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Positive solutions of singular beam equations with the bending term

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Abstract

Using a new technique for dealing with the bending term of beam equations, we consider the existence and multiplicity of positive solutions for a beam equation. Besides achieving new results, upper and lower bounds for these positive solutions will also be provided. The results are shown by using a novel technique and fixed point theories.

Keywords: positive solution; singularity; beam equation with the bending term; fixed point theorem

1 Introduction

In this paper, we shall investigate the existence and multiplicity of positive solutions for the fourth order differential equation with integral boundary conditions

$$\begin{cases} y^{(4)}(t) = \omega(t)F(t, y(t), y''(t)), & 0 < t < 1, \\ y(0) = y(1) = \int_0^1 h(s)y(s) ds, \\ ay''(0) - by'''(0) = \int_0^1 g(s)y''(s) ds, \\ ay''(1) + by'''(1) = \int_0^1 g(s)y''(s) ds, \end{cases} \quad (1.1)$$

where $a, b > 0$, $F: [0, 1] \times R \times R \rightarrow R$ is continuous, and the y'' in F is the bending moment term which represents bending effect. Problem (1.1) is often referred to as the deformation of an elastic beam under a variety of boundary conditions; for details, see [1–18]. Most research papers on beam equations consider nonlinear terms that F in (1.1) involves y only, and derivative-dependent nonlinearities are seldom tackled; see [3–7, 9–11] to name a few.

Some classical tools such as fixed point theorems in cones [5, 8–11, 17, 18], the method of lower and upper solutions [8, 16], and the monotone iterative method [9, 12] and the theory of critical point theory and variational methods [6, 13, 14] have been widely used to study beam equations.

Some new techniques via appropriate transformation are proved to be very effective in studying the solvability of differential equations. Such techniques have attracted the attention of Zhang *et al.* [19] and Wong [20], *etc.* In [19], Zhang *et al.* considered the existence

of positive solutions of the following problems:

$$\begin{cases} x^{(n)}(t) + f(t, x(t), x'(t), \dots, x^{(n-2)}(t)) = \theta, & t \in J, t \neq t_k, k = 1, 2, \dots, m, \\ \Delta x^{(n-1)}|_{t=t_k} = -I_k(x^{(n-2)}(t_k)), & k = 1, 2, \dots, m, \\ x^{(i)}(0) = \theta, & i = 0, 1, \dots, n-3, \\ x^{(n-2)}(0) = x^{(n-2)}(1) = \int_0^1 g(t)x^{(n-2)}(t) dt, \end{cases} \tag{1.2}$$

where $J = [0, 1]$.

Firstly, by means of the transformation

$$x^{(n-2)}(t) = y(t),$$

the authors converted (1.2) into

$$\begin{cases} x^{(n-2)}(t) = y(t), & t \in J, \\ x^{(i)}(0) = \theta, & i = 1, 2, \dots, n-3, \end{cases}$$

and

$$\begin{cases} -y''(t) = f(t, x(t), x'(t), \dots, x^{(n-2)}(t)), & t \in J, t \neq t_k, \\ \Delta y'|_{t=t_k} = -I_k(y(t_k)), & k = 1, 2, \dots, m, \\ y(0) = y(1) = \int_0^1 g(t)y(t) dt. \end{cases} \tag{1.3}$$

Then it follows from Lemma 2.3 in [19] that they converted the results obtained for problem (1.2) to the counterpart for problem (1.3).

In [20], Wong transformed the following problems:

$$\begin{cases} (-1)^m y^{(2m+1)}(t) = F(t, y(t), y'(t)), & 0 < t < 1, \\ y(0) = 0, \quad y^{(2k-1)}(0) = y^{(2k-1)}(1) = 0, & 0 \leq k \leq m, \end{cases} \tag{1.4}$$

into

$$\begin{cases} y'(t) = x(t), & t \in J, \\ y(0) = 0, \end{cases}$$

and

$$\begin{cases} (-1)^m x^{(2m)}(t) = F(t, y(t), y'(t)), & 0 < t < 1, \\ x^{(2k-2)}(0) = x^{(2k-2)}(1) = 0, & 1 \leq k \leq m \end{cases} \tag{1.5}$$

by using $y'(t) = x(t)$. So the existence of a solution of the complementary Lidstone boundary value problem (1.4) follows from the existence of a solution of the Lidstone boundary value problem (1.5). We notice that the above paper requires that F satisfies some assumptions of monotonicity which are essential for the technique used.

Being directly inspired by [19, 20], in the present paper, by using transformation techniques and fixed point theories, the authors shall prove some new and more general results of the existence of at least one or two positive solutions for problem (1.1). The main features of this paper are as follows. Firstly, comparing with [1–18], besides achieving new results,

estimates on the norms of these solutions will also be provided. Secondly, we transform problem (1.1) into a differential system without bending term, *i.e.*, the technique to deal with bending term is completely different from that of [8, 16–18]. Finally, it is pointed out that we do not need any monotone assumption on F , which is weaker than the corresponding assumptions on F in [20].

On the other hand, boundary value problems with integral boundary conditions arise naturally in thermal conduction problems [21], semiconductor problems [22], hydrodynamic problems [23] and so on. It is interesting to point out that such problems include two, three, multi-point and nonlocal boundary value problems as special cases and have been extensively studied in the last ten years (see, for example, [24–30]). Therefore, it is important to study fourth order elasticity problems with integral boundary conditions.

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries and lemmas. In particular, we transform problem (1.1) into a differential system without the bending moment term. In Section 3, the main results will be stated and proved.

2 Preliminaries

To establish the existence of positive solutions for problem (1.1), let us list the following assumptions, which will hold throughout this paper:

- (H₁) $\omega \in C((0, 1), [0, +\infty))$ with $0 < \int_0^1 \omega(s) ds < \infty$ and ω does not vanish on any subinterval of $(0, 1)$;
- (H₂) $F \in C([0, 1] \times [0, +\infty) \times (-\infty, 0], [0, +\infty))$;
- (H₃) $g, h \in L^1[0, 1]$ are nonnegative and $\mu \in [0, a), \nu \in [0, 1)$, where

$$\mu = \int_0^1 g(s) ds, \quad \nu = \int_0^1 h(s) ds. \tag{2.1}$$

Lemma 2.1 [11] *Assume that (H₃) holds. Then, for any $x \in C[0, 1]$, the boundary value problem*

$$\begin{cases} -y''(t) = x(t), & 0 < t < 1, \\ y(0) = y(1) = \int_0^1 h(s)y(s) ds \end{cases} \tag{2.2}$$

has a unique solution y given by

$$y(t) = \int_0^1 H_1(t, s)x(s) ds, \tag{2.3}$$

where

$$H_1(t, s) = G(t, s) + \frac{1}{1-\nu} \int_0^1 G_1(s, \tau)h(\tau) d\tau, \tag{2.4}$$

$$G_1(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.5}$$

Taking into account (2.2) and (2.3), problem (1.1) reduces to the following problem:

$$\begin{cases} x''(t) = -\omega(t)F(t, \int_0^1 H_1(t, s)x(s) ds, -x(t)), & 0 < t < 1, \\ ax(0) - bx'(0) = \int_0^1 g(s)x(s) ds, & ax(1) + bx'(1) = \int_0^1 g(s)x(s) ds. \end{cases} \tag{2.6}$$

Lemma 2.2 [11] *Assume that (H₁)-(H₃) hold. Then boundary value problem (2.6) has a unique solution x given by*

$$x(t) = \int_0^1 H_2(t,s)\omega(s)F\left(s, \int_0^1 H_1(s,\tau)x(\tau) d\tau, -x(s)\right) ds, \tag{2.7}$$

where

$$H_2(t,s) = G_2(t,s) + \frac{1}{a-\mu} \int_0^1 G_2(s,\tau)g(\tau) d\tau, \tag{2.8}$$

$$G_2(t,s) = \frac{1}{d} \begin{cases} (b+as)(b+a(1-t)) & \text{if } 0 \leq s \leq t \leq 1, \\ (b+at)(b+a(1-s)) & \text{if } 0 \leq t \leq s \leq 1, \end{cases} \tag{2.9}$$

$$d = a(2b+a).$$

If problem (2.6) has a solution x^* , then by (2.3) problem (1.1) has a solution given by

$$y^*(t) = \int_0^1 H_1(t,s)x^*(s) ds.$$

So the existence of a solution of problem (1.1) follows from the existence of a solution of problem (2.6).

Lemma 2.3 [11] *For $t, s \in J$, we have the following results:*

$$G_1(t,t)G_1(s,s) \leq G_1(t,s) \leq G_1(s,s) \leq \frac{1}{4}, \tag{2.10}$$

$$\frac{1}{d}b^2 \leq G_2(t,s) \leq G_2(s,s) \leq \frac{1}{d}(b+a)^2. \tag{2.11}$$

Lemma 2.4 [11] *Let (H₃) hold. Then we obtain the following results:*

$$\rho G_1(s,s) \leq H_1(t,s) \leq \gamma G_1(s,s) \leq \frac{1}{4}\gamma, \quad \forall t, s \in J, \tag{2.12}$$

$$\rho_1 \leq H_2(t,s) \leq H_2(s,s) \leq \rho_2, \quad \forall t, s \in J, \tag{2.13}$$

where

$$\begin{aligned} \gamma &= \frac{1}{1-\nu}, & \rho &= \frac{\int_0^1 G(\tau,\tau)h(\tau) d\tau}{1-\nu}, & \rho_1 &= \frac{b^2\gamma_1}{a+2b}, \\ \rho_2 &= \frac{\gamma_1(b+a)^2}{a+2b}, & \gamma_1 &= \frac{1}{a-\mu}. \end{aligned}$$

It is clear from (2.3) that $\|y^*\| \leq \frac{\gamma}{4}\|x^*\|$; moreover, if x^* is positive, so is y^* .

Let $E = C[0,1]$. It is well known that E is a real Banach space with the norm $\|\cdot\|$ defined by $\|x\| = \max_{t \in J} |x(t)|$.

Define an operator $T : E \rightarrow E$ as follows:

$$(Tx)(t) = \int_0^1 H_2(t,s)\omega(s)F\left(s, \int_0^1 H_1(s,\tau)x(\tau) d\tau, -x(s)\right) ds. \tag{2.14}$$

Let C be a cone in E which is defined as

$$C = \{x \in E : x \geq 0, x(t) \geq \delta \|x\|, t \in J\},$$

where

$$\delta = \frac{\rho_1}{\rho_2}. \tag{2.15}$$

It is easy to see that E is a closed convex cone of E .

Lemma 2.5 *Let (H_1) - (H_3) hold. Then we have $T(C) \subset C$, and $T : C \rightarrow C$ is completely continuous.*

Proof In view of condition (2.13), we see that

$$\begin{aligned} \|Tx\| &= \max_{t \in J} \int_0^1 H_2(t,s)\omega(s)F\left(s, \int_0^1 H_1(s,\tau)x(\tau) d\tau, -x(s)\right) ds \\ &\leq \rho_2 \int_0^1 \omega(s)F\left(s, \int_0^1 H_1(s,\tau)x(\tau) d\tau, -x(s)\right) ds. \end{aligned}$$

Moreover, it follows from (2.13) and (2.14) that

$$\begin{aligned} (Tx)(t) &= \int_0^1 H_2(t,s)\omega(s)F\left(s, \int_0^1 H_1(s,\tau)x(\tau) d\tau, -x(s)\right) ds \\ &\geq \rho_1 \int_0^1 \omega(s)F\left(s, \int_0^1 H_1(s,\tau)x(\tau) d\tau, -x(s)\right) ds \\ &\geq \frac{\rho_1}{\rho_2} \rho_2 \int_0^1 \omega(s)F\left(s, \int_0^1 H_1(s,\tau)x(\tau) d\tau, -x(s)\right) ds \\ &\geq \delta \|Tx\|, \quad t \in J. \end{aligned} \tag{2.16}$$

This proves that $T(C) \subset C$.

Next, by standard methods and the Ascoli-Arzelà theorem, one can prove that $T : C \rightarrow C$ is completely continuous. □

To obtain positive solutions of problem (1.1), the following fixed point theorem in cones, which can be found in [31], p.94, is fundamental.

Lemma 2.6 *Let P be a cone in a real Banach space E . Assume Ω_1, Ω_2 are bounded open sets in E with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$. If*

$$A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow P$$

is completely continuous such that either

- (a) $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1$ and $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2$, or
- (b) $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1$ and $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$,

then A has at least one fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3 Main results

In this section, we apply Lemma 2.6 to establish the existence of positive solutions for problem (1.1). We begin by introducing the following conditions on $F(t, u, v)$:

(H₄) There exist two positive constants r, R with $(\frac{\gamma}{4} + 1)r < (\sigma + \delta)R$ such that:

$$F(t, u, v) \leq \frac{1}{\rho_2 \eta} r, \quad \forall t \in J, |u| + |v| \leq \left(\frac{\gamma}{4} + 1\right)r, \tag{3.1}$$

$$F(t, u, v) \geq \frac{1}{\rho_1 \eta \delta} R, \quad \forall t \in J, |u| + |v| \geq (\sigma + \delta)R, \tag{3.2}$$

where

$$\eta = \int_0^1 \omega(s) ds, \quad \sigma = \rho \delta \beta, \quad \beta = \int_0^1 s(1-s) ds = \frac{1}{6}.$$

Theorem 3.1 *Assume that (H₁)-(H₄) hold. Then we have the following conclusions:*

(i) *Problem (2.6) has (at least) one positive solution $x \in C$ such that*

$$\delta r \leq x(t) \leq \frac{1}{\delta} R, \quad t \in J. \tag{3.3}$$

(ii) *Problem (1.1) has (at least) one positive solution y such that*

$$\begin{cases} y(t) = \int_0^1 H_1(t, s)x(s) ds, & t \in J; \\ \|y\| \leq \frac{\gamma}{4} \|x\|; \\ y(t) \geq \sigma \|x\|, & t \in J. \end{cases} \tag{3.4}$$

We further have

$$\sigma \delta r \leq y(t) \leq \frac{\gamma}{4\delta} R, \quad t \in J. \tag{3.5}$$

Proof Let T be the cone preserving, completely continuous operator that was defined by (2.14).

Let $x \in C$ with $\|x\| = r$. Then $0 \leq x(t) \leq r, t \in J$, and $0 \leq \int_0^1 H_1(s, \tau)x(\tau) d\tau \leq \frac{\gamma}{4}r$. And hence, for $x \in C$ with $\|x\| = r$, we have

$$\left| \int_0^1 H_1(s, \tau)x(\tau) d\tau \right| + |-x(s)| \leq \left(\frac{\gamma}{4} + 1\right)r.$$

Then it follows from (3.1) that

$$\begin{aligned} \|Tx\| &= \max_{t \in J} \int_0^1 H_2(t, s)\omega(s)F\left(s, \int_0^1 H_1(s, \tau)x(\tau) d\tau, -x(s)\right) ds \\ &\leq \rho_2 \eta \frac{1}{\rho_2 \eta} r = r. \end{aligned} \tag{3.6}$$

Now if we let $\Omega_1 = \{x \in C : \|x\| < r\}$, then (3.6) shows that

$$\|Tx\| \leq \|x\|, \quad x \in \partial\Omega_1. \tag{3.7}$$

Further, let

$$R_1 = \frac{1}{\delta}R, \tag{3.8}$$

and

$$\Omega_2 = \{x \in C : \|x\| < R_1\}.$$

Then $x \in C$ and $\|x\| = R_1$ imply

$$x(t) \geq \delta x(s), \quad t, s \in J,$$

that is,

$$x(t) \geq \delta R_1 = R, \quad t \in J.$$

Hence, $x(t) \geq R$ for all $t \in J$, and

$$\int_0^1 H_1(s, \tau)x(\tau) d\tau \geq \rho \int_0^1 G(\tau, \tau)x(\tau) d\tau \geq \rho\beta\delta\|x\| = \sigma\|x\|.$$

So

$$\left| \int_0^1 H_1(s, \tau)x(\tau) d\tau \right| + |-x(t)| \geq (\sigma + \delta)R.$$

Using condition (3.2), it follows from $x \in C$ and $\|x\| = R_1$ that

$$\|Tx\| = \max_{t \in J} \int_0^1 H_2(t, s)\omega(s)F\left(s, \int_0^1 H_1(s, \tau)x(\tau) d\tau, -x(s)\right) ds \geq \rho_1\eta \frac{1}{\rho_1\eta\delta}R = \frac{R}{\delta} = R_1,$$

that is, $x \in \partial\Omega_2$ implies

$$\|Tx\| \geq \|x\|. \tag{3.9}$$

It now follows from Lemma 2.6 that problem (2.6) has (at least) one positive solution $x \in \bar{\Omega}_2 \setminus \Omega_1$ satisfying (3.3).

It is observed from (2.3) that problem (1.1) has (at least) one positive solution y such that

$$y = \int_0^1 H_1(t, s)x_1(s) ds, \quad t \in J \text{ and } \|y\| \leq \frac{\gamma}{4}\|x\|.$$

Moreover, since $x \in C$, we get for $t \in J$

$$\begin{aligned} y(t) &= \int_0^1 H_1(t, s)x(s) ds \\ &\geq \rho \int_0^1 G(s, s)x(s) ds \end{aligned}$$

$$\begin{aligned} &\geq \rho\beta\delta\|x\| \\ &= \sigma\|x\|. \end{aligned}$$

Then we get (3.4).

Further, it follows from (3.3) and (3.4) that (3.5) holds. □

In Theorem 3.2 we assume the following condition on $f(t, u, v)$:

(H₅) There exist two positive constants r, R with $(\frac{\gamma}{4} + 1)r < R$ such that:

$$F(t, u, v) \geq \frac{1}{\rho_1\eta(\sigma + \delta)}(|u| + |v|), \quad \forall t \in J, |u| + |v| \leq \left(\frac{\gamma}{4} + 1\right)r, \tag{3.10}$$

$$F(t, u, v) \leq \frac{1}{2\rho_2\eta(\frac{\gamma}{4} + 1)}(|u| + |v|), \quad \forall t \in J, |u| + |v| \geq R, \tag{3.11}$$

and write

$$M = \max_{\forall t \in J, |u| + |v| \leq R} F(t, u, v). \tag{3.12}$$

Theorem 3.2 *Assume that (H₁)-(H₃) and (H₅) hold. Then we have the following conclusions:*

(i) *Problem (2.6) has (at least) one positive solution $x \in C$ such that*

$$\delta r \leq x(t) \leq \max\{2R, 2\rho_2\eta M\}, \quad t \in J. \tag{3.13}$$

(ii) *Problem (1.1) has (at least) one positive solution y such that (3.4) holds. We further have*

$$\sigma \delta r \leq y(t) \leq \max\left\{\frac{\gamma R}{4(\sigma + \delta)}, \frac{\rho_2\eta M\gamma}{2}\right\}, \quad t \in J. \tag{3.14}$$

Proof Let $x \in C$ with $\|x\| = r$. Then $0 \leq x(t) \leq r, t \in J$, and $0 \leq \int_0^1 H_1(s, \tau)x(\tau) d\tau \leq \frac{\gamma}{4}r$. And hence for $x \in C$ with $\|x\| = r$, we have

$$(\sigma + \delta)r \leq \left| \int_0^1 H_1(s, \tau)x(\tau) d\tau \right| + |-x(t)| \leq \left(\frac{\gamma}{4} + 1\right)r,$$

and it follows from condition (H₅) that

$$\begin{aligned} \|Tx\| &= \max_{t \in J} \int_0^1 H_2(t, s)\omega(s)F\left(s, \int_0^1 H_1(s, \tau)x(\tau) d\tau, -x(s)\right) ds \\ &\geq \rho_1\eta \frac{1}{\rho_1\eta(\sigma + \delta)}(\sigma + \delta)r = r, \end{aligned} \tag{3.15}$$

that is, $x \in \partial\Omega_1$ implies that

$$\|Tx\| \geq \|x\|. \tag{3.16}$$

Next, we turn to (3.11) and (3.12). From (3.11) and (3.12), we have

$$F(t, u, v) \leq M + \frac{1}{2\rho_2\eta(\frac{\gamma}{4} + 1)} (|u| + |v|), \quad (t, u, v) \in J \times [0, \infty) \times (-\infty, 0]. \tag{3.17}$$

Further, let

$$R_2 > \max \left\{ 2\rho_2\eta M, \frac{R}{\sigma + \delta} \right\}, \tag{3.18}$$

and

$$\Omega_3 = \{x \in C : \|x\| < R_2\}.$$

Notice that for $x \in \partial\Omega_3$ we have

$$R < (\sigma + \delta)R_2 \leq \left| \int_0^1 H_1(s, \tau)x(\tau) d\tau \right| + |-x(t)| \leq \left(\frac{1}{4}\gamma + 1\right)R_2.$$

Thus, for $x \in \partial\Omega_3$, it follows from (3.17) that

$$\begin{aligned} \|Tx\| &= \max_{t \in J} \int_0^1 H_2(t, s)\omega(s)F\left(s, \int_0^1 H_1(s, \tau)x(\tau) d\tau, -x(s)\right) ds \\ &\leq \rho_2 \int_0^1 \omega(s) \left(M + \frac{1}{2\rho_2\eta}R_2\right) ds \\ &\leq \rho_2\eta \left(M + \frac{1}{2\rho_2\eta}R_2\right) \\ &< R_2, \end{aligned}$$

that is, $x \in \partial\Omega_3$ implies

$$\|Tx\| < \|x\|. \tag{3.19}$$

It now follows from Lemma 2.6 that problem (2.6) has (at least) one positive solution $x \in \bar{\Omega}_3 \setminus \Omega_1$ satisfying (3.13).

It follows from (2.3) that problem (1.1) has (at least) one positive solution y . Similar to the proof of (3.5), one can show that y satisfies (3.14). □

Theorem 3.3 *Assume that (H₁)-(H₃), (3.2) of (H₄) and (3.10) of (H₅) hold. In addition, letting f satisfies the following condition:*

(H₆) *Let l, ζ and L satisfy*

$$0 < l < \left(\frac{\gamma}{4} + 1\right)l < \zeta < \left(\frac{\gamma}{4} + 1\right)\zeta < \delta L < L.$$

If

$$\max_{\forall t \in J, \zeta \leq |u| + |v| \leq (\frac{\gamma}{4} + 1)\zeta} F(t, u, v) < \frac{1}{\rho_2\eta}\zeta,$$

then we have the following conclusions:

(i) Problem (2.6) has (at least) two positive solutions $x_1, x_2 \in C$ such that

$$\delta l \leq x_1(t) < \zeta < \left(\frac{\gamma}{4} + 1\right)\zeta < x_2(t) \leq L, \quad t \in J. \tag{3.20}$$

(ii) Problem (1.1) has (at least) two positive solutions y_1, y_2 such that for $i = 1, 2$,

$$\begin{cases} y_i(t) = \int_0^1 H_1(t,s)x_i(s) ds, & t \in J; \\ \|y_i\| \leq \frac{\gamma}{4}\|x_i\|; \\ y_i(t) \geq \sigma\|x_i\|, & t \in J. \end{cases} \tag{3.21}$$

We further have

$$\begin{cases} y_1(t) > \sigma\delta l, & t \in J; \\ \|y_2\| \leq \frac{\gamma}{4}L. \end{cases} \tag{3.22}$$

Proof If (3.10) of (H_5) holds, similar to the proof of (3.16), we can prove that

$$\|Tx\| \geq \|x\|, \quad x \in C, \|x\| = l. \tag{3.23}$$

If (3.2) of (H_4) holds, similar to the proof of (3.9), we have

$$\|Tx\| \geq \|x\|, \quad x \in C, \|x\| = L. \tag{3.24}$$

Finally, we show that

$$\|Tx\| < \|x\|, \quad x \in C, \|x\| = \zeta. \tag{3.25}$$

In fact, for $x \in C$ with $\|x\| = \zeta$, we have

$$x(t) \leq \|x\| = \zeta,$$

and

$$\sigma\zeta \leq \int_0^1 H_1(s, \tau)x(\tau) d\tau \leq \frac{\gamma}{4}\zeta.$$

Therefore,

$$\zeta < (\sigma + 1)\zeta \leq \left| \int_0^1 H_1(s, \tau)x(\tau) d\tau \right| + |-x(s)| \leq \left(\frac{\gamma}{4} + 1\right)\zeta,$$

and hence it follows from (H_6) that

$$\begin{aligned} \|Tx\| &= \max_{t \in J} \int_0^1 H_2(t,s)\omega(s)F\left(s, \int_0^1 H_1(s, \tau)x(\tau) d\tau, -x(s)\right) ds \\ &< \rho_2\eta \frac{1}{\rho_2\eta}\zeta = \zeta, \end{aligned}$$

which shows that (3.25) holds.

Applying Lemma 2.6 to (3.23), (3.24) and (3.25) yields that problem (2.6) has (at least) two positive solutions x_1, x_2 with $x_1 \in C_{l,\zeta} = \{x \in C, l \leq \|x\| < \zeta\}$, $x_2 \in C_{\zeta,\bar{L}} = \{x \in C, (\frac{\zeta}{4} + 1)\zeta < \|x\| \leq \bar{L}\}$. Hence, since for $x_1 \in C$ we have $x_1(t) \geq \delta \|x_1\|$, $t \in J$, it follows that (3.20) holds.

Similar to the proof of (3.4) and (3.5), one can show that (3.21) and (3.22) hold. \square

Remark 3.1 In Theorems 3.1-3.3, we generalize the results of [8, 16–18] in three main directions as follows:

- (1) Upper and lower bounds for these positive solutions are given.
- (2) Estimates on the norms of positive solutions are considered.
- (3) The method to deal with the bending term of beam equations in this paper is completely different from that of [8, 16–18], which opens a new technique to study the beam equations with the bending term.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

XZ completed the main study and carried out the results of this article. MF checked the proofs and verified the calculation. All the authors read and approved the final manuscript.

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