2).

The existence of the unique solution of (1.5) will be discussed in $L^2(0,T;L^2(\Omega))$, which does not change while p is varying from $\frac{2N}{N+1}$ to $+\infty$ for $N\geq 1$. Hence, the result is different from that on (1.2) in [19]. Our main tool in this paper will be Theorem 1.5 (stated in Section 1.2). Consequently, the proof of our result is different from and comparatively simplified with respect to that of [19].

Actually, (1.5) is very general and it includes the following special cases. The related work can be found in [19–21] and the references cited therein.

Example 1.1 If we set $\alpha(t)=1+t(1+t^2)^{-\frac{1}{2}}$, $t\geq 0$, then it is obvious that $\alpha:\mathbb{R}^+\cup\{0\}\to\mathbb{R}^+$ is a continuous nonlinear mapping, $\alpha(t)\leq 2$ and $\lim_{t\to+\infty}\alpha(t)=2$. Moreover,

$$pt\alpha'(t)+(p-1)\alpha(t)=\frac{pt}{(1+t^2)^{\frac{3}{2}}}+(p-1)+(p-1)\frac{t}{\sqrt{1+t^2}}>0.$$

So, if $\lambda_1\equiv\lambda_2\equiv\lambda$, then (1.5) becomes the following parabolic capillarity equation:

$$\begin{cases}\frac{\partial u}{\partial t}-\operatorname{div}[(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}})|\nabla u|^{p-2}\nabla u]+\lambda|u|^{r_1-2}u+\lambda|u|^{r_2-2}u\\ \qquad +g(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u)=f(x,t),\quad (x,t)\in\Omega\times(0,T),\\ -\langle\vartheta,(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}})|\nabla u|^{p-2}\nabla u\rangle\in\beta_x(u(x,t)),\quad (x,t)\in\Gamma\times(0,T),\\ u(x,0)=u(x,T),\quad x\in\Omega.\end{cases}\tag{1.6}$$

Example 1.2 For $1<p\leq 2$, if we set $\alpha(t)=(C+t^{\frac{2}{p}})^{\frac{p-2}{2}}t^{\frac{2-p}{p}}$, $t>0$, where $C\geq 0$, then it is obvious that $\alpha:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous nonlinear mapping, $\alpha(t)\leq 1$ and $\lim_{t\to+\infty}\alpha(t)=1$. Moreover,

$$\begin{aligned}pt\alpha'(t)+(p-1)\alpha(t)&=\bigl(C+t^{\frac{2}{p}}\bigr)^{\frac{p}{2}-2}t^{\frac{2}{p}-2}\bigl[tC(2-p)+(p-1)t\bigl(C+t^{\frac{2}{p}}\bigr)\bigr]\\ &=\bigl(C+t^{\frac{2}{p}}\bigr)^{\frac{p}{2}-2}t^{\frac{2}{p}-2}\bigl[Ct+(p-1)t^{\frac{2}{p}+1}\bigr]>0.\end{aligned}$$

If $\lambda_2\equiv 0$, then (1.5) becomes the following parabolic equation with generalized p-Laplacian:

$$\begin{cases}\frac{\partial u}{\partial t}-\operatorname{div}[(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u]+\lambda_1|u|^{r_1-2}u+g(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u)\\ \qquad =f(x,t),\quad (x,t)\in\Omega\times(0,T),\\ -\langle\vartheta,(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle\in\beta_x(u(x,t)),\quad (x,t)\in\Gamma\times(0,T),\\ u(x,0)=u(x,T),\quad x\in\Omega.\end{cases}\tag{1.7}$$

Example 1.3 If, in (1.7), $C(x)\equiv 0$, then (1.7) becomes the following parabolic p-Laplacian equation:

$$\begin{cases}\frac{\partial u}{\partial t}-\triangle_p u+\lambda_1|u|^{r_1-2}u+g(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u)=f(x,t),\quad (x,t)\in\Omega\times(0,T),\\ -\langle\vartheta,|\nabla u|^{p-2}\nabla u\rangle\in\beta_x(u(x,t)),\quad (x,t)\in\Gamma\times(0,T),\\ u(x,0)=u(x,T),\quad x\in\Omega.\end{cases}\tag{1.8}$$

Example 1.4 For $s \leq 0$, if we set $\alpha(t) = (1 + t^{\frac{2}{p}})^{\frac{s}{2}} t^{\frac{m-p+1}{p}}$, $t > 0$, where $m \geq 0$, $m + s + 1 = p$, then it is obvious that $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nonlinear mapping, $\alpha(t) \leq 1$ and $\lim_{t\to+\infty} \alpha(t) = 1$. Moreover,

$$pt\alpha'(t) + (p-1)\alpha(t) = t^{\frac{m-p+1}{p}} \left(1 + t^{\frac{2}{p}}\right)^{\frac{s}{2}-1} \left[m + (p-1)t^{\frac{2}{p}}\right] > 0.$$

So, if $\lambda_2 \equiv 0$, then (1.5) becomes the following parabolic curvature equation:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}[(1 + |\nabla u|^2)^{\frac{s}{2}} |\nabla u|^{m-1} \nabla u] + \lambda_1 |u|^{r_1 - 1} u + g(x, u, \frac{\partial u}{\partial t}, \varepsilon \nabla u) \\ \qquad = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\ -\langle \vartheta, (1 + |\nabla u|^2)^{\frac{s}{2}} |\nabla u|^{m-1} \nabla u\rangle \in \beta_x(u(x,t)), \quad (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u(x,T), \quad x \in \Omega. \end{cases} \tag{1.9}$$

1.2 Preliminaries

Let X be a real Banach space with a strictly convex dual space X^*. We shall use $(\cdot,\cdot)$ to denote the generalized duality pairing between X and X^*. We shall use "$\to$" and "w-lim" to denote strong and weak convergence, respectively. Let "$X \hookrightarrow Y$" denote the space X embedded continuously in space Y. For any subset G of X, we denote by $\operatorname{int} G$ its interior and $\overline{G}$ its closure, respectively. For two subsets G_1 and G_2 in X, if $\overline{G_1} = \overline{G_2}$ and $\operatorname{int} G_1 = \operatorname{int} G_2$, then we say G_1 is *almost equal* to G_2, which is denoted by $G_1 \simeq G_2$. A mapping $T : X \to X^*$ is said to be *hemi-continuous* on X [22] if $w\text{-}\lim_{t\to 0} T(x + ty) = Tx$ for any $x, y \in X$.

A function Φ is called a *proper convex function* on X [22] if Φ is defined from X to $(-\infty, +\infty]$, not identically $+\infty$, such that $\Phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\Phi(x) + \lambda\Phi(y)$, whenever $x, y \in X$ and $0 \leq \lambda \leq 1$.

A function $\Phi : X \to (-\infty, +\infty]$ is said to be *lower-semicontinuous* on X [22] if $\liminf_{y\to x} \Phi(y) \geq \Phi(x)$, for any $x \in X$.

Given a proper convex function Φ on X and a point $x \in X$, we denote by $\partial\Phi(x)$ the set of all $x^* \in X^*$ such that $\Phi(x) \leq \Phi(y) + (x - y, x^*)$, for every $y \in X$. Such element x^* is called the *subgradient* of Φ at x, and $\partial\Phi(x)$ is called the *subdifferential* of Φ at x [22].

Let J_r denote the *duality mapping* from X into 2^{X^*}, which is defined by

$$J_r(x) = \left\{f \in X^* : (x,f) = \|x\|^r, \|f\| = \|x\|^{r-1}\right\}, \quad \forall x \in X,$$

where $r > 1$ is a constant. We use J to denote the usual *normalized duality mapping*. It is known that, in general, $J_r(x) = \|x\|^{r-2} J(x)$, for all $x \neq 0$. Since X^* is strictly convex, J is a single-valued mapping [23].

A multi-valued mapping $A : X \to 2^X$ is said to be *accretive* [22, 23] if $(v_1 - v_2, J_r(u_1 - u_2)) \geq 0$, for any $u_i \in D(A)$ and $v_i \in Au_i$, $i = 1, 2$. The accretive mapping A is said to be *m-accretive* if $R(I + \lambda A) = X$ for some $\lambda > 0$.

A multi-valued operator $B : X \to 2^{X^*}$ is said to be *monotone* [24] if its graph $G(B)$ is a monotone subset of $X \times X^*$ in the sense that $(u_1 - u_2, w_1 - w_2) \geq 0$, for any $[u_i, w_i] \in G(B)$, $i = 1, 2$. Further, B is called *strictly monotone* if $(u_1 - u_2, w_1 - w_2) \geq 0$ and the equality holds if and only if $u_1 = u_2$. The monotone operator B is said to be *maximal monotone* if $G(B)$ is maximal among all monotone subsets of $X \times X^*$ in the sense of inclusion. Also, B is maximal monotone if and only if $R(B + \lambda J) = X^*$, for any $\lambda > 0$. The mapping B is said to be *coercive* [24] if $\lim_{n\to+\infty}(x_n, x_n^*)/\|x_n\| = +\infty$ for all $[x_n, x_n^*] \in G(B)$ such that $\lim_{n\to+\infty}\|x_n\| = +\infty$.

Let $B : X \to 2^{X^*}$ be a maximal monotone operator such that $[0, 0] \in G(B)$, then the equation $J(u_t - u) + tBu_t \ni 0$ has a unique solution $u_t \in D(B)$ for every $u \in X$ and $t > 0$. The *resolvent* J_t^B and the *Yosida approximation* B_t of B are defined by [24]

$$J_t^B u = u_t,$$
$$B_t u = -\frac{1}{t}J(u_t - u),$$

for every $u \in X$ and $t > 0$. (Hence, $[J_t^B u, B_t u] \in G(B)$.)

Definition 1.1 ([24]) Let C be a closed convex subset of X and let $A : C \to 2^{X^*}$ be a multi-valued mapping. Then A is said to be a *pseudo-monotone operator* provided that

(i) for each $x \in C$, the image Ax is a non-empty closed and convex subset of X^*;
(ii) if $\{x_n\}$ is a sequence in C converging weakly to $x \in C$ and if $f_n \in Ax_n$ is such that $\limsup_{n\to\infty}(x_n - x, f_n) \le 0$, then to each element $y \in C$, there corresponds an $f(y) \in Ax$ with the property that $(x - y, f(y)) \le \liminf_{n\to\infty}(x_n - x, f_n)$;
(iii) for each finite-dimensional subspace K of X, the operator A is continuous from $C \cap K$ to X^* in the weak topology.

Definition 1.2 ([25, 26]) Let $\mathcal{H}$ be a Hilbert space. Let $H : \mathcal{H} \to \mathcal{H}$ be a single-valued mapping and $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued mapping. We say that A is *H-monotone* if A is monotone and $R(H + \lambda A)(\mathcal{H}) = \mathcal{H}$, for every $\lambda > 0$.

Lemma 1.1 ([24]) *If $A : X \to 2^{X^*}$ is a everywhere defined, monotone, and hemi-continuous mapping, then A is maximal monotone. If, moreover, A is coercive, then $R(A) = X^*$.*

Lemma 1.2 ([24]) *If $\Phi : X \to (-\infty, +\infty]$ is a proper convex and lower-semicontinuous function, then $\partial\Phi$ is maximal monotone from X to X^*.*

Lemma 1.3 ([24]) *If A_1 and A_2 are two maximal monotone operators in X such that $(\operatorname{int} D(A_1)) \cap D(A_2) \neq \emptyset$, then $A_1 + A_2$ is maximal monotone.*

Theorem 1.1 ([27]) *Let X be a real Banach space with a strictly convex dual space X^*. Let $J : X \to X^*$ be a duality mapping on X and there exists a function $\eta : X \to [0, +\infty)$ such that for all $u, v \in X$,*

$$\|Ju - Jv\| \le \eta(u - v). \tag{1.10}$$

Let $A, C_1 : X \to 2^X$ be accretive mappings such that

(i) *either both A and C_1 satisfy condition* (1.11), *or $D(A) \subset D(C_1)$ and C_1 satisfies condition* (1.11):

$$\begin{cases} \text{for } u \in D(A) \text{ and } v \in Au, \text{there exists a constant } C(a,f) \text{ such that} \\ \qquad\qquad (v - f, J(u - a)) \ge C(a,f). \end{cases} \tag{1.11}$$

(ii) *$A + C_1$ is m-accretive and boundedly-inversely-compact.*

Let $C_2 : X \to X$ be a bounded continuous mapping such that, for any $y \in X$, there is a constant $C(y)$ satisfying $(C_2(u+y), Ju) \geq -C(y)$ for any $u \in X$. Then the following results hold:

(a) $\overline{[R(A) + R(C_1)]} \subset \overline{R(A + C_1 + C_2)}$;

(b) $\operatorname{int}[R(A) + R(C_1)] \subset \operatorname{int} R(A + C_1 + C_2)$.

Theorem 1.2 ([28]) *Let $T : X \to X^*$ be a bounded and pseudo-monotone operator, K be a closed and convex subset of X. Suppose that Φ is a lower-semicontinuous and convex function defined on K which is not always $+\infty$ such that $\Phi(v) \in (-\infty, +\infty]$ for all $v \in K$. Suppose there exists $v_0 \in K$ such that $\Phi(v_0) < +\infty$ and*

$$\frac{(v - v_0, Tv) + \Phi(v)}{\|v\|} \to \infty, \quad \text{as } \|v\| \to \infty, v \in K.$$

Then there exists $u \in K$ such that

$$(u - v, Tu) \leq \Phi(v) - \Phi(u), \quad \forall v \in K.$$

Theorem 1.3 ([29]) *Let X be a real reflexive Banach space with both X and its dual X^* being convex spaces. Let $S : D(S) \subset X \to X^*$ be a linear maximal monotone operator and $T : X \to X^*$ be a pseudo-monotone and coercive operator. Then, for each $f \in X^*$, there exists an $u \in D(S)$ such that, in the weak sense, $Su + Tu = f$.*

Theorem 1.4 ([30]) *Let X be a real reflexive Banach space with both X and its dual X^* being strictly convex. Let J be the normalized duality mapping. Let A and B be two maximal monotone operators in X. Suppose there exist $0 \leq k < 1$ and $C_1, C_2 > 0$ such that*

$$\big(a, J^{-1}(B_t v)\big) \geq -k\|B_t v\|^2 - C_1\|B_t v\| - C_2 \tag{1.12}$$

for $v \in D(A)$, $a \in Av$ and $t > 0$, where B_t is the Yosida approximation of B. Then $R(A) + R(B) \simeq R(A + B)$.

Theorem 1.5 ([25]) *Let $A : \mathcal{H} \to \mathcal{H}$ be a maximal monotone operator and $H : \mathcal{H} \to \mathcal{H}$ be a bounded, coercive, hemi-contiunuous, and monotone mapping. Then A is H-monotone.*

2 Main results

In this paper, unless otherwise stated, we shall assume that

$$N \geq 1, \qquad \frac{2N}{N+1} < p < +\infty, \qquad 1 \leq r_i \leq \min\{p, p'\}, \quad i = 1, 2,$$
$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \frac{1}{r_1} + \frac{1}{r_1'} = 1, \qquad \frac{1}{r_2} + \frac{1}{r_2'} = 1.$$

In (1.5), Ω is a bounded conical domain of a Euclidean space $\mathbb{R}^N$ with its boundary $\Gamma \in C^1$ [6], T is a positive constant, λ_1, λ_2 and ε are non-negative constants, and ϑ denotes the exterior normal derivative of Γ. We shall assume that Green's formula is available.

Suppose that $g : \Omega \times \mathbb{R}^{N+2} \to \mathbb{R}$ is a given function satisfying the following conditions:

(a) *Carathéodory's conditions*

$$x \to g(x,r) \text{ is measurable on } \Omega, \quad \text{for all } r \in \mathbb{R}^{N+2};$$
$$r \to g(x,r) \text{ is continuous on } \mathbb{R}^{N+2}, \quad \text{for almost all } x \in \Omega.$$

(b) *Growth condition*

$$g(x, s_1, \ldots, s_{N+2}) \le h(x) + k_3 |s_1|^{\min\{p/p',1\}},$$

where $(s_1, s_2, \ldots, s_{N+2}) \in \mathbb{R}^{N+2}$, $h(x) \in L^2(\Omega) \cap L^{p'}(\Omega)$ and k_3 is a positive constant.

(c) *Monotone condition* g is monotone with respect to r_1, *i.e.*,

$$\bigl(g(x, s_1, \ldots, s_{N+2}) - g(x, t_1, \ldots, t_{N+2})\bigr)(s_1 - t_1) \ge 0$$

for all $x \in \Omega$ and $(s_1, \ldots, s_{N+2}), (t_1, \ldots, t_{N+2}) \in \mathbb{R}^{N+2}$.

(d) *Coercive condition*

$$g(x, s_1, \ldots, s_{N+2}) s_1 \ge k_4 s_1^2,$$

where k_4 is a fixed positive constant.

Now, we present our discussion in the sequel.

Lemma 2.1 ([18]) *Let X_0 denote the closed subspace of all constant functions in $W^{1,p}(\Omega)$. Let X be the quotient space $W^{1,p}(\Omega)/X_0$. For $u \in W^{1,p}(\Omega)$, define the mapping $P : W^{1,p}(\Omega) \to X_0$ by*

$$Pu = \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega u \, dx.$$

Then there is a constant $k_5 > 0$ such that for all $u \in W^{1,p}(\Omega)$,

$$\|u - Pu\|_p \le k_5 \|\nabla u\|_{(L^p(\Omega))^N}.$$

Lemma 2.2 *Define the mapping $B : L^p(0,T; W^{1,p}(\Omega)) \to L^{p'}(0,T; (W^{1,p}(\Omega))^*)$ by*

$$\begin{aligned}(w, Bu) = {}& \int_0^T \int_\Omega \biggl\langle \int_\Omega \alpha\bigl(|\nabla u|^p\bigr) |\nabla u|^{p-2} \nabla u, \nabla w \biggr\rangle \, dx \, dt + \lambda_1 \int_0^T \int_\Omega |u|^{r_1 - 2} u w \, dx \, dt \\ &+ \lambda_2 \int_0^T \int_\Omega |u|^{r_2 - 2} u w \, dx \, dt\end{aligned}$$

for any $u, w \in L^p(0,T; W^{1,p}(\Omega))$. Then B is strictly monotone, pseudo-monotone, and coercive.

(Here, $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the Euclidean inner-product and Euclidean norm in $\mathbb{R}^N$, respectively.)

Proof *Step* 1. B is everywhere defined.

For $u, w \in L^p(0,T;W^{1,p}(\Omega))$, we find

$$\begin{aligned}\left|(w,Bu)\right| &\le \int_0^T \int_\Omega k_1|\nabla u|^{p-1}|\nabla w|\,dx\,dt + \lambda_1 \int_0^T \int_\Omega |u|^{r_1-1}|w|\,dx\,dt \\ &\quad + \lambda_2 \int_0^T \int_\Omega |u|^{r_2-1}|w|\,dx\,dt \\ &\le k_1\|u\|^{p/p'}_{L^p(0,T;W^{1,p}(\Omega))}\|w\|_{L^p(0,T;W^{1,p}(\Omega))} + \lambda_1\|w\|_{L^{r_1}(0,T;L^{r_1}(\Omega))}\|u\|^{r_1/r_1'}_{L^{r_1}(0,T;L^{r_1}(\Omega))} \\ &\quad + \lambda_2\|w\|_{L^{r_2}(0,T;L^{r_2}(\Omega))}\|u\|^{r_2/r_2'}_{L^{r_2}(0,T;L^{r_2}(\Omega))}.\end{aligned}$$

Since $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^{r_1}(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^{r_2}(\Omega)$, for $v \in W^{1,p}(\Omega)$, we have $\|v\|_{L^{r_1}(\Omega)} \le k_6\|v\|_{W^{1,p}(\Omega)}$, $\|v\|_{L^{r_2}(\Omega)} \le k_7\|v\|_{W^{1,p}(\Omega)}$, where k_6 and k_7 are positive constants. Hence,

$$\begin{aligned}\left|(w,Bu)\right| &\le k_1\|u\|^{p/p'}_{L^p(0,T;W^{1,p}(\Omega))}\|w\|_{L^p(0,T;W^{1,p}(\Omega))} \\ &\quad + \lambda_1 k_6\|u\|^{r_1/r_1'}_{L^p(0,T;W^{1,p}(\Omega))}\|w\|_{L^p(0,T;W^{1,p}(\Omega))} \\ &\quad + \lambda_2 k_7\|u\|^{r_2/r_2'}_{L^p(0,T;W^{1,p}(\Omega))}\|w\|_{L^p(0,T;W^{1,p}(\Omega))},\end{aligned}$$

which implies that B is everywhere defined.

Step 2. B is strictly monotone.

For $u, v \in L^p(0,T;W^{1,p}(\Omega))$, we have

$$\begin{aligned}(u-v,Bu-Bv) &\ge \int_0^T \int_\Omega \bigl[\alpha\bigl(|\nabla u|^p\bigr)|\nabla u|^{p-1} - \alpha\bigl(|\nabla v|^p\bigr)|\nabla v|^{p-1}\bigr]\bigl(|\nabla u| - |\nabla v|\bigr)\,dx\,dt \\ &\quad + \lambda_1 \int_0^T \int_\Omega \bigl(|u|^{r_1-1} - |v|^{r_1-1}\bigr)\bigl(|u| - |v|\bigr)\,dx\,dt \\ &\quad + \lambda_2 \int_0^T \int_\Omega \bigl(|u|^{r_2-1} - |v|^{r_2-1}\bigr)\bigl(|u| - |v|\bigr)\,dx\,dt.\end{aligned}$$

If we set $f(s) = s^{1-\frac{1}{p}}\alpha(s)$, $s > 0$, then in view of the assumption of α, we have

$$f'(s) = \left[\left(1 - \frac{1}{p}\right)\alpha(s) + s\alpha'(s)\right]s^{-\frac{1}{p}} > 0,$$

which implies that f is strictly monotone. Hence, B is strictly monotone.

Step 3. B is hemi-continuous.

It suffices to show that for any $u, v, w \in L^p(0,T;W^{1,p}(\Omega))$ and $t \in [0,1]$, $(w, B(u+tv) - Bu) \to 0$ as $t \to 0$. Since

$$\begin{aligned}&\bigl|\bigl(w, B(u+tv) - Bu\bigr)\bigr| \\ &\quad\le \int_0^T \int_\Omega \bigl|\alpha\bigl(|\nabla u + t\nabla v|^p\bigr)|\nabla u + t\nabla v|^{p-2}(\nabla u + t\nabla v) \\ &\qquad - \alpha\bigl(|\nabla u|^p\bigr)|\nabla u|^{p-2}\nabla u\bigr||\nabla w|\,dx\,dt\end{aligned}$$

$$+ \lambda_1 \int_0^T \int_\Omega \big||u+tv|^{r_1-2}(u+tv) - |u|^{r_1-2}u\big||w|\,dx\,dt$$
$$+ \lambda_2 \int_0^T \int_\Omega \big||u+tv|^{r_2-2}(u+tv) - |u|^{r_2-2}u\big||w|\,dx\,dt,$$

by Lebesque's dominated convergence theorem and noting that α is continuous, we find

$$\lim_{t\to 0}\big(w, B(u+tv) - Bu\big) = 0.$$

Hence, B is hemi-continuous.

Step 4. B is coercive.

We shall first show that for $u \in L^p(0,T;W^{1,p}(\Omega))$,

$$\|u\|_{L^p(0,T;W^{1,p}(\Omega))} \le k_8\left(\int_0^T\int_\Omega |\nabla u|^p\,dx\,dt\right)^{\frac{1}{p}} + k_9, \tag{2.1}$$

where k_8 and k_9 are positive constants.

In fact, using Lemma 2.1, we know that, for $u \in L^p(0,T;W^{1,p}(\Omega))$,

$$\left\|u - \frac{1}{\operatorname{meas}(\Omega)}\int_\Omega u\,dx\right\|_{L^p(\Omega)} \le k_5\left(\int_\Omega |\nabla u|^p\,dx\right)^{\frac{1}{p}}.$$

Thus,

$$\begin{aligned}
&\left\|u - \frac{1}{\operatorname{meas}(\Omega)}\int_\Omega u\,dx\right\|^p_{W^{1,p}(\Omega)} \\
&\quad = \left\|u - \frac{1}{\operatorname{meas}(\Omega)}\int_\Omega u\,dx\right\|^p_{L^p(\Omega)} + \left\|\nabla\left(u - \frac{1}{\operatorname{meas}(\Omega)}\int_\Omega u\,dx\right)\right\|^p_{(L^p(\Omega))^N} \\
&\quad \le \left(k_5^p + 1\right)\int_\Omega |\nabla u|^p\,dx.
\end{aligned}$$

Since

$$\left\|u - \frac{1}{\operatorname{meas}(\Omega)}\int_\Omega u\,dx\right\|_{W^{1,p}(\Omega)} \ge \|u\|_{W^{1,p}(\Omega)} - \left\|\frac{1}{\operatorname{meas}(\Omega)}\int_\Omega u\,dx\right\|_{W^{1,p}(\Omega)},$$

we have

$$\|u\|_{W^{1,p}(\Omega)} \le \left\|u - \frac{1}{\operatorname{meas}(\Omega)}\int_\Omega u\,dx\right\|_{W^{1,p}(\Omega)} + \text{Const}.$$

Therefore,

$$\begin{aligned}
\|u\|_{L^p(0,T;W^{1,p}(\Omega))} &\le \left\|u - \frac{1}{\operatorname{meas}(\Omega)}\int_\Omega u\,dx\right\|_{L^p(0,T;W^{1,p}(\Omega))} + k_9 \\
&\le \left(k_5^p + 1\right)^{\frac{1}{p}}\left(\int_0^T\int_\Omega |\nabla u|^p\,dx\,dt\right)^{\frac{1}{p}} + k_9.
\end{aligned}$$

If we set $k_8 = (k_5^p + 1)^{\frac{1}{p}}$, then (2.1) is true.

Since $\lim_{t\to+\infty}\alpha(t) = k_2 > 0$, there exists sufficiently large $K > 0$ such that $\alpha(t) > \frac{l}{2}$ whenever $t > K$. Now, for $u \in L^p(0,T;W^{1,p}(\Omega))$, let $\|u\|_{L^p(0,T;W^{1,p}(\Omega))} \to +\infty$. Using (2.1), we find

$$\begin{aligned}
&\frac{(u,Bu)}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}}\\
&\quad= \frac{\int_0^T\int_\Omega \alpha(|\nabla u|^p)|\nabla u|^p\,dx\,dt}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} + \lambda_1\frac{\int_0^T\int_\Omega |u|^{r_1}\,dx\,dt}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} + \lambda_2\frac{\int_0^T\int_\Omega |u|^{r_2}\,dx\,dt}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}}\\
&\quad> \frac{1}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}}\left[\frac{l}{2}\int_0^T\int_\Omega|\nabla u|^p\,dx\,dt + \lambda_1\int_0^T\int_\Omega |u|^{r_1}\,dx\,dt\right.\\
&\qquad\left. + \lambda_2\int_0^T\int_\Omega |u|^{r_2}\,dx\,dt\right]\\
&\quad> \frac{1}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}}\frac{l}{2}\int_0^T\int_\Omega|\nabla u|^p\,dx\,dt \to +\infty.
\end{aligned}$$

This completes the proof. □

Lemma 2.3 *The mapping $\Phi : L^p(0,T;W^{1,p}(\Omega)) \to \mathbb{R}$ defined by*

$$\Phi(u) = \int_0^T\int_\Gamma \varphi_x\bigl(u|_\Gamma(x,t)\bigr)\,d\Gamma(x)\,dt,$$

for any $u \in L^p(0,T;W^{1,p}(\Omega))$, is proper, convex, and lower-semicontinuous on $L^p(0,T;W^{1,p}(\Omega))$. Moreover, the subdifferential $\partial\Phi$ of Φ is maximal monotone in view of Lemma 1.2.

Proof The proof is similar to that of Lemma 3.1 in [27]. □

Lemma 2.4 ([19]) *Define $S : D(S) \to L^{p'}(0,T;(W^{1,p}(\Omega))^*)$ by*

$$Su(x,t) = \frac{\partial u}{\partial t},$$

where

$$D(S) = \left\{u \in L^p\bigl(0,T;W^{1,p}(\Omega)\bigr)\,\middle|\,\frac{\partial u}{\partial t} \in L^{p'}\bigl(0,T;\bigl(W^{1,p}(\Omega)\bigr)^*\bigr), u(x,0) = u(x,T)\right\}.$$

The mapping S is linear maximal monotone.

Definition 2.1 Define a mapping $A : L^2(0,T;L^2(\Omega)) \to 2^{L^2(0,T;L^2(\Omega))}$ by

$$Au = \bigl\{w(x) \in L^2\bigl(0,T;L^2(\Omega)\bigr)|w(x) \in Bu + \partial\Phi(u) + Su\bigr\}$$

for $u \in D(A) = \{u \in L^2(0,T;L^2(\Omega))|$ there exists a $w(x) \in L^2(0,T;L^2(\Omega))$ such that $w(x) \in Bu + \partial\Phi(u) + Su\}$.

Lemma 2.5 *Define the mapping $F : L^p(0,T;W^{1,p}(\Omega)) \to L^{p'}(0,T;(W^{1,p}(\Omega))^*)$ by*

$$(v,Fu) = \int_0^T\int_\Omega g\left(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u\right)v(x,t)\,dx\,dt$$

for $u(x,t), v(x,t) \in L^p(0,T;W^{1,p}(\Omega))$. Then F is everywhere defined.

Proof *Step* 1. For $u(x,t) \in L^p(0,T;W^{1,p}(\Omega))$, $x \to g(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u)$ is measurable on Ω.

From the fact that $u(x,t), \frac{\partial u}{\partial x_i} \in L^p(\Omega)$, $i = 1,2,\ldots,N$, we see that $x \to (u, \frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_N})$ is measurable on Ω. Combining with the fact that g satisfies Carathéodory's conditions, we know that $x \to g(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u)$ is measurable on Ω.

Step 2. F is everywhere defined.

For $u,v \in L^p(0,T;W^{1,p}(\Omega))$, we have

$$\begin{aligned}\bigl|(v,Fu)\bigr| &\le \int_0^T \int_\Omega \bigl|h(x)\bigr|\bigl|v(x,t)\bigr|\,dx\,dt + k_3\int_0^T\int_\Omega \bigl|u(x,t)\bigr|^{p/p'}\bigl|v(x,t)\bigr|\,dx\,dt \\ &\le \bigl(T^{\frac{1}{p'}}\bigl\|h(x)\bigr\|_{L^{p'}(\Omega)} + k_3\|u\|^{p/p'}_{L^p(0,T;W^{1,p}(\Omega))}\bigr)\|v\|_{L^p(0,T;W^{1,p}(\Omega))},\end{aligned}$$

which implies that F is everywhere defined.

This completes the proof. □

Definition 2.2 Define the mapping $H : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ by

$$Hu(x) = \bigl\{v(x) \in L^2\bigl(0,T;L^2(\Omega)\bigr) | v(x) = Fu(x)\bigr\}$$

for $u \in D(H) = \{u(x) \in L^2(0,T;L^2(\Omega))|$ there exists $v(x) \in L^2(0,T;L^2(\Omega))$ such that $v(x) = Fu(x)\}$, where F is the same as in Lemma 2.5.

Lemma 2.6 *The mapping* $H : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ *defined in Definition* 2.2 *is bounded, coercive, hemi-continuous, and monotone.*

Proof *Step* 1. H is bounded.

From condition (b) of g, we know that

$$\begin{aligned}\|Hu\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T\int_\Omega \biggl|g\biggl(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u\biggr)\biggr|^2\,dx\,dt \\ &\le k_{10}\|u\|^2_{L^2(0,T;L^2(\Omega))} + k_{11}\bigl\|h(x)\bigr\|^2_{L^2(\Omega)},\end{aligned}$$

where k_{10} and k_{11} are positive constants. This implies that H is bounded.

Step 2. H is coercive.

From condition (d) of g, we know that

$$\begin{aligned}(u,Hu) &= \int_0^T\int_\Omega g\biggl(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u\biggr)u\,dx\,dt \ge k_4\int_0^T\int_\Omega |u|^2\,dx\,dt \\ &= k_4\|u\|^2_{L^2(0,T;L^2(\Omega))} \to +\infty,\end{aligned}$$

as $\|u\|_{L^2(0,T;L^2(\Omega))} \to +\infty$. Hence, H is coercive.

Step 3. H is hemi-continuous.

Since g satisfies condition (a), we have, for any $w(x,t) \in L^2(0,T;L^2(\Omega))$,

$$\begin{aligned}&\bigl(w,H(u+tv) - Hu\bigr) \\ &\quad= \int\int_\Omega\biggl[g\biggl(x,u+tv,\frac{\partial u}{\partial t} + t\frac{\partial v}{\partial t},\varepsilon(\nabla u + t\nabla v)\biggr) - g\biggl(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u\biggr)\biggr]w\,dx\,dt \to 0,\end{aligned}$$

as $t \to 0$, which implies that H is hemi-continuous.

Step 4. H is monotone.

In view of condition (c) of g, we have

$$\begin{aligned}&(u-v,Hu-Hv)\\&\quad=\int_0^T\int_\Omega\left[g\left(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u\right)-g\left(x,v,\frac{\partial v}{\partial t},\varepsilon\nabla v\right)\right]\big(u(x,t)-v(x,t)\big)\,dx\,dt\geq 0,\end{aligned}$$

which implies that H is monotone.

This completes the proof. □

Lemma 2.7 *For all* $u,v\in L^p(0,T;W^{1,p}(\Omega))$, *we have*

$$\big(v,\partial\Phi(u)\big)=\int_0^T\int_\Omega\beta_x\big(u|_\Gamma(x,t)\big)v|_\Gamma(x,t)\,d\Gamma(x)\,dt.$$

Moreover, $0\in\partial\Phi(0)$.

Proof The idea of the proof mainly comes from Proposition 3.2(ii) in [27]. For completeness, we give the outline of the proof as follows.

Define the mapping $G:L^p(0,T;L^p(\Gamma))\to L^{p'}(0,T;L^{p'}(\Gamma))$ by $Gu=\beta_x(u)$, for any $u\in L^p(0,T;L^p(\Gamma))$. Also, define the mapping $K:L^p(0,T;W^{1,p}(\Omega))\to L^p(0,T;L^p(\Gamma))$ by $K(v)=v|_\Gamma$, for any $v\in L^p(0,T;W^{1,p}(\Omega))$. Then $K^*GK=\partial\Phi$, where Φ is the same as in Lemma 2.3.

In fact, it is obvious that G is continuous. For $u(x,t),v(x,t)\in L^p(0,T;L^p(\Gamma))$, we have $(u-v,Gu-Gv)=\int_0^T\int_\Gamma(\beta_x(u)-\beta_x(v))(u-v)\,d\Gamma(x)\,dt\geq 0$, since β_x is monotone. Thus, G is monotone. In view of Lemma 1.1, $G:L^p(0,T;L^p(\Gamma))\to L^{p'}(0,T;L^{p'}(\Gamma))$ is maximal monotone.

Define $\Psi:L^p(0,T;L^p(\Gamma))\to\mathbb{R}$ by $\Psi(u)=\int_0^T\int_\Gamma\varphi_x(u)\,d\Gamma(x)\,dt$. It is easy to see that Ψ is a proper, convex, and lower-semicontinuous function on $L^p(0,T;L^p(\Gamma))$, which implies that $\partial\Psi:L^p(0,T;L^p(\Gamma))\to L^{p'}(0,T;L^{p'}(\Gamma))$ is maximal monotone in view of Lemma 1.2. Since

$$\begin{aligned}\Psi(u)-\Psi(v)&=\int_0^T\int_\Gamma\big[\varphi_x(u)-\varphi_x(v)\big]\,d\Gamma(x)\,dt\\&\geq\int_0^T\int_\Gamma\beta_x(v)(u-v)\,d\Gamma(x)\,dt=(Gv,u-v)\end{aligned}$$

for all $u(x,t),v(x,t)\in L^p(0,T;L^p(\Gamma))$, we have $Gv\in\partial\Psi(v)$. So $G=\partial\Psi$.

Now, it is clear that $K^*GK:L^p(0,T;W^{1,p}(\Omega))\to L^{p'}(0,T;(W^{1,p}(\Omega))^*)$ is maximal monotone since both K and G are continuous. Finally, for any $u,v\in L^p(0,T;W^{1,p}(\Omega))$, we have

$$\begin{aligned}\Phi(v)-\Phi(u)&=\Psi(Kv)-\Psi(Ku)\\&=\int_0^T\int_\Gamma\big[\varphi_x\big(v|_\Gamma(x,t)\big)-\varphi_x\big(u|_\Gamma(x,t)\big)\big]\,d\Gamma(x)\,dt\\&\geq\int_0^T\int_\Gamma\beta_x\big(u|_\Gamma(x,t)\big)\big(v|_\Gamma(x,t)-u|_\Gamma(x,t)\big)\,d\Gamma(x)\,dt\\&=(GKu,Kv-Ku)=\big(K^*GKu,v-u\big).\end{aligned}$$

Hence, we get $K^*GK\subset\partial\Phi$ and so $K^*GK=\partial\Phi$.

It now follows that for all $u, v \in L^p(0,T;W^{1,p}(\Omega))$,

$$(v, \partial\Phi(u)) = \int_0^T \int_\Omega \beta_x\big(u|_\Gamma(x,t)\big) v|_\Gamma(x,t)\, d\Gamma(x)\, dt.$$

Moreover, $0 \in \partial\Phi(0)$ since $0 \in \beta_x(0)$. This completes the proof. □

Lemma 2.8 *The mapping $A: L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ defined in Definition* 2.1 *is maximal monotone.*

Proof Noting Lemmas 2.2-2.4, we can easily get the result that A is monotone.

Next, we shall show that $R(I+A) = L^2(0,T;L^2(\Omega))$, which ensures that A is maximal monotone.

Case 1. $p \geq 2$. We define $\overline{F}: L^p(0,T;W^{1,p}(\Omega)) \to L^{p'}(0,T;(W^{1,p}(\Omega))^*)$ by

$$\overline{F}u = u, (v, \overline{F}u)_{L^p(0,T;W^{1,p}(\Omega))\times L^{p'}(0,T;(W^{1,p}(\Omega))^*)} = (v,u)_{L^2(0,T;L^2(\Omega))},$$

where $(\cdot,\cdot)_{L^2(0,T;L^2(\Omega))}$ denotes the inner-product of $L^2(0,T;L^2(\Omega))$. Then $\overline{F}$ is everywhere defined, monotone and hemi-continuous, which implies that $\overline{F}$ is maximal monotone in view of Lemma 1.1. Combining with the facts of Lemmas 1.3, 2.2-2.4, we have $R(B + \partial\Phi + S + \overline{F}) = L^{p'}(0,T;(W^{1,p}(\Omega))^*)$.

For $f \in L^2(0,T;L^2(\Omega)) \subset L^{p'}(0,T;(W^{1,p}(\Omega))^*)$, there exists $u \in L^p(0,T;W^{1,p}(\Omega)) \subset L^2(0,T;L^2(\Omega))$ such that

$$f = Bu + \partial\Phi(u) + Su + \overline{F}u = Au + u,$$

which implies that $R(I+A) = L^2(0,T;L^2(\Omega))$.

Case 2. $\frac{2N}{N+1} < p < 1$, then $p' \geq 2$. Similar to Lemma 2.2, we define $\widehat{B}: L^{p'}(0,T;W^{1,p}(\Omega)) \to L^p(0,T;(W^{1,p}(\Omega))^*)$ by

$$\begin{aligned}(w, \widehat{B}u) = {} & \int_0^T \int_\Omega \Big\langle \int_\Omega \alpha\big(|\nabla u|^p\big)|\nabla u|^{p-2}\nabla u, \nabla w \Big\rangle\, dx\, dt + \lambda_1 \int_0^T \int_\Omega |u|^{r_1-2}uw\, dx\, dt \\ & + \lambda_2 \int_0^T \int_\Omega |u|^{r_2-2}uw\, dx\, dt\end{aligned}$$

for any $u, w \in L^{p'}(0,T;W^{1,p}(\Omega))$. Then $\widehat{B}$ is maximal monotone and coercive. Similar to Lemma 2.3, define the mapping $\widehat{\Phi}: L^{p'}(0,T;W^{1,p}(\Omega)) \to \mathbb{R}$ by

$$\widehat{\Phi}(u) = \int_0^T \int_\Gamma \varphi_x\big(u|_\Gamma(x,t)\big)\, d\Gamma(x)\, dt,$$

for any $u \in L^{p'}(0,T;W^{1,p}(\Omega))$, then $\partial\widehat{\Phi}$ is maximal monotone. Similar to Lemma 2.4, define $\widehat{S}: D(\widehat{S}) = \{u \in L^{p'}(0,T;W^{1,p}(\Omega)) | \frac{\partial u}{\partial t} \in L^p(0,T;(W^{1,p}(\Omega))^*), u(x,0) = u(x,T)\} \to L^p(0,T;(W^{1,p}(\Omega))^*)$ by

$$\widehat{S}u(x,t) = \frac{\partial u}{\partial t}.$$

Then $\widehat{S}$ is linear maximal monotone. Similar to Case 1, define $\overline{\overline{F}} : L^{p'}(0,T;W^{1,p}(\Omega)) \to L^p(0,T;(W^{1,p}(\Omega))^*)$ by

$$\overline{\overline{F}}u = u, \quad (v, \overline{\overline{F}}u)_{L^{p'}(0,T;W^{1,p}(\Omega))\times L^p(0,T;(W^{1,p}(\Omega))^*)} = (v,u)_{L^2(0,T;L^2(\Omega))},$$

then we have $R(\widehat{B} + \partial\widehat{\Phi} + \widehat{S} + \overline{\overline{F}}) = L^p(0,T;(W^{1,p}(\Omega))^*)$. So, for $f \in L^2(0,T;L^2(\Omega)) \subset L^p(0,T;(W^{1,p}(\Omega))^*)$, there exists $u \in L^{p'}(0,T;W^{1,p}(\Omega)) \subset L^2(0,T;L^2(\Omega))$ such that

$$f = \widehat{B}u + \partial\widehat{\Phi}(u) + \widehat{S}u + \overline{\overline{F}}u = Au + u,$$

which implies that $R(I + A) = L^2(0,T;L^2(\Omega))$. □

Theorem 2.1 *For $f(x,t) \in L^2(0,T;L^2(\Omega))$, the nonlinear parabolic equation* (1.5) *has a unique solution $u(x,t)$ in $L^2(0,T;L^2(\Omega))$, i.e.,*

(a) $\frac{\partial u}{\partial t} - \operatorname{div}[\alpha(|\nabla u|^p)|\nabla u|^{p-2}\nabla u] + \lambda_1|u|^{r_1-2}u + \lambda_2|u|^{r_2-2}u + g(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u) = f(x,t)$, *a.e.* $(x,t) \in \Omega \times (0,T)$;

(b) $-\langle\vartheta, \alpha(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\rangle \in \beta_x(u(x,t))$, *a.e.* $x \in \Gamma \times (0,T)$;

(c) $u(x,0) = u(x,T)$, $x \in \Omega$.

Proof We split our proof into two steps.

Step 1. There exists a unique $u(x,t)$ which satisfies $Hu + \lambda Au = f$, where $f(x,t) \in L^2(0,T;L^2(\Omega))$ is a given function.

From Theorem 1.5, Lemmas 2.6 and 2.8, we know that A is H-monotone. Thus, $R(H + \lambda A) = L^2(0,T;L^2(\Omega))$. Then, for $f(x,t) \in L^2(0,T;L^2(\Omega))$ in (1.5), there exists $u(x,t) \in L^2(0,T;L^2(\Omega))$ such that $Hu(x,t) + \lambda Au(x,t) = f(x,t)$. Next, we shall prove that $u(x,t)$ is unique.

Suppose that $u(x,t)$ and $v(x,t)$ satisfy $Hu + \lambda Au = f$ and $Hv + \lambda Av = f$, respectively. Then $0 \le \lambda(u - v, Au - Av) = -(u - v, Hu - Hv) \le 0$, which ensures that

$$0 = (u - v, Au - Av) = (u - v, Bu - Bv) + \bigl(u - v, \partial\Phi(u) - \partial\Phi(v)\bigr) + (u - v, Su - Sv).$$

Using Lemmas 2.2, 2.3, and 2.4, we have $(u - v, Bu - Bv) = 0$, which implies that $u(x,t) = v(x,t)$, since B is strictly monotone.

Step 2. If $u(x,t) \in L^2(0,T;L^2(\Omega))$ satisfies $f = Hu + Au$, then $u(x,t)$ is the solution of (1.5).

Since $\Phi(u + \varphi) = \Phi(u)$ for any $\varphi \in C_0^\infty(\Omega \times (0,T))$, we have $(\varphi, \partial\Phi(u)) = 0$. Then, for $\varphi \in C_0^\infty(\Omega \times (0,T))$, we have

$$(\varphi, f - Hu) = (\varphi, Bu) + \bigl(\varphi, \partial\Phi(u)\bigr) + (\varphi, Su) = (\varphi, Bu) + (\varphi, Su).$$

So

$$\begin{aligned}
&\int_0^T \int_\Omega \left(f - g\left(x, u, \frac{\partial u}{\partial t}, \varepsilon\nabla u\right)\right)\varphi\,dx\,dt \\
&\quad = \int_0^T \int_\Omega \bigl\langle \alpha\bigl(|\nabla u|^p\bigr)|\nabla u|^{p-2}\nabla u, \nabla\varphi\bigr\rangle\,dx\,dt + \lambda_1 \int_0^T \int_\Omega |u|^{r_1-2}u\varphi\,dx\,dt \\
&\qquad + \lambda_2 \int_0^T \int_\Omega |u|^{r_2-2}u\varphi\,dx\,dt + \int_0^T \int_\Omega \frac{\partial u}{\partial t}\varphi\,dx\,dt
\end{aligned}$$

$$= -\int_0^T \int_\Omega \operatorname{div}\left[\alpha\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right]\varphi\,dx\,dt + \lambda_1 \int_0^T \int_\Omega |u|^{r_1-2}u\varphi\,dx\,dt$$
$$+ \lambda_2 \int_0^T \int_\Omega |u|^{r_2-2}u\varphi\,dx\,dt + \int_0^T \int_\Omega \frac{\partial u}{\partial t}\varphi\,dx\,dt,$$

which implies that the equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\left[\alpha\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right] + \lambda_1|u|^{r_1-2}u$$
$$+ \lambda_2|u|^{r_2-2}u + g\left(x, u, \frac{\partial u}{\partial t}, \varepsilon\nabla u\right) = f(x,t), \quad \text{a.e. } x \in \Omega \times (0,T), \tag{2.2}$$

is true.

By using (2.2) and Green's formula, we have

$$\int_0^T \int_\Gamma \left\langle \vartheta, \alpha\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right\rangle v|_\Gamma\,d\Gamma(x)\,dt$$
$$= \int_0^T \int_\Omega \operatorname{div}\left[\alpha\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right]v\,dx\,dt + \int_0^T \int_\Omega \left\langle \alpha\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u, \nabla v\right\rangle dx\,dt$$
$$= \left(v, \frac{\partial u}{\partial t} + \lambda_1|u|^{r_1-2}u + \lambda_2|u|^{r_2-2}u + g\left(x, u, \frac{\partial u}{\partial t}, \varepsilon\nabla u\right) - f\right)$$
$$+ \left(v, Bu - \lambda_1|u|^{r_1-2}u - \lambda_2|u|^{r_2-2}u\right) = (v, Su + Bu + Hu - f) = \left(v, -\partial\Phi(u)\right)$$
$$= -\int_0^T \int_\Gamma \beta_x\left(u|_\Gamma(x)\right)v|_\Gamma(x)\,d\Gamma(x)\,dt. \tag{2.3}$$

Then

$$-\left\langle \vartheta, \alpha\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right\rangle \in \beta_x\left(u(x,t)\right), \quad \text{a.e. on } \Gamma \times (0,T). \tag{2.4}$$

From the definition of S, we can easily obtain $u(x,0) = u(x,T)$ for all $x \in \Omega$. Combining with (2.2) and (2.4) we see that u is the unique solution of (1.5).

This completes the proof. □

Lemma 2.9 *Define* $\widetilde{B} : L^p(0,T;W^{1,p}(\Omega)) \to L^{p'}(0,T;(W^{1,p}(\Omega))^*)$ *by* $\widetilde{B}u \equiv Bu - f(x,t)$, *for* $u \in L^p(0,T;W^{1,p}(\Omega))$. *Then* $\widetilde{B}$ *is maximal monotone.*

Proof Similar to the proof of Lemma 2.2, we know that $\widetilde{B}$ is everywhere defined, monotone, and hemi-continuous. It follows that $\widetilde{B}$ is maximal monotone. □

Definition 2.3 Define a mapping $\widetilde{A} : L^2(0,T;L^2(\Omega)) \to 2^{L^2(0,T;L^2(\Omega))}$ by

$$\widetilde{A}u = \left\{w(x) \in L^2\left(0,T;L^2(\Omega)\right) | w(x) \in \widetilde{B}u + \partial\Phi(u) + Su\right\}$$

for $u \in D(\widetilde{A}) = \{u \in L^2(0,T;L^2(\Omega)) |$there exists $w(x) \in L^2(0,T;L^2(\Omega))$ such that $w(x) \in \widetilde{B}u + \partial\Phi(u) + Su\}$.

Definition 2.4 Let $\mathcal{H}$ be a Hilbert space and A be a H-monotone operator. The resolvent operator of A, $R^H_{A,\lambda} : \mathcal{H} \to \mathcal{H}$, is defined by

$$R^H_{A,\lambda}(u) = (H + \lambda A)^{-1}u, \quad \forall u \in \mathcal{H}.$$

Theorem 2.2 *$u(x,t) = R^H_{\widetilde{A},1}(0)$ if and only if $u(x,t) \in L^2(0,T;L^2(\Omega))$ is the solution of* (1.5).

Proof Let $u(x,t)$ be the solution of (1.5). Then, using Green's formula and Lemma 2.7, we have

$$\begin{aligned}
&\bigl(v,(H+\widetilde{A})u\bigr)\\
&\quad= \int_0^T \int_\Omega \bigl\langle \alpha\bigl(|\nabla u|^p\bigr)|\nabla u|^{p-2}\nabla u, \nabla v\bigr\rangle\,dx\,dt + \lambda_1 \int_0^T \int_\Omega |u|^{r_1-1}uv\,dx\,dt\\
&\qquad + \lambda_2 \int_0^T \int_\Omega |u|^{r_2-1}uv\,dx\,dt - \int_0^T \int_\Omega f(x,t)v(x,t)\,dx\,dt\\
&\qquad + \int_0^T \int_\Omega g\biggl(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u\biggr)v(x,t)\,dx\,dt + \bigl(v,\partial\Phi(u)\bigr) + \int_0^T \int_\Omega \frac{\partial u}{\partial t}v\,dx\,dt\\
&\quad= -\int_0^T \int_\Omega \operatorname{div}\bigl[\alpha\bigl(|\nabla u|^p\bigr)|\nabla u|^{p-2}\nabla u\bigr]v\,dx\,dt\\
&\qquad + \int_0^T \int_\Gamma \bigl\langle\vartheta, \alpha\bigl(|\nabla u|^p\bigr)|\nabla u|^{p-2}\nabla u\bigr\rangle v|_\Gamma\,d\Gamma(x)\,dt\\
&\qquad + \lambda_1 \int_0^T \int_\Omega |u|^{r_1-1}uv\,dx\,dt + \lambda_2 \int_0^T \int_\Omega |u|^{r_2-1}uv\,dx\,dt\\
&\qquad - \int_0^T \int_\Omega f(x,t)v(x,t)\,dx\,dt + \int_0^T \int_\Omega g\biggl(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u\biggr)v(x,t)\,dx\,dt\\
&\qquad + \int_0^T \int_\Gamma \beta_x(u|_\Gamma)v|_\Gamma\,d\Gamma(x)\,dt + \int_0^T \int_\Omega \frac{\partial u}{\partial t}v\,dx\,dt\\
&\quad= \int_0^T \int_\Gamma \bigl\langle\vartheta, \alpha\bigl(|\nabla u|^p\bigr)|\nabla u|^{p-2}\nabla u\bigr\rangle v|_\Gamma\,d\Gamma(x)\,dt + \int_0^T \int_\Gamma \beta_x(u|_\Gamma)v|_\Gamma\,d\Gamma(x)\,dt\\
&\quad= -\int_0^T \int_\Gamma \beta_x(u|_\Gamma)v|_\Gamma\,d\Gamma(x)\,dt + \int_0^T \int_\Gamma \beta_x(u|_\Gamma)v|_\Gamma\,d\Gamma(x)\,dt = 0.
\end{aligned}$$

Thus, $u(x,t) = R^H_{\widetilde{A},1}(0)$.

If $u(x,t) = R^H_{\widetilde{A},1}(0)$, then noting Lemma 2.7, we have for $\varphi \in C_0^\infty(\Omega \times (0,T))$,

$$\begin{aligned}
0 &= \int_0^T \int_\Omega \frac{\partial u}{\partial t}\varphi\,dx\,dt + \int_0^T \int_\Omega \bigl\langle \alpha\bigl(|\nabla u|^p\bigr)|\nabla u|^{p-2}\nabla u, \nabla\varphi\bigr\rangle\,dx\,dt\\
&\quad + \lambda_1 \int_0^T \int_\Omega |u|^{r_1-2}u\,dx\,dt + \lambda_2 \int_0^T \int_\Omega |u|^{r_2-2}u\,dx\,dt\\
&\quad - \int_0^T \int_\Omega f\varphi\,dx\,dt + \int_0^T \int_\Omega g\biggl(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u\biggr)\varphi\,dx\,dt,
\end{aligned}$$

which implies that the equation

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \operatorname{div}\bigl[\alpha\bigl(|\nabla u|^p\bigr)|\nabla u|^{p-2}\nabla u\bigr] + \lambda_1|u|^{r_1-2}u\\
&\quad + \lambda_2|u|^{r_2-2}u + g\biggl(x,u,\frac{\partial u}{\partial t},\varepsilon\nabla u\biggr) = f(x,t), \quad \text{a.e. } (x,t) \in \Omega \times (0,T),
\end{aligned}$$

is true.

Similar to the last part of Theorem 2.1, we know that $-\langle\vartheta,\alpha(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\rangle \in \beta_x(u(x,t))$. From the definition of S, we know that $u(x,0) = u(x,T)$ for all $x \in \Omega$, which implies that $u(x,t)$ is the solution of (1.5). This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Hebei University of Economics and Business, Shijiazhuang, 050061, China. [2]Department of Mathematics, Texas A&M University - Kingsville, Kingsville, TX 78363, USA. [3]Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, 21589, Saudi Arabia. [4]School of Electrical and Electronic Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore, 639798, Singapore.

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