Lin and Tang *Boundary Value Problems* (2015) 2015:111
DOI 10.1186/s13661-015-0351-3

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

New conditions on homoclinic solutions for a subquadratic second order Hamiltonian system

Xiaoyan Lin[1*] and Xianhua Tang[2]

[*]Correspondence:
xiaoyanlin98@hotmail.com
[1]Department of Mathematics, Huaihua College, Huaihua, Hunan 418008, P.R. China
Full list of author information is available at the end of the article

Abstract
In this paper, we deal with the second-order Hamiltonian system

$$(*)\quad \ddot{u}-L(t)u+\nabla W(t,u)=0.$$

We establish some criteria which guarantee that the above system has at least one or infinitely many homoclinic solutions under the assumption that $W(t,x)$ is subquadratic at infinity and $L(t)$ is a real symmetric matrix and satisfies

$$\liminf_{|t|\to+\infty}\Big[|t|^{\nu-2}\inf_{|x|=1}(L(t)x,x)\Big]>0$$

for some constant $\nu<2$. In particular, $L(t)$ and $W(t,x)$ are allowed to be sign-changing.

MSC: 34C37; 58E05; 70H05

Keywords: homoclinic orbit; Hamiltonian systems; subquadratic potential; indefinite sign

1 Introduction

Consider the second-order Hamiltonian system

$$\ddot{u}-L(t)u+\nabla W(t,u)=0, \tag{1.1}$$

where $t\in\mathbb{R}$, $u\in\mathbb{R}^N$, $L\in C(\mathbb{R},\mathbb{R}^{N\times N})$ is a symmetric matrix-valued function, $W\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ and $\nabla W(t,x)=\nabla_x W(t,x)$. As usual [1], we say that a solution $u(t)$ of system (1.1) is homoclinic (to 0) if $u(t)\to 0$ as $t\to\pm\infty$. In addition, if $u(t)\not\equiv 0$ then $u(t)$ is called a nontrivial homoclinic solution.

The existence and multiplicity of nontrivial homoclinic solutions for problem (1.1) have been extensively investigated in the literature with the aid of critical point theory and variational methods (see, for example, [2–17]). Most of them treat the case where $W(t,x)$ is superquadratic as $|x|\to\infty$.

Compared to the superquadratic case, as far as the authors are aware, there are a few papers [17–20] concerning the case where $W(t,x)$ has subquadratic growth at infinity. In these papers, since $L(t)$ is positive definite, the energy functional associated with system

(1.1) is bounded from below, techniques based on the genus properties have been well applied. In particular, Clark's theorem is an effective tool to prove the existence and multiplicity of homoclinic solutions for system (1.1). However, if $L(t)$ is not global positive definite on $\mathbb{R}$, the problem is far more difficult as 0 is a saddle point rather than a local minimum of the energy functional, which is strongly indefinite and it is not easy to prove the boundedness of the Palais-Smale sequence.

In [21], Ding studied the existence of homoclinic solutions of system (1.1) under the case when $L(t)$ is not global positive definite on $\mathbb{R}$ and $W(t,x)$ is subquadratic at infinity. He obtained the following result.

Theorem A ([21]) *Assume that L and W satisfy the following conditions*:

(A1) *There exists a constant $\nu < 2$ such that*

$$|t|^{\nu-2} \inf_{|x|=1} \bigl(L(t)x,x\bigr) \to +\infty \quad \textit{as } |t| \to +\infty;$$

(A2) $0 < \inf_{t\in\mathbb{R},|x|=1} W(t,x) \le \sup_{t\in\mathbb{R},|x|=1} W(t,x) < +\infty$;

(A3) *There exists a constant μ with $1 < \mu \in ((4-\nu)/(3-\nu), 2)$ such that*

$$0 < \bigl(\nabla W(t,x),x\bigr) \le \mu W(t,x), \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N \setminus \{0\};$$

(A4) *There exist three constants $a_1, r_1 > 0$ and $1 < \mu_1 \in (2/(3-\nu), \mu]$ such that*

$$W(t,x) \ge a_1|x|^{\mu_1}, \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N, |x| \ge r_1;$$

(A5) $W(t,0) \equiv 0$ *and there exist three constants $a_2, r_2 > 0$ and $1 < \mu_2 \in (2/(3-\nu), \mu]$ such that*

$$\bigl|\nabla W(t,x)\bigr| \le a_2|x|^{\mu_2-1}, \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N, |x| \le r_2.$$

Then system (1.1) *has at least one nontrivial homoclinic solution. Moreover, if $W(t,x)$ is also even with respect to x, then system* (1.1) *has infinitely many homoclinic solutions*.

In Theorem A, assumptions (A2)-(A5) imply that there exist positive constants a_*, a^*, b_* and b^* such that

$$b_*|x|^{\mu} \le W(t,x) \le a_*|x|^{\mu_2}, \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N, |x| \le 1, \tag{1.2}$$

$$a^*|x|^{\mu_1} \le W(t,x) \le b^*|x|^{\mu}, \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N, |x| \ge 1. \tag{1.3}$$

However, there are many potential functions $W(t,x)$ satisfying (1.2) and (1.3), but not (A3). For example $W(t,x) = (1+\sin^2 t)(2|x|^{5/4} - 3|x|^{3/2} + 2|x|^{7/4})$ is such a potential function. In particular, Theorem A is only applicable when the potential $W(t,x)$ is positive definite.

In the present paper, we will use new tricks to generalize and improve Theorem A. For example, we can replace (A1) by a weaker one (L_ν):

(L_ν) *There exists a constant $\nu < 2$ such that*

$$\liminf_{|t|\to+\infty}\Bigl[|t|^{\nu-2} \inf_{|x|=1} \bigl(L(t)x,x\bigr)\Bigr] > 0.$$

We also relax (A3) and (A4) in Theorem A to two of the following weaker assumptions:

(W3) *There exist constants* $b_1, b_2 > 0$ *and* $\max\{1, 2/(3-\nu)\} < \gamma_5 \leq \gamma_4 < 2$ *such that*

$$2W(t,x) - \big(\nabla W(t,x), x\big) \geq \begin{cases} b_1|x|^{\gamma_4}, & |x| \leq 1, \\ b_2|x|^{\gamma_5}, & |x| \geq 1, \end{cases} \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W4) *There exist constants* $b_3, b_4 > 0$ *and* $\max\{1, 2/(3-\nu)\} < \gamma_7 \leq \gamma_6 < 2$ *such that*

$$W(t,x) \geq \begin{cases} b_3|x|^{\gamma_6}, & |x| \leq 1, \\ b_4|x|^{\gamma_7}, & |x| \geq 1, \end{cases} \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W3′) *There exist constants* $b_5 > 0$, $b_6 \geq 0$ *and* $\max\{1, 2/(3-\nu)\} < \gamma_9 < \gamma_8 < 2$ *such that*

$$2W(t,x) - \big(\nabla W(t,x), x\big) \geq b_5|x|^{\gamma_8} - b_6|x|^{\gamma_9}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W4′) *There exist constants* $b_7 > 0$, $b_8 \geq 0$ *and* $\max\{1, 2/(3-\nu)\} < \gamma_{10} < \gamma_{11} < 2$

$$W(t,x) \geq b_7|x|^{\gamma_{10}} - b_8|x|^{\gamma_{11}}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W4″) $\lim_{|x|\to 0} \frac{W(t,x)}{|x|^2} = \infty$ *uniformly in* $t \in \mathbb{R}$.

Our main results are the following four theorems.

Theorem 1.1 *Assume that L and W satisfy* (L_ν), (W3), (W4) *and the following conditions*:

(W1) *There exist constants* $\max\{1, 2/(3-\nu)\} < \gamma_1 < \gamma_2 < 2$ *and* $a_1, a_2 \geq 0$ *such that*

$$\big|W(t,x)\big| \leq a_1|x|^{\gamma_1} + a_2|x|^{\gamma_2}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W2) *There exists a function* $\varphi \in C([0,+\infty), [0,+\infty))$ *such that*

$$\big|\nabla W(t,x)\big| \leq \varphi\big(|x|\big), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $\varphi(s) = O(s^{\gamma_3 - 1})$ *as* $s \to 0^+$, $\max\{1, 2/(3-\nu)\} < \gamma_3 < 2$.

Then system (1.1) *possesses at least one nontrivial homoclinic solution.*

Theorem 1.2 *Assume that L and W satisfy* (L_ν), (W1), (W2), (W3), (W4) *and the following condition*:

(W5) $W(t,-x) = W(t,x)$, $\forall (t,x) \in \mathbb{R} \times \mathbb{R}^N$.

Then system (1.1) *possesses infinitely many nontrivial homoclinic solutions.*

Theorem 1.3 *Assume that L and W satisfy* (L_ν), (W1), (W2), (W3′) *and* (W4). *Then system* (1.1) *possesses at least one nontrivial homoclinic solution.*

Theorem 1.4 *Assume that L and W satisfy* (L_ν), (W1), (W2), (W3′), (W4′) *and* (W5). *Then system* (1.1) *possesses infinitely many nontrivial homoclinic solutions.*

Corollary 1.5 *The conclusion of Theorem* 1.4 *also holds if* (W4′) *is replaced by* (W4″).

Remark 1.6 Our results can be applied to the following potential functions:

$$W(t,x) = \left(1 + \sin^2 t\right)\left(|x|^{5/4} - 3|x|^{3/2} + |x|^{7/4}\right) \tag{1.4}$$

and

$$W(t,x) = d_1|x|^{\tau_1} - \sum_{i=2}^{m-1} d_i|x|^{\tau_i} + d_m|x|^{\tau_m}, \tag{1.5}$$

where $m \geq 4$, $1 < \tau_1 < \tau_2 < \cdots < \tau_m < 2$ and $d_i > 0$ for $i = 1, 2, \ldots, m$. Note that the above potential functions are with indefinite signs, and hence Theorem A is not applicable. See Examples 4.1 and 4.2 in Section 4.

The remainder of this paper is organized as follows. In Section 2, we first define a Hilbert space E and describe its space structure. Then we state the critical point theorems needed for the proofs of our main results. The proofs of our main results are given in Section 3. Some examples to illustrate our results are given in Section 4.

Throughout this paper, we denote the norm of $L^p(\mathbb{R}, \mathbb{R}^N)$ by $\|u\|_p = (\int_{\mathbb{R}} |u|^s \, dt)^{1/p}$ for $p \geq 1$, and positive constants, possibly different in different places, by $C_1, C_2, \ldots$.

2 Preliminaries

In this section, we first make the following weaker assumption on $L(t)$:

(L) *The smallest eigenvalue of* $L(t) \to +\infty$ *as* $|t| \to +\infty$, *i.e.*,

$$\lim_{|t|\to+\infty}\left[\inf_{|x|=1}\left(L(t)x, x\right)\right] = +\infty.$$

In order to establish our existence results via the critical point theory, we first describe some properties of the space on which the variational functional associated with (1.1) is defined.

In what follows $L(t)$ is assumed to satisfy assumption (L). We denote by I_N the identity matrix of order N, I the identity operator. Let $\{\mathcal{E}(\lambda) : -\infty < \lambda < +\infty\}$ and $|\mathcal{A}|$ be the spectral family and the absolute value of $\mathcal{A}$, respectively, and $|\mathcal{A}|^{1/2}$ be the square root of $|\mathcal{A}|$. Set $U = I - \mathcal{E}(0) - \mathcal{E}(0-)$. Then U commutes with $\mathcal{A}$, $|\mathcal{A}|$ and $|\mathcal{A}|^{1/2}$, and $\mathcal{A} = U|\mathcal{A}|$ is the polar decomposition of $\mathcal{A}$ (see [22, 23]). Let $E = \mathfrak{D}(|\mathcal{A}|^{1/2})$, the domain of $|\mathcal{A}|^{1/2}$, and define on E the inner product

$$(u, v)_0 = \left(|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v\right)_2 + (u, v)_2, \quad \forall u, v \in E$$

and the norm

$$\|u\|_0 = \sqrt{(u, u)_0}, \quad \forall u \in E,$$

where, as usual, $(\cdot, \cdot)_2$ denotes the inner product of L^2. Then E is a Hilbert space. Clearly, $C_0^\infty \equiv C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ is dense in E.

By (L), $L(t)$ is bounded from below and so there is $l_0 > 0$ such that

$$l(t) + l_0 \geq 1, \quad \forall t \in \mathbb{R}, \tag{2.1}$$

where, and in the sequel,

$$l(t) = \inf_{x\in\mathbb{R}^N,|x|=1} \bigl(L(t)x,x\bigr). \tag{2.2}$$

Set

$$E_* = \left\{u \in W^{1,2}\bigl(\mathbb{R},\mathbb{R}^N\bigr) : \int_{\mathbb{R}} \bigl[|\dot u|^2 + \bigl(\bigl(L(s)+l_0 I_N\bigr)u,u\bigr)\bigr]\,\mathrm{d}s < +\infty\right\},$$

$$(u,v)_* = \int_{\mathbb{R}} \bigl[(\dot u,\dot v) + \bigl(\bigl(L(s)+l_0 I_N\bigr)u,v\bigr)\bigr]\,\mathrm{d}s, \quad \forall u,v \in E_*$$

and

$$\|u\|_* = \left\{\int_{\mathbb{R}} \bigl[|\dot u|^2 + \bigl(\bigl(L(s)+l_0 I_N\bigr)u,u\bigr)\bigr]\,\mathrm{d}s\right\}^{1/2}, \quad \forall u \in E_*.$$

Then E_* is also a Hilbert space with the above inner product $(\cdot,\cdot)_*$ and the norm $\|\cdot\|_*$.

Lemma 2.1 ([15]) *For $u \in E_*$,*

$$\|u\|_\infty \le \frac{1}{\sqrt{2}}\|u\|_* = \frac{1}{\sqrt{2}}\left\{\int_{\mathbb{R}} \bigl[|\dot u|^2 + \bigl(\bigl(L(s)+l_0 I_N\bigr)u,u\bigr)\bigr]\,\mathrm{d}s\right\}^{1/2}, \tag{2.3}$$

$$\bigl|u(t)\bigr| \le \left\{\int_t^\infty \frac{1}{\sqrt{l(s)+l_0}} \bigl[|\dot u|^2 + \bigl(\bigl(L(s)+l_0 I_N\bigr)u,u\bigr)\bigr]\,\mathrm{d}s\right\}^{1/2}, \quad \forall t \in \mathbb{R} \tag{2.4}$$

and

$$\bigl|u(t)\bigr| \le \left\{\int_{-\infty}^t \frac{1}{\sqrt{l(s)+l_0}} \bigl[|\dot u|^2 + \bigl(\bigl(L(s)+l_0 I_N\bigr)u,u\bigr)\bigr]\,\mathrm{d}s\right\}^{1/2}, \quad \forall t \in \mathbb{R}. \tag{2.5}$$

Lemma 2.2 *Suppose that $L(t)$ satisfies* (L). *Then E is compactly embedded in $L^p(\mathbb{R},\mathbb{R}^N)$ for $2 \le p \le \infty$, and*

$$\|u\|_p^p \le 2^{(2-p)/2}\|u\|_*^p, \qquad \int_{|t|>T} \bigl|u(t)\bigr|^p\,\mathrm{d}t \le \frac{2^{(2-p)/2}}{\min_{|s|\ge T}[l(s)+l_0]}\|u\|_*^p, \quad \forall T>0. \tag{2.6}$$

Proof In fact, the first part of Lemma 2.2 was proved in [21]. Here, we give the proof of the second part. From (2.1), (2.2) and (2.3), we have

$$\|u\|_p^p \le \|u\|_\infty^{p-2} \int_{\mathbb{R}} \bigl|u(t)\bigr|^2\,\mathrm{d}t \le \|u\|_\infty^{p-2} \int_{\mathbb{R}} \bigl(\bigl(L(t)+l_0 I_N\bigr)u,u\bigr)\,\mathrm{d}t \le 2^{(2-p)/2}\|u\|_*^p \tag{2.7}$$

and

$$\begin{aligned}
\int_{|t|>T} \bigl|u(t)\bigr|^p\,\mathrm{d}t &\le \|u\|_\infty^{p-2} \int_{|t|>T} \bigl|u(t)\bigr|^2\,\mathrm{d}t \\
&\le \|u\|_\infty^{p-2} \int_{|t|>T} \frac{((L(t)+l_0 I_N)u,u)}{l(t)+l_0}\,\mathrm{d}t \\
&\le \frac{\|u\|_\infty^{p-2}}{\min_{|s|\ge T}[l(s)+l_0]}\|u\|_*^2 \le \frac{2^{(2-p)/2}}{\min_{|s|\ge T}[l(s)+l_0]}\|u\|_*^p.
\end{aligned}$$

□

By (L), there exists a constant $\alpha \in \mathbb{R}$ such that

$$\big(L(t)x, x\big) > \alpha |x|^2, \quad \forall x \in \mathbb{R}^N \setminus \{0\}. \tag{2.8}$$

Analogous to the proof of [24], Lemma 2.4, we can prove the following lemma by using Lemma 2.2.

Lemma 2.3 *Suppose that $L(t)$ satisfies* (L). *Let*

$$E^- = \mathcal{E}(0-)E, \qquad E^0 = \big[\mathcal{E}(0) - \mathcal{E}(0-)\big]E, \qquad E^+ = \big[\mathcal{E}(+\infty) - \mathcal{E}(0)\big]E. \tag{2.9}$$

Then $E = E^- \oplus E^0 \oplus E^+$, and E^-, E^0 and E^+ are orthogonal with respect to the inner products $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_2$ on E. Furthermore, the following hold:

$$\dim\big(\mathcal{E}(M)E\big) < +\infty, \quad \forall M \geq 0, \tag{2.10}$$

$$E^0 = \operatorname{Ker}(\mathcal{A}), \qquad \mathcal{A}u^- = -|\mathcal{A}|u^-, \qquad \mathcal{A}u^+ = |\mathcal{A}|u^+, \quad \forall u \in \mathfrak{D}(\mathcal{A}) \tag{2.11}$$

and

$$u = u^- + u^0 + u^+, \quad \forall u \in E, \tag{2.12}$$

where

$$\begin{aligned} &u^- = \mathcal{E}(0-)u \in E^-, \qquad u^0 = \big[\mathcal{E}(0) - \mathcal{E}(0-)\big]u \in E^0, \\ &u^+ = \big[\mathcal{E}(+\infty) - \mathcal{E}(0)\big]u \in E^+. \end{aligned} \tag{2.13}$$

In view of Lemma 2.3, we introduce on E the following inner product:

$$(u,v) = \big(|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v\big)_2 + \big(u^0, v^0\big)_2$$

and the norm

$$\|u\|^2 = (u,u) = \big\||\mathcal{A}|^{1/2}u\big\|_2^2 + \big\|u^0\big\|_2^2,$$

where $u = u^- + u^0 + u^+$, $v = v^- + v^0 + v^+ \in E^- \oplus E^0 \oplus E^+ = E$. Then it is easy to check the following lemma.

Lemma 2.4 *Suppose that $L(t)$ satisfies* (L). *Then E^-, E^0 and E^+ are orthogonal with respect to the inner product $(\cdot,\cdot)$ on E.*

Analogous to the proof of [24], Lemma 2.1, Lemma 2.6, we can prove the following lemma.

Lemma 2.5 *Suppose that $L(t)$ satisfies* (L). *Then the norms $\|\cdot\|_0$, $\|\cdot\|_*$ and $\|\cdot\|$ on E are equivalent. Hence, there exists $\beta > 0$ such that*

$$\|u\|_* \leq \beta \|u\|, \quad \forall u \in E. \tag{2.14}$$

By virtue of (L_ν), there exist two constants $T_0 > 0$ and $M_0 > 0$ such that

$$|t|^{\nu-2} l(t) = |t|^{\nu-2} \inf_{|x|=1} \bigl(L(t)x, x\bigr) \geq M_0, \quad \forall |t| \geq T_0,$$

which implies

$$|t|^{\nu-2} \bigl(L(t)x, x\bigr) \geq M_0 |x|^2, \quad \forall |t| \geq T_0, x \in \mathbb{R}^N. \tag{2.15}$$

Lemma 2.6 *Suppose that $L(t)$ satisfies* (L_ν). *Then, for $1 \leq p \in (2/(3-\nu), 2)$, E is compactly embedded in $L^p(\mathbb{R}, \mathbb{R}^N)$; moreover,*

$$\int_{|t|\geq T} \bigl|u(t)\bigr|^p \,\mathrm{d}t \leq \frac{K(p)}{T^\kappa} \|u\|_*^p, \quad \forall u \in E, T \geq T_0 \tag{2.16}$$

and

$$\|u\|_p^p \leq \biggl[\biggl(\int_{|t|\leq T} \bigl[l(t) + l_0\bigr]^{-p/(2-p)} \,\mathrm{d}t \biggr)^{1-\frac{p}{2}} + \frac{K(p)}{T^\kappa} \biggr] \|u\|_*^p, \quad \forall u \in E, T \geq T_0, \tag{2.17}$$

where

$$\kappa = \frac{(3-\nu)p - 2}{2} > 0, \qquad K(p) = \biggl[\frac{2(2-p)}{(3-\nu)p-2} \biggr]^{1-\frac{p}{2}} M_0^{-p/2}. \tag{2.18}$$

Proof For $1 \leq p \in (2/(3-\nu), 2)$, we set $r = [(3-\nu)p - 2]/(2-p)$. Then $r > 0$. For $u \in E$ and $T \geq T_0$, it follows from (2.15) and the Hölder inequality that

$$\begin{aligned} \int_{|t|\geq T} \bigl|u(t)\bigr|^p \,\mathrm{d}t &\leq \biggl(\int_{|t|\geq T} |t|^{-(2-\nu)p/(2-p)} \,\mathrm{d}t \biggr)^{1-\frac{p}{2}} \biggl(\int_{|t|\geq T} |t|^{2-\nu} \bigl|u(t)\bigr|^2 \,\mathrm{d}t \biggr)^{\frac{p}{2}} \\ &\leq \biggl(\frac{2}{rT^r} \biggr)^{1-\frac{p}{2}} \biggl[\frac{1}{M_0} \int_{|t|\geq T} \bigl(L(t)u(t), u(t)\bigr) \,\mathrm{d}t \biggr]^{\frac{p}{2}} \\ &\leq \frac{2^{(2-p)/2}}{M_0^{p/2} r^{(2-p)/2} T^\kappa} \|u\|_*^p = \frac{K(p)}{T^\kappa} \|u\|_*^p. \end{aligned}$$

From (2.2) and (2.16), one has

$$\begin{aligned} \|u\|_p^p &= \int_{|t|\leq T} \bigl|u(t)\bigr|^p \,\mathrm{d}t + \int_{|t|>T} \bigl|u(t)\bigr|^p \,\mathrm{d}t \\ &\leq \biggl(\int_{|t|\leq T} \bigl[l(t) + l_0\bigr]^{-p/(2-p)} \,\mathrm{d}t \biggr)^{1-\frac{p}{2}} \biggl(\int_{|t|\leq T} \bigl[l(t) + l_0\bigr] \bigl|u(t)\bigr|^2 \,\mathrm{d}t \biggr)^{\frac{p}{2}} + \frac{K(p)}{T^\kappa} \|u\|_*^p \\ &\leq \biggl(\int_{|t|\leq T} \bigl[l(t) + l_0\bigr]^{-p/(2-p)} \,\mathrm{d}t \biggr)^{1-\frac{p}{2}} \|u\|_*^p + \frac{K(p)}{T^\kappa} \|u\|_*^p. \end{aligned}$$

For $1 \leq p \in (2/(3-\nu), 2)$, applying (2.16), we can prove that E is compactly embedded in $L^p(\mathbb{R}, \mathbb{R}^N)$ by a standard argument. □

Lemma 2.7 ([25]) *Let X be real Banach space, Q and S be two closed subsets of X, and S and ∂Q link. Suppose that $f \in C^1(X, \mathbb{R})$ satisfy the (PS)-condition, and that*

(i) *there exist two constants* $\eta > \zeta$ *such that* $\sup_{x\in\partial Q} f(x) \le \zeta < \eta \le \inf_{x\in S} f(x)$;
(ii) $\sup_{x\in Q} f(x) < +\infty$.
Then f *possesses a critical value* $c \ge \eta$.

Lemma 2.8 ([21], Lemma 2.4) *Let* X *be an infinite dimensional Banach space and* $f \in C^1(X,\mathbb{R})$ *be even, satisfy the* (PS)*-condition, and* $f(0) = 0$. *If* $X = X_1 \oplus X_2$, *where* X_1 *is finite dimensional, and* f *satisfies*
(i) f *is bounded from below on* X_2;
(ii) *for each finite dimensional subspace* $\tilde{X} \subset X$, *there are positive constants* $\rho = \rho(\tilde{X})$ *and* $\sigma = \sigma(\tilde{X})$ *such that* $f|_{B_\rho\cap\tilde{X}} \le 0$ *and* $f|_{\partial B_\rho\cap\tilde{X}} \le -\sigma$, *where* $B_\rho = \{x \in X : \|x\| = \rho\}$.
Then f *possesses infinitely many nontrivial critical points.*

Lemma 2.9 ([26]) *Let* X *be a real Banach space and* $f \in C^1(X,\mathbb{R})$ *satisfy the* (PS)*-condition. If* f *is bounded from below, then* $c = \inf_X f$ *is a critical value of* f.

3 Proofs of theorems

Lemma 3.1 *Assume that* (L_ν) *and* (W1) *hold. Then, for* $u \in E$,

$$\int_{\mathbb{R}} \bigl|W(t,u)\bigr|\,\mathrm{d}t \le \phi_1(T)\|u\|^{\gamma_1} + \phi_2(T)\|u\|^{\gamma_2}, \quad T \ge T_0, \tag{3.1}$$

where

$$\phi_1(T) = a_1\beta^{\gamma_1}\biggl[\biggl(\int_{|t|\le T}\bigl[l(t)+l_0\bigr]^{-\gamma_1/(2-\gamma_1)}\,\mathrm{d}t\biggr)^{1-\frac{\gamma_1}{2}} + \frac{K(\gamma_1)}{T^{\kappa_1}}\biggr], \tag{3.2}$$

$$\phi_2(T) = a_2\beta^{\gamma_2}\biggl[\biggl(\int_{|t|\le T}\bigl[l(t)+l_0\bigr]^{-\gamma_2/(2-\gamma_2)}\,\mathrm{d}t\biggr)^{1-\frac{\gamma_2}{2}} + \frac{K(\gamma_2)}{T^{\kappa_2}}\biggr] \tag{3.3}$$

and

$$\kappa_1 = \frac{(3-\nu)\gamma_1 - 2}{2}, \qquad \kappa_2 = \frac{(3-\nu)\gamma_2 - 2}{2}. \tag{3.4}$$

Proof For $T \ge T_0$, it follows from (2.14), (2.17), (3.2), (3.3) and (W1) that

$$\begin{aligned}
\int_{\mathbb{R}} \bigl|W(t,u)\bigr|\,\mathrm{d}t &\le a_1\int_{\mathbb{R}}\bigl|u(t)\bigr|^{\gamma_1}\,\mathrm{d}t + a_2\int_{\mathbb{R}}\bigl|u(t)\bigr|^{\gamma_2}\,\mathrm{d}t \\
&\le a_1\biggl[\biggl(\int_{|t|\le T}\bigl[l(t)+l_0\bigr]^{-\gamma_1/(2-\gamma_1)}\,\mathrm{d}t\biggr)^{1-\frac{\gamma_1}{2}} + \frac{K(\gamma_1)}{T^{\kappa_1}}\biggr]\|u\|_*^{\gamma_1} \\
&\quad + a_2\biggl[\biggl(\int_{|t|\le T}\bigl[l(t)+l_0\bigr]^{-\gamma_2/(2-\gamma_2)}\,\mathrm{d}t\biggr)^{1-\frac{\gamma_2}{2}} + \frac{K(\gamma_2)}{T^{\kappa_2}}\biggr]\|u\|_*^{\gamma_2} \\
&\le a_1\beta^{\gamma_1}\biggl[\biggl(\int_{|t|\le T}\bigl[l(t)+l_0\bigr]^{-\gamma_1/(2-\gamma_1)}\,\mathrm{d}t\biggr)^{1-\frac{\gamma_1}{2}} + \frac{K(\gamma_1)}{T^{\kappa_1}}\biggr]\|u\|^{\gamma_1} \\
&\quad + a_2\beta^{\gamma_2}\biggl[\biggl(\int_{|t|\le T}\bigl[l(t)+l_0\bigr]^{-\gamma_2/(2-\gamma_2)}\,\mathrm{d}t\biggr)^{1-\frac{\gamma_2}{2}} + \frac{K(\gamma_2)}{T^{\kappa_2}}\biggr]\|u\|^{\gamma_2} \\
&= \phi_1(T)\|u\|^{\gamma_1} + \phi_2(T)\|u\|^{\gamma_2}.
\end{aligned}$$

□

Analogous to the proof of [17], Lemma 2.2, we can prove the following lemma.

Lemma 3.2 *Assume that* (L_ν), (W1) *and* (W2) *hold. Then the functional* $f : E \to \mathbb{R}$ *defined by*

$$\Phi(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) - \int_{\mathbb{R}} W(t,u)\,\mathrm{d}t \tag{3.5}$$

is well defined and of class $C^1(E,\mathbb{R})$ *and*

$$\big\langle \Phi'(u), v\big\rangle = \|u^+\|^2 - \|u^-\|^2 - \int_{\mathbb{R}} \big(\nabla W(t,u), v\big)\,\mathrm{d}t. \tag{3.6}$$

Furthermore, the critical points of Φ *in* E *are classical solutions of* (1.1) *with* $u(\pm\infty) = 0$.

Proof of Theorem 1.1 In view of Lemma 3.2, $\Phi \in C^1(E,\mathbb{R})$. In what follows, we divide the rest of the proof of Theorem 1.1 into four steps.

Step 1. Φ satisfies the (PS)-condition.

Assume that $\{u_n\}_{n\in\mathbb{N}} \subset E$ is a (PS)-sequence: $\{\Phi(u_n)\}_{k\in\mathbb{N}}$ is bounded and $\|\Phi'(u_n)\| \to 0$ as $n \to +\infty$. In the sequel we write for any $u \in E$

$$u^1(t) = \begin{cases} u(t) & \text{if } |u(t)| < 1, \\ 0 & \text{if } |u(t)| \geq 1; \end{cases} \qquad u^2(t) = \begin{cases} 0 & \text{if } |u(t)| < 1, \\ u(t) & \text{if } |u(t)| \geq 1. \end{cases} \tag{3.7}$$

Then, by (3.5), (3.6), (3.7) and (W3), we get

$$\begin{aligned} \big\langle \Phi'(u_n), u_n\big\rangle - 2\Phi(u_n) &= \int_{\mathbb{R}} \big[2W(t,u_n) - \big(\nabla W(t,u_n), u_n\big)\big]\,\mathrm{d}t \\ &\geq b_1 \int_{\mathbb{R}} \big|u_n^1\big|^{\gamma_4}\,\mathrm{d}t + b_2 \int_{\mathbb{R}} \big|u_n^2\big|^{\gamma_5}\,\mathrm{d}t \\ &= b_1 \big\|u_n^1\big\|_{\gamma_4}^{\gamma_4} + b_2 \big\|u_n^2\big\|_{\gamma_5}^{\gamma_5}. \end{aligned}$$

It follows that there exists a constant $C_1 > 0$ such that

$$b_1 \big\|u_n^1\big\|_{\gamma_4}^{\gamma_4} + b_2 \big\|u_n^2\big\|_{\gamma_5}^{\gamma_5} \leq C_1\big(1 + \|u_n\|\big). \tag{3.8}$$

Since $\dim(E^- \oplus E^0) < +\infty$, there exists a constant $C_2 > 0$ such that

$$\begin{aligned} \big\|u_n^- + u_n^0\big\|_2^2 &= \big(u_n^- + u_n^0, u_n\big)_2 \\ &= \big(u_n^- + u_n^0, u_n^1\big)_2 + \big(u_n^- + u_n^0, u_n^2\big)_2 \\ &\leq \big\|u_n^- + u_n^0\big\|_{\gamma_4'} \big\|u_n^1\big\|_{\gamma_4} + \big\|u_n^- + u_n^0\big\|_{\gamma_5'} \big\|u_n^2\big\|_{\gamma_5} \\ &\leq C_2 \big\|u_n^- + u_n^0\big\|_2 \big(\big\|u_n^1\big\|_{\gamma_4} + \big\|u_n^2\big\|_{\gamma_5}\big), \end{aligned} \tag{3.9}$$

where $\gamma_4' = \gamma_4/(\gamma_4 - 1)$ and $\gamma_5' = \gamma_5/(\gamma_5 - 1)$. Combining (3.8) with (3.9), one has

$$\big\|u_n^- + u_n^0\big\|^2 \leq C_3 \big\|u_n^- + u_n^0\big\|_2^2 \leq C_4\big(1 + \|u_n\|^{2/\gamma_4} + \|u_n\|^{2/\gamma_5}\big). \tag{3.10}$$

Choose $T_2 > T_0$, it follows from (3.1) that

$$\int_{\mathbb{R}} W(t,u_n)\,\mathrm{d}t \le \phi_1(T_2)\|u_n\|^{\gamma_1} + \phi_2(T_2)\|u_n\|^{\gamma_2}. \tag{3.11}$$

From (3.5), (3.10) and (3.11), we obtain

$$\begin{aligned}
\|u_n\|^2 &= \|u_n^- + u_n^0\|^2 + \|u_n^+\|^2 \\
&= \|u_n^- + u_n^0\|^2 + 2\Phi(u_n) + \|u_n^-\|^2 + 2\int_{\mathbb{R}} W(t,u_n)\,\mathrm{d}t \\
&\le 2C_4\bigl(1 + \|u_n\|^{2/\gamma_4} + \|u_n\|^{2/\gamma_5}\bigr) + 2\Phi(u_n) \\
&\quad + 2\phi_1(T_2)\|u_n\|^{\gamma_1} + 2\phi_2(T_2)\|u_n\|^{\gamma_2} \\
&\le C_5\bigl(1 + \|u_n\|^{\gamma_1} + \|u_n\|^{\gamma_2} + \|u_n\|^{2/\gamma_4} + \|u_n\|^{2/\gamma_5}\bigr).
\end{aligned} \tag{3.12}$$

Since $1 < \gamma_1 < \gamma_2 < 2$, $1 < \gamma_5 \le \gamma_4 < 2$, it follows from (3.12) that $\{\|u_n\|\}$ is bounded, and so $\{\|u_n\|_*\}$ is bounded. Choose a constant $\Lambda > 0$ such that

$$\|u_n\|_\infty \le \frac{1}{\sqrt{2}}\|u_n\|_* \le \Lambda, \quad n \in \mathbb{N}. \tag{3.13}$$

Passing to a subsequence if necessary, it can be assumed that $u_n \rightharpoonup u_0$ in E. Hence $u_n \to u_0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$; moreover, it is easy to verify that $\{u_n(t)\}$ converges to $u_0(t)$ point-wise for all $t \in \mathbb{R}$. Hence, (3.13) yields that $\|u_0\|_\infty \le \Lambda$. By (W2), there exists $M_3 > 0$ such that

$$\nabla W(t,x) \le M_3|x|^{\gamma_3-1}, \quad \forall x \in \mathbb{R}^N, |x| \le \Lambda. \tag{3.14}$$

For any given number $\varepsilon > 0$, we can choose $T_3 > T_0$ such that

$$\frac{K(\gamma_3)[(\sqrt{2}\Lambda)^{\gamma_3} + \|u_0\|_*^{\gamma_3}]}{T_3^{\kappa_3}} < \varepsilon. \tag{3.15}$$

Hence, from (2.16), (3.13), (3.14) and (3.15) we have that

$$\begin{aligned}
\int_{|t|>T_3} \bigl|\nabla W(t,u_n) - \nabla W(t,u_0)\bigr||u_n - u_0|\,\mathrm{d}t &\le 2M_3 \int_{|t|>T_3} \bigl(|u_k(t)|^{\gamma_3} + |u_0(t)|^{\gamma_3}\bigr)dt \\
&\le \frac{2M_3K(\gamma_3)}{T_3^{\kappa_3}}\bigl(\|u_k\|_*^{\gamma_3} + \|u_0\|_*^{\gamma_3}\bigr) \\
&\le \frac{2M_3K(\gamma_3)}{T_3^{\kappa_3}}\bigl[(\sqrt{2}\Lambda)^{\gamma_3} + \|u_0\|_*^{\gamma_3}\bigr] \\
&\le 2M_3\varepsilon, \quad n \in \mathbb{N}.
\end{aligned} \tag{3.16}$$

On the other hand, since $u_n \to u_0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$, it follows from the continuity of $\nabla W(t,x)$ that

$$\int_{-T_3}^{T_3} \bigl|\nabla W(t,u_n) - \nabla W(t,u_0)\bigr||u_n - u_0|\,\mathrm{d}t = o(1). \tag{3.17}$$

Since ε is arbitrary, combining (3.16) with (3.17) we get

$$\int_{\mathbb{R}} \big(\nabla W(t,u_n) - \nabla W(t,u_0), u_n - u_0\big)\,\mathrm{d}t = o(1). \tag{3.18}$$

It follows from (3.6) that

$$\begin{aligned} \big\langle \Phi'(u_n) - \Phi'(u_0), u_n - u_0\big\rangle = {} & \|u_n^+ - u_0^+\|^2 - \|u_n^- - u_0^-\|^2 \\ & - \int_{\mathbb{R}} \big(\nabla W(t,u_n) - \nabla W(t,u_0), u_n - u_0\big)\,\mathrm{d}t. \end{aligned} \tag{3.19}$$

Since $\langle \Phi'(u_n) - \Phi'(u_0), u_n - u_0\rangle = o(1)$, it follows from (3.18) and (3.19) that

$$\|u_n^+ - u_0^+\|^2 - \|u_n^- - u_0^-\|^2 = o(1). \tag{3.20}$$

Since $u_n \rightharpoonup u_0$ in E and $\dim(E^- \oplus E^0) < +\infty$, it follows that

$$\|u_n^0 - u_0^0\|^2 + \|u_n^- - u_0^-\|^2 = o(1). \tag{3.21}$$

Combining (3.20) with (3.21), we have

$$\|u_n - u_0\|^2 = \|u_n^+ - u_0^+\|^2 + \|u_n^0 - u_0^0\|^2 + \|u_n^- - u_0^-\|^2 = o(1).$$

Hence, Φ satisfies the (PS)-condition.

Step 2. $\Phi(u) \to +\infty$ as $\|u\| \to +\infty$ and $u \in E^+$.

It follows from (3.1) that

$$\int_{\mathbb{R}} W(t,u)\,\mathrm{d}t \le \phi_1(T_2)\|u\|^{\gamma_1} + \phi_2(T_2)\|u\|^{\gamma_2}, \quad \forall u \in E. \tag{3.22}$$

Hence, for $u \in E^+$, it follows from (3.5) and (3.22) that

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t,u)\,\mathrm{d}t \\ &\ge \frac{1}{2}\|u\|^2 - \phi_1(T_2)\|u\|^{\gamma_1} - \phi_2(T_2)\|u\|^{\gamma_2} \to +\infty \end{aligned}$$

as $\|u\| \to +\infty$ and $u \in E^+$, since $1 < \gamma_1 < \gamma_2 < 2$.

Step 3. Taking $e \in E^+$ with $\|e\| = 1$, there exist $s_0 \in (0,1)$ and $\sigma_0 > 0$ such that

$$\Phi(u) \le -\sigma_0, \quad \forall u \in S_e := E^- \oplus E^0 \oplus s_0 e. \tag{3.23}$$

Set $X = E^- \oplus E^0 \oplus \mathbb{R}e$. For $u = u^- + u^0 + se \in X$, by (3.5), (3.7) and (W4),

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\big(\|se\|^2 - \|u^-\|^2\big) - \int_{\mathbb{R}} W(t,u)\,\mathrm{d}t \\ &\le \frac{s^2}{2} - b_3\|u^1\|_{\gamma_6}^{\gamma_6} - b_4\|u^2\|_{\gamma_7}^{\gamma_7}. \end{aligned} \tag{3.24}$$

On the other hand, one sees that

$$s^2\|e\|_2^2 = (se, se)_2 = (se, u)_2 = \left(se, u^1\right)_2 + \left(se, u^2\right)_2 \le |s|\left(\|e\|_{\gamma_6'}\left\|u^1\right\|_{\gamma_6} + \|e\|_{\gamma_7'}\left\|u^2\right\|_{\gamma_7}\right),$$

where $\gamma_6' = \gamma_6/(\gamma_6 - 1) > \gamma_6$ and $\gamma_7' = \gamma_7/(\gamma_7 - 1) > \gamma_7$. Hence,

$$s \le C_6\left(\left\|u^1\right\|_{\gamma_6} + \min\left\{\left\|u^2\right\|_{\gamma_7}, 1\right\}\right), \quad \forall s \in (0,1). \tag{3.25}$$

Combining (3.24) with (3.25), we have

$$\begin{aligned}
\Phi(u) &\le \frac{s^2}{2} - b_3\left\|u^1\right\|_{\gamma_6}^{\gamma_6} - b_4\left\|u^2\right\|_{\gamma_7}^{\gamma_7} \\
&\le \frac{s^2}{2} - \min\{b_3, b_4\}\left[\left\|u^1\right\|_{\gamma_6}^{\gamma_6} + \left(\min\left\{\left\|u^2\right\|_{\gamma_7}, 1\right\}\right)^{\gamma_7}\right] \\
&\le \frac{s^2}{2} - 2^{1-\gamma_6}\min\{b_3, b_4\}\left[\left\|u^1\right\|_{\gamma_6} + \left(\min\left\{\left\|u^2\right\|_{\gamma_7}, 1\right\}\right)\right]^{\gamma_6} \\
&\le \frac{s^2}{2} - 2^{1-\gamma_6}\min\{b_3, b_4\}C_6^{-\gamma_6}s^{\gamma_6} \\
&= \frac{s^2}{2} - C_7 s^{\gamma_6}, \quad \forall u = u^- + u^0 + se \in X, s \in (0,1),
\end{aligned}$$

which implies that there exist $s_0 \in (0,1)$ and $\sigma_0 > 0$ such that (3.23) holds.

Step 4. If $E^- \oplus E^0 = \{0\}$, then Lemmas 2.9 and 3.2, Steps 1-3 imply that Φ has a minimum (< 0) which yields a homoclinic solution for system (1.1).

If $E^- \oplus E^0 \ne \{0\}$, by Step 2, one can take $C_8 > 0$ and $r > s_0$ large such that

$$\Phi(u) \ge -C_8, \quad \forall u \in E^+$$

and

$$\Phi(u) \ge 0, \quad \forall u \in E^+ \text{ with } \|u\| \ge r.$$

Let $Q = B_r \cap E^+$. Since S_e and ∂Q link, by Lemma 2.7, $-\Phi$ has a critical point $u^* \in E$ with $\Phi(u^*) \le -\sigma_0$, which is a nontrivial homoclinic solution of system (1.1). □

Proof of Theorem 1.2 Set $X = E$, $X_1 = E^- \oplus E^0$ and $X_2 = E^+$. In view of Lemma 3.2 and Steps 1 and 2 in the proof of Theorem 1.1, $X = X_1 \oplus X_2$, $\dim X_1 < +\infty$, $\Phi \in C^1(X, \mathbb{R})$, Φ satisfies the (PS)-condition and is bounded from below on X_2. Obviously, (W1) and (W5) imply $\Phi(0) = 0$ and Φ is even. Next, we prove that assumption (ii) in Lemma 2.8 holds.

Let $\tilde{X} \subset X$ be any finite dimensional subspace. Then there exist constants $c_0 = c(\tilde{X}) > 0$ and $c_* = c(\tilde{X}) > 0$ such that

$$c_0\|u\| \le \|u\|_{\gamma_6}, \|u\|_{\gamma_7}, \|u\|_\infty \le c_*\|u\|, \quad \forall u \in \tilde{X}. \tag{3.26}$$

Since $\gamma_6 \ge \gamma_7$, it follows from (3.7) and (3.26) that

$$\begin{aligned}
\|u\|_{\gamma_6}^{\gamma_6} &= \left\|u^1\right\|_{\gamma_6}^{\gamma_6} + \left\|u^2\right\|_{\gamma_6}^{\gamma_6} \le \left\|u^1\right\|_{\gamma_6}^{\gamma_6} + \|u\|_\infty^{\gamma_6-\gamma_7}\left\|u^2\right\|_{\gamma_7}^{\gamma_7} \\
&\le \left\|u^1\right\|_{\gamma_6}^{\gamma_6} + \left\|u^2\right\|_{\gamma_7}^{\gamma_7}, \quad \forall u \in \tilde{X}, c_*\|u\| < 1.
\end{aligned} \tag{3.27}$$

From (3.5), (3.7), (3.26), (3.27) and (W4), one has

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \int_{\mathbb{R}} W(t,u)\,\mathrm{d}t \\ &\leq \frac{1}{2}\|u\|^2 - b_3\|u^1\|_{\gamma_6}^{\gamma_6} - b_4\|u^2\|_{\gamma_7}^{\gamma_7} \\ &\leq \frac{1}{2}\|u\|^2 - \min\{b_3,b_4\}\|u\|_{\gamma_6}^{\gamma_6} \\ &\leq \frac{1}{2}\|u\|^2 - c_0^{\gamma_6}\min\{b_3,b_4\}\|u\|^{\gamma_6}, \quad \forall u \in \tilde{X}, c_*\|u\| < 1.\end{aligned}$$

Since $1 < \gamma_6 < 2$, the above implies that there exist $\rho = \rho(b_3,b_4,c_0) = \rho(\tilde{X}) \in (0, c_*^{-1})$ and $\sigma = \sigma(b_3,b_4,c_0) = \sigma(\tilde{X}) > 0$ such that

$$\Phi(u) \leq 0, \quad \forall u \in B_\rho \cap \tilde{X}; \qquad \Phi(u) \leq -\sigma, \quad \forall u \in \partial B_\rho \cap \tilde{X}.$$

Hence assumption (ii) in Lemma 2.8 holds. By Lemma 2.8, Φ has infinitely many (pairs) critical points which are homoclinic solutions for system (1.1). □

Proof of Theorem 1.3 In the proof of Theorem 1.1, assumption (W3) is used only in Step 1 to prove that a (PS)-sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ is bounded. Therefore, we only prove that any (PS)-sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ is also bounded by using (W3′) instead of (W3). From (3.5), (3.6) and (W3′), we have

$$\begin{aligned}\langle\Phi'(u_n),u_n\rangle - 2\Phi(u_n) &= \int_{\mathbb{R}}\left[2W(t,u_n) - \left(\nabla W(t,u_n),u_n\right)\right]\mathrm{d}t \\ &\geq b_5\int_{\mathbb{R}}|u_n(t)|^{\gamma_8}\,\mathrm{d}t - b_6\int_{\mathbb{R}}|u_n|^{\gamma_9}\,\mathrm{d}t \\ &= b_5\|u_n\|_{\gamma_8}^{\gamma_8} - b_6\|u_n\|_{\gamma_9}^{\gamma_9}.\end{aligned}$$

It follows that there exists a constant $C_9 > 0$ such that

$$b_5\|u_n\|_{\gamma_8}^{\gamma_8} - b_6\|u_n\|_{\gamma_9}^{\gamma_9} \leq C_9\left(1 + \|u_n\|\right). \tag{3.28}$$

Since $\dim(E^- \oplus E^0) < +\infty$, there exists a constant $C_{10} > 0$ such that

$$\|u_n^- + u_n^0\|_2^2 = \left(u_n^- + u_n^0, u_n\right)_2 \leq \|u_n^- + u_n^0\|_{\gamma_8'}\|u_n\|_{\gamma_8} \leq C_{10}\|u_n^- + u_n^0\|_2\|u_n\|_{\gamma_8}, \tag{3.29}$$

where $\gamma_8' = \gamma_8/(\gamma_8 - 1)$. Combining (3.28) with (3.29), one has

$$\|u_n^- + u_n^0\|^2 \leq C_{11}\|u_n^- + u_n^0\|_2^2 \leq C_{12}\left(1 + \|u_n\|^{2/\gamma_8} + \|u_n\|^{2\gamma_9/\gamma_8}\right). \tag{3.30}$$

From (3.5), (3.11) and (3.30), we obtain

$$\begin{aligned}\|u_n\|^2 &= \|u_n^- + u_n^0\|^2 + \|u_n^+\|^2 \\ &= \|u_n^- + u_n^0\|^2 + 2\Phi(u_n) + \|u_n^-\|^2 + 2\int_{\mathbb{R}} W(t,u_n)\,\mathrm{d}t\end{aligned}$$

$$\begin{aligned}&\leq 2C_{12}\bigl(1+\|u_n\|^{2/\gamma_8}+\|u_n\|^{2\gamma_9/\gamma_8}\bigr)+2\Phi(u_n)\\&\quad+2\phi_1(T_2)\|u_n\|^{\gamma_1}+2\phi_2(T_2)\|u_n\|^{\gamma_2}\\&\leq C_{13}\bigl(1+\|u_n\|^{\gamma_1}+\|u_n\|^{\gamma_2}+\|u_n\|^{2/\gamma_8}+\|u_n\|^{2\gamma_9/\gamma_8}\bigr).\end{aligned}$$

Since $1<\gamma_1<\gamma_2<2$, $1<\gamma_9<\gamma_8<2$, it follows that $\{\|u_n\|\}$ is bounded. The proof is complete. □

Proof of Theorem 1.4 Set $X=E$, $X_1=E^-\oplus E^0$ and $X_2=E^+$. In view of Lemma 3.2 and Steps 1 and 2 in the proof of Theorem 1.1, $X=X_1\oplus X_2$, $\dim X_1<+\infty$, $\Phi\in C^1(X,\mathbb{R})$, Φ satisfies the (PS)-condition and is bounded from below on X_2. Obviously, (W1) and (W5) imply $\Phi(0)=0$ and Φ is even. Next, we prove that assumption (ii) in Lemma 2.8 holds.

Let $\tilde{X}\subset X$ be any finite dimensional subspace. Then there exist constants $c_0=c(\tilde{X})>0$ and $c_*=c(\tilde{X})>0$ such that

$$c_0\|u\|\leq\|u\|_{\gamma_{10}},\|u\|_{\gamma_{11}}\leq c_*\|u\|,\quad\forall u\in\tilde{X}.\tag{3.31}$$

From (3.5), (3.31) and (W4′), one has

$$\begin{aligned}\Phi(u)&=\frac{1}{2}\bigl(\|u^+\|^2-\|u^-\|^2\bigr)-\int_{\mathbb{R}}W(t,u)\,\mathrm{d}t\\&\leq\frac{1}{2}\|u\|^2-b_7\|u\|_{L^{\gamma_{10}}}^{\gamma_{10}}+b_8\|u\|_{L^{\gamma_{11}}}^{\gamma_{11}}\\&\leq\frac{1}{2}\|u\|^2-b_7c_0^{\gamma_{10}}\|u\|^{\gamma_{10}}+b_8c_*^{\gamma_{11}}\|u\|^{\gamma_{11}},\quad\forall u\in\tilde{X}.\end{aligned}$$

Since $1<\gamma_{10}<\gamma_{11}<2$, the above implies that there exist $\rho=\rho(b_7,b_8,c_0)=\rho(\tilde{X})>0$ and $\sigma=\sigma(b_7,b_8,c_0)=\sigma(\tilde{X})>0$ such that

$$\Phi(u)\leq 0,\quad\forall u\in B_\rho\cap\tilde{X};\qquad\Phi(u)\leq-\sigma,\quad\forall u\in\partial B_\rho\cap\tilde{X}.$$

Hence assumption (ii) in Lemma 2.8 holds. By Lemma 2.8, Φ has infinitely many (pairs) critical points which are homoclinic solutions for system (1.1). □

In the proof of Theorem 1.4, (W4′) is used in the last part to verify assumption (ii) of Lemma 2.8. It is easy to see that it also holds by using (W4″) instead of (W4′). So we omit the proof of Corollary 1.5.

4 Examples

In this section, we give two examples to illustrate our results.

Example 4.1 In system (1.1), let $L(t)=(|t|^{4/5}-1)I_N$, and $W(t,x)$ be as in (1.4). Then $L(t)$ satisfies (L_ν) with $\nu=6/5$, and

$$\begin{aligned}&\nabla W(t,x)=\bigl(1+\sin^2 t\bigr)\biggl(\frac{5}{4}|x|^{-3/4}x-\frac{9}{2}|x|^{-1/2}x+\frac{7}{4}|x|^{-1/4}x\biggr),\\&\bigl|W(t,x)\bigr|\leq 5\bigl(|x|^{5/4}+|x|^{7/4}\bigr),\quad\forall(t,x)\in\mathbb{R}\times\mathbb{R}^N,\end{aligned}$$

$$\left|\nabla W(t,x)\right| \leq \frac{5|x|^{1/4} + 18|x|^{1/2} + 7|x|^{3/4}}{2}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$
$$2W(t,x) - \left(\nabla W(t,x), x\right) \geq \frac{1}{4}|x|^{7/4} - 3|x|^{3/2}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N$$

and

$$W(t,x) \geq |x|^{5/4} - 6|x|^{3/2}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Thus all conditions of Theorem 1.4 are satisfied with

$$\frac{5}{4} = \gamma_1 = \gamma_3 = \gamma_{10} < \gamma_9 = \gamma_{11} = \frac{3}{2} < \gamma_8 = \gamma_2 = \frac{7}{4}; \qquad a_1 = a_2 = 5;$$
$$b_5 = \frac{1}{4}, \qquad b_6 = 3, \qquad b_7 = 1, \qquad b_8 = 6; \qquad \varphi(s) = \frac{5s^{1/4} + 18s^{1/2} + 7s^{3/4}}{2}.$$

Hence, by Theorem 1.4, system (1.1) has infinitely many nontrivial homoclinic solutions.

Example 4.2 In system (1.1), let $L(t) = (|t|^{\varrho} - 1)I_N$, let $W(t,x)$ be as in (1.5). Set

$$\lambda_i = \frac{\tau_m - \tau_i}{\tau_m - \tau_1}, \qquad \mu_i = \frac{\tau_i - \tau_1}{\tau_{m-1} - \tau_1}, \qquad \theta_j = \frac{\tau_m - \tau_j}{\tau_m - \tau_2}, \quad i = 2, \ldots, m-1; j = 3, \ldots, m-1;$$
$$a_1 = d_1 + \sum_{i=2}^{m-1} \lambda_i d_i, \qquad a_2 = d_m + \sum_{i=2}^{m-1} (1 - \lambda_i) d_i;$$
$$b_6 = (2 - \tau_{m-1}) d_{m-1} + \sum_{i=3}^{m-2} (2 - \tau_i) \mu_i d_i \left[m(2 - \tau_i) d_i (1 - \mu_i)\right]^{(1-\mu_i)/\mu_i} |x|^{\tau_{m-1}}$$

and

$$b_8 = d_2 + \sum_{j=3}^{m-1} \theta_j d_j \left[m d_j (1 - \theta_j)\right]^{(1-\theta_j)/\theta_j}.$$

Note that

$$|x|^{\tau_i} \leq \lambda_i |x|^{\tau_1} + (1 - \lambda_i)|x|^{\tau_m}, \quad i = 2, \ldots, m-1,$$
$$|x|^{\tau_i} \leq \frac{1}{m(2 - \tau_i) d_i} |x|^{\tau_1} + \mu_i \left[m(2 - \tau_i) d_i (1 - \mu_i)\right]^{(1-\mu_i)/\mu_i} |x|^{\tau_{m-1}}, \quad i = 2, \ldots, m-2$$

and

$$|x|^{\tau_j} \leq \theta_j \left[m d_j (1 - \theta_j)\right]^{(1-\theta_j)/\theta_j} |x|^{\tau_2} + \frac{1}{m d_j} |x|^{\tau_m}, \quad j = 3, \ldots, m-1.$$

Then $L(t)$ satisfies (L_ν) with $\nu = 2 - \varrho < 3 - 2/\tau_1$, and

$$\left|W(t,x)\right| \leq a_1 |x|^{\tau_1} + a_2 |x|^{\tau_m}, \qquad \left|\nabla W(t,x)\right| \leq \sum_{i=1}^{m} \tau_i d_i |x|^{\tau_i - 1}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$
$$2W(t,x) - \left(\nabla W(t,x), x\right) \geq (2 - \tau_m) d_m |x|^{\tau_m} - b_6 |x|^{\tau_{m-1}}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N$$

and

$$W(t,x) \geq d_1|x|^{\tau_1} - b_8|x|^{\tau_2}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Theorem 1.4 applies with

$$\tau_1 = \gamma_1 = \gamma_3 = \gamma_{10} < \gamma_{11} = \tau_2 < \gamma_9 = \tau_{m-1} < \gamma_8 = \gamma_2 = \tau_m;$$

$$b_5 = (2 - \tau_m)d_m, \qquad b_7 = d_1; \qquad \varphi(s) = \sum_{i=1}^{m} \tau_i d_i s^{\tau_i - 1},$$

and system (1.1) has infinitely many nontrivial homoclinic solutions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results, and they read and approved the final manuscript.

Author details
[1]Department of Mathematics, Huaihua College, Huaihua, Hunan 418008, P.R. China. [2]School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, P.R. China.

Acknowledgements
This work is partially supported by the NNSF (No: 11471137) of China.

Received: 29 January 2015 Accepted: 17 May 2015 Published online: 26 June 2015

References
1. Ambrosetti, A, Rabinowitz, PH: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**(4), 349-381 (1973)
2. Ambrosetti, A, Coti Zelati, V: Multiple homoclinic orbits for a class of conservative systems. Rend. Semin. Mat. Univ. Padova **89**, 177-194 (1993)
3. Caldiroli, P, Montecchiari, P: Homoclinic orbits for second order Hamiltonian systems with potential changing sign. Commun. Appl. Nonlinear Anal. **1**(2), 97-129 (1994)
4. Coti Zelati, V, Ekeland, I, Sere, E: A variational approach to homoclinic orbits in Hamiltonian systems. Math. Ann. **288**(1), 133-160 (1990)
5. Coti Zelati, V, Rabinowitz, PH: Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials. J. Am. Math. Soc. **4**, 693-727 (1991)
6. Ding, YH, Girardi, M: Periodic and homoclinic solutions to a class of Hamiltonian systems with the potentials changing sign. Dyn. Syst. Appl. **2**(1), 131-145 (1993)
7. Flavia, A: Periodic and homoclinic solutions to a class of Hamiltonian systems with indefinite potential in sign. Boll. Unione Mat. Ital., B (7) **10**(2), 303-324 (1996)
8. Izydorek, M, Janczewska, J: Homoclinic solutions for a class of second order Hamiltonian systems. J. Differ. Equ. **219**(2), 375-389 (2005)
9. Korman, P, Lazer, AC: Homoclinic orbits for a class of symmetric Hamiltonian systems. Electron. J. Differ. Equ. **1994**, 1 (1994)
10. Lv, Y, Tang, C: Existence of even homoclinic orbits for a class of Hamiltonian systems. Nonlinear Anal. **67**(7), 2189-2198 (2007)
11. Omana, W, Willem, M: Homoclinic orbits for a class of Hamiltonian systems. Differ. Integral Equ. **5**(5), 1115-1120 (1992)
12. Paturel, E: Multiple homoclinic orbits for a class of Hamiltonian systems. Calc. Var. Partial Differ. Equ. **12**(2), 117-143 (2001)
13. Rabinowitz, PH: Homoclinic orbits for a class of Hamiltonian systems. Proc. R. Soc. Edinb., Sect. A **114**(1-2), 33-38 (1990)
14. Rabinowitz, PH, Tanaka, K: Some results on connecting orbits for a class of Hamiltonian systems. Math. Z. **206**(3), 473-499 (1991)
15. Tang, XH, Lin, XY: Homoclinic solutions for a class of second-order Hamiltonian systems. J. Math. Anal. Appl. **354**(2), 539-549 (2009)
16. Tang, XH, Lin, XY: Existence of infinitely many homoclinic orbits in Hamiltonian systems. Proc. R. Soc. Edinb. A **141**, 1103-1119 (2011)
17. Tang, XH, Lin, XY: Infinitely many homoclinic orbits for Hamiltonian systems with indefinite sign subquadratic potentials. Nonlinear Anal. **74**, 6314-6325 (2011)
18. Lv, Y, Tang, C: Homoclinic orbits for second-order Hamiltonian systems with subquadratic potentials. Chaos Solitons Fractals **57**, 137-145 (2013)

19. Zhang, Z, Yuan, R: Homoclinic solutions for a class of non-autonomous subquadratic second order Hamiltonian systems. Nonlinear Anal. **71**, 4125-4130 (2009)
20. Zhang, Z, Yuan, R: Homoclinic solutions for some second order non-autonomous systems. Nonlinear Anal. **71**, 5790-5798 (2009)
21. Ding, YH: Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems. Nonlinear Anal. **25**(11), 1095-1113 (1995)
22. Edmunds, DE, Evans, WD: Spectral Theory and Differential Operators. Clarendon Press, Oxford (1987)
23. Sun, J, Wang, Z: Spectral Analysis for Linear Operators. Science Press, Beijing (2005) (in Chinese)
24. Tang, XH: Non-Nehari manifold method for superlinear Schrödinger equation. Taiwan. J. Math. **18**, 1957-1979 (2014)
25. Chang, KC: Critical Point Theory and Applications. Shanghai Scientific and Technology Press, Shanghai (1986) (in Chinese)
26. Mawhin, J, Willem, M: Critical Point Theory and Hamiltonian Systems. Applied Mathematical Sciences, vol. 74. Springer, New York (1989)