Ntouyas et al. *Boundary Value Problems* (2015) 2015:92
DOI 10.1186/s13661-015-0356-y

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence of solutions for fractional differential inclusions with integral boundary conditions

Sotiris K Ntouyas[1,2], Sina Etemad[3] and Jessada Tariboon[4,5]*

*Correspondence: jessadat@kmutnb.ac.th
[4]Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand
[5]Centre of Excellence in Mathematics, CHE, Si Ayutthaya Rd., Bangkok, 10400, Thailand
Full list of author information is available at the end of the article

Abstract
In this paper, we study the existence of solutions for a new class of boundary value problems for nonlinear fractional differential inclusions with mixed type integral boundary conditions. The cases when the multifunction has convex as well as non-convex values are considered. Our results rely on the standard tools of fixed point theory and are well illustrated with the aid of an example.

MSC: 34A08; 34A60; 34B10; 34B15

Keywords: fractional differential inclusion; boundary value problem; existence; fixed point theorem

1 Introduction and preliminaries

Fractional differential equations and inclusions are generalizations of ordinary differential equations and inclusions to arbitrary non-integer orders. Fractional differential equations and inclusions appear naturally in a number of fields such as physics, engineering, biophysics, chemistry, biology, economics, control theory, *etc.* Recently, many papers have been published about fractional differential equations and inclusions by researchers which apply the fixed point theory in their existence theorems. For instance, one can find a lot of papers in this field (see [1–24] and the references therein).

Let $\alpha > 0$, $n-1 < \alpha < n$, $n = [\alpha]+1$, and $u \in C([a,b],\mathbb{R})$. The Caputo derivative of fractional order α for the function u is defined by ${}^{c}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}u^{(n)}(\tau)\,d\tau$ (see for more details [22, 24–28]). Also, the Riemann-Liouville fractional order integral of the function u is defined by $I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau$ $(t > 0)$ whenever the integral exists [22, 24–28]. In [27], it has been proved that the general solution of the fractional differential equation ${}^{c}D^{\alpha}u(t) = 0$ is given by $u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$, where $c_0, \ldots, c_{n-1}$ are real constants and $n = [\alpha]+1$. Also, for each $T > 0$ and $u \in C([0,T])$ we have

$$I^{\alpha}\,{}^{c}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_0, \ldots, c_{n-1}$ are real constants and $n = [\alpha]+1$ [27].

Now, we review some definitions and notations about multifunctions [29, 30].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_{b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) :$

Y is compact and convex}. A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \bigcup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_{\mathrm{b}}(X)$ (*i.e.*, $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_{\mathrm{b}}(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, *i.e.*, $u_n \to u_*$, $y_n \to y_*$, $y_n \in G(u_n)$ imply $y_* \in G(u_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$. A multivalued map $G : J \to \mathcal{P}_{\mathrm{cl}}(\mathbb{R})$ is said to be measurable if, for every $y \in \mathbb{R}$, the function $t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable.

Consider the Pompeiu-Hausdorff metric $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\Bigl\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\Bigr\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. A multivalued operator $N : X \to \mathcal{P}_{\mathrm{cl}}(X)$ is called contraction if there exists $\gamma \in (0,1)$ such that $H_d(N(x),N(y)) \le \gamma d(x,y)$ for each $x, y \in X$.

Let $2 < \alpha \le 3$, $1 < p \le 2$, $t \in J = [0,1]$, $0 < \xi < 1$, and $\gamma, \eta \in \mathbb{R}$. In this paper, we study the existence of solutions for the following fractional differential inclusion:

$$^{\mathrm{c}}D^{\alpha} u(t) \in F\bigl(t, u(t), u'(t), u''(t), {}^{\mathrm{c}}D^{p} u(t)\bigr) \tag{1.1}$$

via the integral boundary value conditions

$$\begin{aligned} &u(0) + u(1) = 0, \qquad {}^{\mathrm{c}}D^{p} u(1) = \gamma \int_0^1 u(\tau)\,d\tau, \\ &u'(0) + u''(\xi) = \eta \int_0^1 u(\tau)\,d\tau, \end{aligned} \tag{1.2}$$

where $F : J \times \mathbb{R}^4 \to \mathcal{P}(\mathbb{R})$ is a compact-valued multifunction.

We mention that the investigated fractional differential inclusion is a generalization of a huge class of classical ordinary differential inclusions which can be found in applications in engineering and physics *etc.* The boundary conditions (1.2) used in this manuscript, are as general as possible and we recover several cases of fractional nonlinear differential inclusions for many particular cases of the parameters.

We say that $F : J \times \mathbb{R}^4 \to \mathcal{P}(\mathbb{R})$ is a Carathéodory multifunction whenever $t \mapsto F(t,u,v,z,w)$ is measurable for all $u,v,z,w \in \mathbb{R}$ and $(u,v,z,w) \mapsto F(t,u,v,z,w)$ is upper semicontinuous for almost all $t \in J$ ([31] and [29]). Also, a Carathéodory multifunction $F : J \times \mathbb{R}^4 \to \mathcal{P}(\mathbb{R})$ is called L^1-Carathéodory whenever for each $\rho > 0$ there exists $\phi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\bigl\|F(t,u,v,z,w)\bigr\| = \sup_{t\in J}\bigl\{|s| : s \in F(t,u,v,z,w)\bigr\} \le \phi_\rho(t)$$

for all $|u|, |v|, |z|, |w| \leq \rho$, and for almost all $t \in J$ ([31] and [29]). Define the set of selections of F by

$$S_{F,u} := \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t, u(t), u'(t), u''(t), {}^{c}D^{p}u(t)) \text{ for almost all } t \in J\}.$$

We define the graph of a function G to be the set $\mathrm{Gr}(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and recall two results for closed graphs and upper semicontinuity.

Lemma 1.1 (Proposition 1.2 in [29]) *If $G : X \to \mathcal{P}_{\mathrm{cl}}(Y)$ is u.s.c., then $\mathrm{Gr}(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $u_n \to u_*$, $y_n \to y_*$, and $y_n \in G(u_n)$, then $y_* \in G(u_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semicontinuous.*

Lemma 1.2 ([32]) *Let X be a separable Banach space. Let $F : [0,1] \times X^4 \to \mathcal{P}_{\mathrm{cp,c}}(X)$ be an L^1-Carathéodory function. Then the operator*

$$\Theta \circ S_F : C(J,X) \to \mathcal{P}_{\mathrm{cp,c}}(C(J,X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator.

Now we state some known fixed point theorems which are needed in the sequel.

Lemma 1.3 (Nonlinear alternative for Kakutani maps [33]) *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{\mathrm{cp,c}}(C)$ is a upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Lemma 1.4 ([34]) *Let (X,d) be a complete metric space. If $N : X \to \mathcal{P}_{\mathrm{cl}}(X)$ is a contraction, then* $\mathrm{Fix}\, N \neq \emptyset$.

2 Main results

Now, we are ready to prove our main results. Let $X = \{u : u, u', u'', {}^{c}D^{p}u \in C(J,\mathbb{R})\}$ endowed with the norm $\|u\| = \sup_{t\in J}|u(t)| + \sup_{t\in J}|u'(t)| + \sup_{t\in J}|u''(t)| + \sup_{t\in J}|{}^{c}D^{p}u(t)|$. Then $(X, \|\cdot\|)$ is a Banach space [35].

Lemma 2.1 *Let $y \in L^1(J,\mathbb{R})$. Then the integral solution of the linear problem*

$$\begin{cases} {}^{c}D^{\alpha}u(t) = y(t), \\ u(0) + u(1) = 0, \qquad {}^{c}D^{p}u(1) = \gamma \int_0^1 u(\tau)\,d\tau, \\ u'(0) + u''(\xi) = \eta \int_0^1 u(\tau)\,d\tau, \end{cases} \tag{2.1}$$

is given by

$$\begin{aligned} u(t) = {} & I^{\alpha}y(t) + \left(\frac{1}{2} - t\right)I^{\alpha-2}y(\xi) + A(t)I^{\alpha}y(1) \\ & + B(t)I^{\alpha-p}y(1) + C(t)I^{\alpha+1}y(1), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
A(t) &= -\frac{1}{2}\bigl(1 + C(t)\bigr),\\
B(t) &= \frac{1}{4}\Gamma(3-p)\bigl(-1 + 4t - 2t^2\bigr) + \frac{\Gamma(3-p)}{12}C(t),\\
C(t) &= \frac{3}{12 + \gamma\Gamma(3-p)}\bigl[\gamma\Gamma(3-p)\bigl(1 - 4t + 2t^2\bigr) + 2\eta(2t-1)\bigr].
\end{aligned} \tag{2.3}$$

Proof It is well known that the solution of equation ${}^{c}D^{\alpha}u(t) = y(t)$ can be written as

$$u(t) = I^{\alpha}y(t) + c_0 + c_1 t + c_2 t^2,$$

where $c_0, c_1, c_2 \in \mathbb{R}$ are arbitrary constants. Then we get

$$u'(t) = I^{\alpha-1}y(t) + c_1 + 2c_2 t, \qquad u''(t) = I^{\alpha-2}y(t) + 2c_2$$

and

$${}^{c}D^{p}u(t) = I^{\alpha-p}y(t) + c_2\frac{2t^{2-p}}{\Gamma(3-p)}, \quad 1 < p \le 2.$$

By using the boundary conditions $u(0) + u(1) = 0$, ${}^{c}D^{p}u(1) = \gamma\int_0^1 u(\tau)\,d\tau$, $u'(0) + u''(\xi) = \eta\int_0^1 u(\tau)\,d\tau$, we obtain

$$\begin{aligned}
c_0 &= -\frac{1}{2}I^{\alpha}y(1) + I^{\alpha-2}y(\xi) - \frac{\Gamma(3-p)}{4}I^{\alpha-p}y(1) + \frac{\gamma\Gamma(3-p) - 2\eta}{4}\int_0^1 u(\tau)\,d\tau,\\
c_1 &= -I^{\alpha-2}y(\xi) + \Gamma(3-p)I^{\alpha-p}y(1) + \bigl(\eta - \gamma\Gamma(3-p)\bigr)\int_0^1 u(\tau)\,d\tau
\end{aligned}$$

and

$$c_2 = -\frac{\Gamma(3-p)}{2}I^{\alpha-p}y(1) + \frac{\gamma\Gamma(3-p)}{2}\int_0^1 u(\tau)\,d\tau.$$

Then

$$\begin{aligned}
u(t) &= I^{\alpha}y(t) - \frac{1}{2}I^{\alpha}y(1) + \left(\frac{1}{2} - t\right)I^{\alpha-2}y(\xi) + \frac{\Gamma(3-p)(-1 + 4t - 2t^2)}{4}I^{\alpha-p}y(1)\\
&\quad + \frac{1}{4}\bigl[\gamma\Gamma(3-p)\bigl(1 - 4t + 2t^2\bigr) + 2\eta(2t-1)\bigr]\int_0^1 u(\tau)\,d\tau.
\end{aligned} \tag{2.4}$$

Letting $A = \int_0^1 u(\tau)\,d\tau$, we have

$$\begin{aligned}
A = \int_0^1 u(\tau)\,d\tau &= \int_0^1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\,dt + c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2\\
&= I^{\alpha+1}y(1) + c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2,
\end{aligned}$$

or after substituting c_0, c_1, and c_2,

$$A = \frac{12}{12 + \gamma\Gamma(3-p)}\left[I^{\alpha+1}y(1) - \frac{1}{2}I^{\alpha}y(1) + \frac{\Gamma(3-p)}{12}I^{\alpha-p}y(1)\right].$$

Substituting the value A in (2.4), we get (2.2). The proof is completed. □

Remark 2.2 Throughout this paper, the following relations hold:

$$\begin{aligned}
&\bigl|A(t)\bigr| \le \frac{1}{2}(1 + C_1), \qquad \bigl|B(t)\bigr| \le \Gamma(3-p)\left(\frac{7}{4} + \frac{1}{12}C_1\right),\\
&\bigl|C(t)\bigr| \le \frac{3}{12 + \gamma\Gamma(3-p)}\bigl[7\gamma\Gamma(3-p) + 2\eta\bigr] := C_1,\\
&\bigl|A'(t)\bigr| \le \frac{1}{2}C_1', \qquad \bigl|B'(t)\bigr| \le \Gamma(3-p)\left(1 + \frac{1}{12}C_1'\right),\\
&\bigl|C'(t)\bigr| \le \frac{12[\gamma\Gamma(3-p) + \eta]}{12 + \gamma\Gamma(3-p)} := C_1',\\
&\bigl|A''(t)\bigr| \le \frac{1}{2}C_1'', \qquad \bigl|B''(t)\bigr| \le \Gamma(3-p)\left(1 + \frac{1}{12}C_1''\right),\\
&\bigl|C''(t)\bigr| \le \frac{12\gamma\Gamma(3-p)}{12 + \gamma\Gamma(3-p)} := C_1'',\\
&\bigl|{}^{c}D^{p}A(t)\bigr| \le D_1, \qquad \bigl|{}^{c}D^{p}B(t)\bigr| \le 1 + \frac{\Gamma(3-p)}{12}D_1,\\
&\bigl|{}^{c}D^{p}C(t)\bigr| \le \frac{12\gamma}{12 + \gamma\Gamma(3-p)} := D_1.
\end{aligned}$$

Definition 2.3 A function $u \in C^2(J,\mathbb{R})$ is called a solution for the problem (1.1)-(1.2) if there exists a function $v \in L^1(J,\mathbb{R})$ with $v(t) \in F(t,u(t),u'(t),u''(t),{}^{c}D^{p}u(t))$ for almost all $t \in J$, $u(0) + u(1) = 0$, ${}^{c}D^{p}u(1) = \gamma\int_0^1 u(\tau)\,d\tau$, $u'(0) + u''(\xi) = \eta\int_0^1 u(\tau)\,d\tau$, and

$$u(t) = I^{\alpha}v(t) + \left(\frac{1}{2} - t\right)I^{\alpha-1}v(\xi) + A(t)I^{\alpha}v(1) + B(t)I^{\alpha-p}v(1) + C(t)I^{\alpha+1}v(1) \tag{2.5}$$

for all $t \in J$.

For the sake of brevity, we set

$$\begin{aligned}
\Lambda_1 &= \frac{1}{\Gamma(\alpha+1)} + \frac{1}{2}\frac{\xi^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{1}{2}(1 + C_1)\frac{1}{\Gamma(\alpha+1)}\\
&\quad + \Gamma(3-p)\left(\frac{7}{4} + \frac{1}{12}C_1\right)\frac{1}{\Gamma(\alpha-p+1)} + \frac{1}{2}C_1\frac{1}{\Gamma(\alpha+2)}, \qquad (2.6)\\
\Lambda_2 &= \frac{1}{\Gamma(\alpha)} + \frac{\xi^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{1}{2}C_1'\frac{1}{\Gamma(\alpha+1)}\\
&\quad + \Gamma(3-p)\left(1 + \frac{1}{12}C_1'\right)\frac{1}{\Gamma(\alpha-p+1)} + \frac{1}{2}C_1'\frac{1}{\Gamma(\alpha+2)}, \qquad (2.7)\\
\Lambda_3 &= \frac{1}{\Gamma(\alpha-1)} + C_1''\frac{1}{\Gamma(\alpha+1)}\\
&\quad + \Gamma(3-p)\left(1 + \frac{1}{12}C_1''\right)\frac{1}{\Gamma(\alpha-p+1)} + \frac{1}{2}C_1''\frac{1}{\Gamma(\alpha+2)} \qquad (2.8)
\end{aligned}$$

and

$$\Lambda_4 = \frac{1}{\Gamma(\alpha-p+1)} + D_1\frac{1}{\Gamma(\alpha+1)} + \left(1+\frac{\Gamma(3-p)}{12}D_1\right)\frac{1}{\Gamma(\alpha-p+1)} + \frac{1}{2}D_1\frac{1}{\Gamma(\alpha+2)}. \tag{2.9}$$

Theorem 2.4 *Suppose that*:

(H_1) $F : J \times \mathbb{R}^4 \to \mathcal{P}_{\mathrm{cp,c}}(\mathbb{R})$ *is a* L^1-*Carathéodory multifunction*.

(H_2) *There exist continuous nondecreasing functions* $\psi_i : [0,\infty) \to (0,\infty)$ *and functions* $p_i \in C(J,\mathbb{R}^+)$, $1 \le i \le 4$, *such that*

$$\|F(t,x_1,x_2,x_3,x_4)\| := \sup\{|v| : v \in F(t,x_1,x_2,x_3,x_4)\} \le \sum_{i=1}^{4} p_i(t)\psi_i(|x_i|)$$

for each $(t,x_i) \in J \times \mathbb{R}$, $1 \le i \le 4$.

(H_3) *There exists a constant* $M > 0$ *such that*

$$\frac{M}{\sum_{i=1}^{4}\Lambda_i\|p_i\|\psi_i(M)} > 1,$$

where Λ_i, $1 \le i \le 4$ *are defined by* (2.6)-(2.9).

Then the inclusion boundary value problem (1.1)-(1.2) *has at least one solution*.

Proof To transform the problem (1.1)-(1.2) into a fixed point problem, we define an operator $\mathcal{N} : X \to \mathcal{P}(X)$ as

$$\mathcal{N}(u) = \left\{ h \in X : h(t) = \left\{ \begin{array}{l} I^{\alpha}v(t) + (\frac{1}{2}-t)I^{\alpha-1}v(\xi) + A(t)I^{\alpha}v(1) \\ \quad + B(t)I^{\alpha-p}v(1) + C(t)I^{\alpha+1}v(1), t \in J, v \in S_{F,u} \end{array} \right\} \right\}.$$

We will show that $\mathcal{N}$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that $\mathcal{N}$ *is convex for each* $u \in X$. This step is obvious since $S_{F,u}$ is convex (F has convex values), and therefore we omit the proof.

In the second step, we show that $\mathcal{N}$ *maps bounded sets* (*balls*) *into bounded sets in* X. For a positive number r, let $B_r = \{u \in X : \|u\| \le r\}$ be a bounded ball in X. Then, for each $h \in \mathcal{N}(u)$ and $u \in B_r$, there exists $v \in S_{F,u}$ such that

$$h(t) = I^{\alpha}v(t) + \left(\frac{1}{2}-t\right)I^{\alpha-1}v(\xi) + A(t)I^{\alpha}v(1) + B(t)I^{\alpha-p}v(1) + C(t)I^{\alpha+1}v(1), \quad t \in J.$$

Then we have

$$\begin{aligned} |h(t)| &\le I^{\alpha}|v(t)| + \left|\frac{1}{2}-t\right|I^{\alpha-2}|v(\xi)| + |A(t)|I^{\alpha}|v(1)| \\ &\quad + |B(t)|I^{\alpha-p}|v(1)| + |C(t)|I^{\alpha+1}|v(1)| \\ &\le I^{\alpha}\big[p_1(t)\psi_1(|u(t)|) + p_2(t)\psi_2(|u'(t)|) \end{aligned}$$

$$
\begin{aligned}
&+ p_3(t)\psi_3\big(\big|u''(t)\big|\big) + p_4(t)\psi_4\big(\big|{}^{c}D^{p}u(t)\big|\big)\big] \\
&+ \frac{1}{2}I^{\alpha-1}\big[p_1(\xi)\psi_1\big(\big|u(\xi)\big|\big) + p_2(\xi)\psi_2\big(\big|u'(\xi)\big|\big) \\
&+ p_3(\xi)\psi_3\big(\big|u''(\xi)\big|\big) + p_4(\xi)\psi_4\big(\big|{}^{c}D^{p}u(\xi)\big|\big)\big] \\
&+ \frac{1}{2}(1+C_1)I^{\alpha}\big[p_1(1)\psi_1\big(\big|u(1)\big|\big) + p_2(1)\psi_2\big(\big|u'(1)\big|\big) \\
&+ p_3(1)\psi_3\big(\big|u''(1)\big|\big) + p_4(1)\psi_4\big(\big|{}^{c}D^{p}u(1)\big|\big)\big] \\
&+ \Gamma(3-p)\left(\frac{7}{4}+\frac{1}{12}C_1\right)I^{\alpha-p}\big[p_1(1)\psi_1\big(\big|u(1)\big|\big) + p_2(1)\psi_2\big(\big|u'(1)\big|\big) \\
&+ p_3(1)\psi_3\big(\big|u''(1)\big|\big) + p_4(1)\psi_4\big(\big|{}^{c}D^{p}u(1)\big|\big)\big] \\
&+ C_1 I^{\alpha+1}\big[p_1(1)\psi_1\big(\big|u(1)\big|\big) + p_2(1)\psi_2\big(\big|u'(1)\big|\big) \\
&+ p_3(1)\psi_3\big(\big|u''(1)\big|\big) + p_4(1)\psi_4\big(\big|{}^{c}D^{p}u(1)\big|\big)\big] \\
\le{}& \big[\|p_1\|\psi_1(r) + \|p_2\|\psi_2(r) + \|p_3\|\psi_3(r) + \|p_4\|\psi_4(r)\big]\left\{\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2}\frac{\xi^{\alpha-2}}{\Gamma(\alpha-1)}\right. \\
&\left. + \frac{1}{2}(1+C_1)\frac{1}{\Gamma(\alpha+1)} + \Gamma(3-p)\left(\frac{7}{4}+\frac{1}{12}C_1\right)\frac{1}{\Gamma(\alpha-p+1)} + \frac{1}{2}C_1\frac{1}{\Gamma(\alpha+2)}\right\} \\
={}& \Lambda_1 \sum_{i=1}^{4}\|p_i\|\psi_i(r)
\end{aligned}
$$

for all $t \in J$. In a similar manner we obtain

$$
\begin{aligned}
\big|h'(t)\big| \le{}& \big[\|p_1\|\psi_1(r) + \|p_2\|\psi_2(r) + \|p_3\|\psi_3(r) + \|p_4\|\psi_4(r)\big]\left\{\frac{1}{\Gamma(\alpha)} + \frac{\xi^{\alpha-2}}{\Gamma(\alpha-1)}\right. \\
&\left. + \frac{1}{2}C_1'\frac{1}{\Gamma(\alpha+1)} + \Gamma(3-p)\left(1+\frac{1}{12}C_1'\right)\frac{1}{\Gamma(\alpha-p+1)} + \frac{1}{2}C_1'\frac{1}{\Gamma(\alpha+2)}\right\} \\
={}& \Lambda_2 \sum_{i=1}^{4}\|p_i\|\psi_i(r), \\
\big|h''(t)\big| \le{}& \big[\|p_1\|\psi_1(r) + \|p_2\|\psi_2(r) + \|p_3\|\psi_3(r) + \|p_4\|\psi_4(r)\big]\left\{\frac{1}{\Gamma(\alpha-1)} + C_1''\frac{1}{\Gamma(\alpha+1)}\right. \\
&\left. + \Gamma(3-p)\left(1+\frac{1}{12}C_1''\right)\frac{1}{\Gamma(\alpha-p+1)} + \frac{1}{2}C_1''\frac{1}{\Gamma(\alpha+2)}\right\} \\
={}& \Lambda_3 \sum_{i=1}^{4}\|p_i\|\psi_i(r)
\end{aligned}
$$

and

$$
\begin{aligned}
\big|{}^{c}D^{p}h(t)\big| \le{}& \big[\|p_1\|\psi_1(r) + \|p_2\|\psi_2(r) + \|p_3\|\psi_3(r) + \|p_4\|\psi_4(r)\big]\left\{\frac{1}{\Gamma(\alpha-p+1)}\right. \\
&\left. + D_1\frac{1}{\Gamma(\alpha+1)} + \left(1+\frac{\Gamma(3-p)}{12}D_1\right)\frac{1}{\Gamma(\alpha-p+1)} + \frac{1}{2}D_1\frac{1}{\Gamma(\alpha+2)}\right\} \\
={}& \Lambda_4 \sum_{i=1}^{4}\|p_i\|\psi_i(r)
\end{aligned}
$$

for all $t \in J$. Thus we get

$$\|h\| \le (\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4) \sum_{i=1}^{4} \|p_i\| \psi_i(r) = \sum_{i=1}^{4} \Lambda_i \|p_i\| \psi_i(r),$$

which implies that $\mathcal{N}$ maps bounded sets into bounded sets in X.

Now, we prove that $\mathcal{N}$ maps bounded sets into equi-continuous subsets of X. Suppose that $u \in B_r$ and $t_1, t_2 \in J$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&\bigl|h(t_2) - h(t_1)\bigr| \\
&\quad \le \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} v(\tau)\, d\tau - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} v(\tau)\, d\tau \right| \\
&\qquad + |t_2 - t_1| I^{\alpha-2} \bigl|v(\xi)\bigr| + \bigl|A(t_2) - A(t_1)\bigr| I^{\alpha} \bigl|v(1)\bigr| + \bigl|B(t_2) - B(t_1)\bigr| I^{\alpha-p} \bigl|v(1)\bigr| \\
&\qquad + \bigl|C(t_2) - C(t_1)\bigr| I^{\alpha+1} \bigl|v(1)\bigr| \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \bigl[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\bigr] \bigl|v(\tau)\bigr|\, d\tau + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \bigl|v(\tau)\bigr|\, d\tau \\
&\qquad + |t_2 - t_1| I^{\alpha-2} \bigl|v(\xi)\bigr| + \bigl|A(t_2) - A(t_1)\bigr| I^{\alpha} \bigl|v(1)\bigr| + \bigl|B(t_2) - B(t_1)\bigr| I^{\alpha-p} \bigl|v(1)\bigr| \\
&\qquad + \bigl|C(t_2) - C(t_1)\bigr| I^{\alpha+1} \bigl|v(1)\bigr| \\
&\quad \le \sum_{i=1}^{4} \|p_i\| \psi_i(r) \biggl\{ \bigl(t_2^{\alpha} - t_1^{\alpha}\bigr) \frac{1}{\Gamma(\alpha+1)} + |t_2 - t_1| \frac{\xi^{\alpha-2}}{\Gamma(\alpha-1)} + \bigl|A(t_2) - A(t_1)\bigr| \frac{1}{\Gamma(\alpha+1)} \\
&\qquad + \bigl|B(t_2) - B(t_1)\bigr| \frac{1}{\Gamma(\alpha-p+1)} + \bigl|C(t_2) - C(t_1)\bigr| \frac{1}{\Gamma(\alpha+2)} \biggr\}.
\end{aligned}$$

Proceeding as above we have

$$\begin{aligned}
&\bigl|h'(t_2) - h'(t_1)\bigr| \\
&\quad \le \left| \frac{1}{\Gamma(\alpha-1)} \int_0^{t_2} (t_2 - s)^{\alpha-2} v(\tau)\, d\tau - \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1 - s)^{\alpha-2} v(\tau)\, d\tau \right| \\
&\qquad + \bigl|A'(t_2) - A'(t_1)\bigr| I^{\alpha} \bigl|v(1)\bigr| + \bigl|B'(t_2) - B'(t_1)\bigr| I^{\alpha-p} \bigl|v(1)\bigr| \\
&\qquad + \bigl|C'(t_2) - C'(t_1)\bigr| I^{\alpha+1} \bigl|v(1)\bigr| \\
&\quad \le \sum_{i=1}^{4} \|p_i\| \psi_i(r) \biggl\{ \bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr) \frac{1}{\Gamma(\alpha)} + \bigl|A'(t_2) - A'(t_1)\bigr| \frac{1}{\Gamma(\alpha+1)} \\
&\qquad + \bigl|B'(t_2) - B'(t_1)\bigr| \frac{1}{\Gamma(\alpha-p+1)} + \bigl|C'(t_2) - C'(t_1)\bigr| \frac{1}{\Gamma(\alpha+2)} \biggr\}, \\
&\bigl|h''(t_2) - h''(t_1)\bigr| \\
&\quad \le \sum_{i=1}^{4} \|p_i\| \psi_i(r) \biggl\{ \bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr) \frac{1}{\Gamma(\alpha-1)} + \bigl|A''(t_2) - A''(t_1)\bigr| \frac{1}{\Gamma(\alpha+1)} \\
&\qquad + \bigl|B''(t_2) - B''(t_1)\bigr| \frac{1}{\Gamma(\alpha-p+1)} + \bigl|C''(t_2) - C''(t_1)\bigr| \frac{1}{\Gamma(\alpha+2)} \biggr\}
\end{aligned}$$

and

$$\begin{aligned}
&\left|{}^{c}D^{p}h(t_2)-{}^{c}D^{p}h(t_1)\right| \\
&\quad\le \sum_{i=1}^{4}\|p_i\|\psi_i(r)\biggl\{\bigl(t_2^{\alpha-p}-t_1^{\alpha-p}\bigr)\frac{1}{\Gamma(\alpha-p+1)}+|t_2-t_1|\frac{12\gamma}{12+\gamma\Gamma(3-p)}\frac{1}{\Gamma(\alpha+1)} \\
&\qquad+|t_2-t_1|\left(1+\frac{12\gamma}{12+\gamma\Gamma(3-p)}\right)\frac{1}{\Gamma(\alpha-p+1)} \\
&\qquad+|t_2-t_1|\frac{1}{2}\frac{\Gamma(3-p)\gamma}{12+\gamma\Gamma(3-p)}\frac{1}{\Gamma(\alpha+2)}\biggr\}.
\end{aligned}$$

Obviously the right-hand side of the above inequalities tends to zero independently of $u \in B_r$ as $t_2 - t_1 \to 0$. Therefore it follows by the Ascoli-Arzelá theorem that $\mathcal{N} : X \to \mathcal{P}(X)$ is completely continuous.

In our next step, we show that $\mathcal{N}$ is upper semicontinuous. It is well known by Lemma 1.1 that $\mathcal{N}$ will be upper semicontinuous if we prove that it has a closed graph, since $\mathcal{N}$ is already shown to be completely continuous. Thus we will prove that $\mathcal{N}$ *has a closed graph*. Let $u_n \to u_*$, $h_n \in \mathcal{N}(u_n)$, and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{N}(u_*)$. Associated with $h_n \in \mathcal{N}(u_n)$, there exists $v_n \in S_{F,u_n}$ such that, for each $t \in J$,

$$h_n(t) = I^{\alpha}v_n(t) + \left(\frac{1}{2}-t\right)I^{\alpha-2}v_n(\xi) + A(t)I^{\alpha}v_n(1) + B(t)I^{\alpha-p}v_n(1) + C(t)I^{\alpha+1}v_n(1).$$

Thus it suffices to show that there exists $v_* \in S_{F,u_*}$ such that, for each $t \in J$,

$$h_*(t) = I^{\alpha}v_*(t) + \left(\frac{1}{2}-t\right)I^{\alpha-2}v_*(\xi) + A(t)I^{\alpha}v_*(1) + B(t)I^{\alpha-p}v_*(1) + C(t)I^{\alpha+1}v_*(1).$$

Let us consider the linear operator $\Theta : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ given by

$$f \mapsto \Theta(v)(t) = I^{\alpha}v(t) + \left(\frac{1}{2}-t\right)I^{\alpha-1}v(\xi) + A(t)I^{\alpha}v(1) + B(t)I^{\alpha-p}v(1) + C(t)I^{\alpha+1}v(1).$$

Observe that

$$\begin{aligned}
\|h_n(t)-h_*(t)\| = \biggl\| & I^{\alpha}\bigl(v_n(t)-v_*(t)\bigr) + \left(\frac{1}{2}-t\right)I^{\alpha-2}\bigl(v_n(\xi)-v_*(\xi)\bigr) \\
&+ A(t)I^{\alpha}\bigl(v_n(1)-v_*(1)\bigr) + B(t)I^{\alpha-p}\bigl(v_n(1)-v_*(1)\bigr) \\
&+ C(t)I^{\alpha+1}\bigl(v_n(1)-v_*(1)\bigr)\biggr\| \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, it follows by Lemma 1.2 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,u_n})$. Since $u_n \to u_*$, therefore, we have

$$h_*(t) = I^{\alpha}v_*(t) + \left(\frac{1}{2}-t\right)I^{\alpha-2}v_*(\xi) + A(t)I^{\alpha}v_*(1) + B(t)I^{\alpha-p}v_*(1) + C(t)I^{\alpha+1}v_*(1)$$

for some $v_* \in S_{F,u_*}$.

Finally, we show that there exists an open set $U \subseteq X$ with $u \notin \mathcal{N}(u)$ for any $\lambda \in (0,1)$ and all $u \in \partial U$. Let $\lambda \in (0,1)$ and $u \in \lambda\mathcal{N}(u)$. Then there exists $v \in L^1(J,\mathbb{R})$ with $v \in S_{F,u}$ such that, for $t \in J$, we have

$$u(t) = \lambda I^\alpha v(t) + \lambda\left(\frac{1}{2} - t\right)I^{\alpha-1}v(\xi) + \lambda A(t)I^\alpha v(1) + \lambda B(t)I^{\alpha-p}v(1) + \lambda C(t)I^{\alpha+1}v(1).$$

Using the computations of the second step above we have

$$\begin{aligned}\|u\| \le{}& (\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4)\bigl[\|p_1\|\psi_1\bigl(\|u\|\bigr) + \|p_2\|\psi_2\bigl(\|u\|\bigr)\\ &+ \|p_3\|\psi_3\bigl(\|u\|\bigr) + \|p_4\|\psi_4\bigl(\|u\|\bigr)\bigr],\end{aligned}$$

which implies that

$$\frac{\|u\|}{\sum_{i=1}^4 \Lambda_i\|p_i\|\psi_i(\|u\|)} \le 1.$$

In view of (H_3), there exists M such that $\|u\| \ne M$. Let us set

$$U = \bigl\{u \in X : \|x\| < M\bigr\}.$$

Note that the operator $\mathcal{N} : \overline{U} \to \mathcal{P}(X)$ is upper semicontinuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u \in \lambda\mathcal{N}(u)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 1.3), we deduce that $\mathcal{N}$ has a fixed point $u \in \overline{U}$ which is a solution of the problem (1.1)-(1.2). This completes the proof. □

In the next theorem, we prove the existence of solution for the inclusion boundary value problem (1.1)-(1.2) when the multifunction F is non-convex valued.

Theorem 2.5 *Assume that*:

(H_4) $F : J \times \mathbb{R}^4 \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot,u,v,z,w) : J \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $u,v,z,w \in \mathbb{R}$.

(H_5) *For almost all* $t \in J$ *and* $u_1,u_2,u_3,u_4,w_1,w_2,w_3,w_4 \in \mathbb{R}$ *we have*

$$\begin{aligned}&H_d\bigl(F(t,u_1,u_2,u_3,u_4),F(t,w_1,w_2,w_3,w_4)\bigr)\\ &\quad\le m(t)\bigl(|u_1 - w_1| + |u_2 - w_2| + |u_3 - w_3| + |u_4 - w_4|\bigr)\end{aligned}$$

with $m \in C(J,\mathbb{R}^+)$ *and* $d(0,F(t,0,0,0,0)) \le m(t)$, *for almost all* $t \in J$.

Then the boundary value problem (1.1)-(1.2) *has at least one solution on* J *if* $\|m\|\sum_{i=1}^4 \Lambda_i < 1$.

Proof Observe that the set $S_{F,u}$ is nonempty for each $u \in X$ by the assumption (H_4), so F has a measurable selection (see Theorem III.6 in [36]). Now we show that the operator $\mathcal{N}$, defined in the beginning of proof of Theorem 2.4, satisfies the assumptions of Lemma 1.4.

To show that $\mathcal{N}(u) \in \mathcal{P}_{cl}(X)$ for each $u \in X$, let $\{u_n\}_{n\geq 0} \in \mathcal{N}(u)$ be such that $u_n \to u$ ($n \to \infty$) in X. Then $u \in X$ and there exists $v_n \in S_{F,u_n}$ such that, for each $t \in J$,

$$u_n(t) = I^\alpha v_n(t) + \left(\frac{1}{2} - t\right) I^{\alpha-2} v_n(\xi) + A(t) I^\alpha v_n(1) + B(t) I^{\alpha-p} v_n(1) + C(t) I^{\alpha+1} v_n(1).$$

As F has compact values, we pass onto a subsequence (if necessary) to find that v_n converges to v in $L^1(J, \mathbb{R})$. Thus, $v \in S_{F,u}$ and for each $t \in J$, we have

$$u_n(t) \to u(t) = I^\alpha v(t) + \left(\frac{1}{2} - t\right) I^{\alpha-1} v(\xi) + A(t) I^\alpha v(1) + B(t) I^{\alpha-p} v(1) + C(t) I^{\alpha+1} v(1).$$

Hence, $u \in \mathcal{N}(u)$.

Next we show that $\mathcal{N}$ is a contractive multifunction with constant $\delta = \|m\| \sum_{i=1}^4 \Lambda_i < 1$. Let $u, w \in X$ and $h_1 \in \mathcal{N}(u)$. Then there exists $v_1 \in S_{F,u}$ such that, for each $t \in J$,

$$h_1(t) = I^\alpha v_1(t) + \left(\frac{1}{2} - t\right) I^{\alpha-2} v_1(\xi) + A(t) I^\alpha v_1(1) + B(t) I^{\alpha-p} v_1(1) + C(t) I^{\alpha+1} v_1(1).$$

By (H_5), we have

$$\begin{aligned} &H_d\big(F\big(t, u(t), u'(t), u''(t), {}^cD^p u(t)\big), F\big(t, w(t), w'(t), w''(t), {}^cD^p w(t)\big)\big) \\ &\quad \leq m(t)\big(\big|u(t) - w(t)\big| + \big|u'(t) - w'(t)\big| + \big|u''(t) - w''(t)\big| + \big|{}^cD^p u(t) - {}^cD^p w(t)\big|\big), \end{aligned}$$

so there exists $z \in F(t, u(t), u'(t), u''(t), {}^cD^p u(t))$ such that

$$\big|v_1(t) - z\big| \leq m(t)\big(\big|u(t) - w(t)\big| + \big|u'(t) - w'(t)\big| + \big|u''(t) - w''(t)\big| + \big|{}^cD^p u(t) - {}^cD^p w(t)\big|\big)$$

for almost all $t \in J$. Define the multifunction $U : J \to \mathcal{P}(\mathbb{R})$ by

$$\begin{aligned} U(t) = \big\{&z \in \mathbb{R} : \big|v_1(t) - z\big| \leq m(t)\big(\big|u(t) - w(t)\big| + \big|u'(t) - w'(t)\big| + \big|u''(t) - w''(t)\big| \\ &+ \big|{}^cD^p u(t) - {}^cD^p w(t)\big|\big) \text{ for almost all } t \in J\big\}. \end{aligned}$$

It is easy to check that the multifunction $U(\cdot) \cap F(\cdot, u(\cdot), u'(\cdot), u''(\cdot), {}^cD^p u(\cdot))$ is measurable. Hence, we can choose $v_2 \in S_{F,u}$ such that

$$\big|v_1(t) - v_2(t)\big| \leq m(t)\big(\big|u(t) - w(t)\big| + \big|u'(t) - w'(t)\big| + \big|u''(t) - w''(t)\big| + \big|{}^cD^p u(t) - {}^cD^p w(t)\big|\big)$$

for almost all $t \in J$. Consider $h_2 \in \mathcal{N}(u)$ which is defined by

$$h_2(t) = I^\alpha v_2(t) + \left(\frac{1}{2} - t\right) I^{\alpha-2} v_2(\xi) + A(t) I^\alpha v_2(1) + B(t) I^{\alpha-p} v_2(1) + C(t) I^{\alpha+1} v_2(1).$$

Thus,

$$\begin{aligned} \big|h_1(t) - h_2(t)\big| &= I^\alpha \big|v_1(t) - v_2(t)\big| + \left|\frac{1}{2} - t\right| I^{\alpha-2} \big|v_1(1) - v_2(1)\big| + \big|A(t)\big| I^\alpha \big|v_1(1) - v_2(1)\big| \\ &\quad + \big|B(t)\big| I^{\alpha-p} \big|v_1(1) - v_2(1)\big| + \big|C(t)\big| I^{\alpha+1} \big|v_1(1) - v_2(1)\big| \end{aligned}$$

$$\begin{aligned}
&\le \|m\| \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{1}{2}\frac{\xi^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{1}{2}(1+C_1)\frac{1}{\Gamma(\alpha+1)} \right. \\
&\quad \left. + \Gamma(3-p)\left(\frac{7}{4} + \frac{1}{12}C_1\right)\frac{1}{\Gamma(\alpha-p+1)} + \frac{1}{2}C_1\frac{1}{\Gamma(\alpha+2)} \right\} \|u-w\| \\
&= \|m\| \Lambda_1 \|u-w\|.
\end{aligned}$$

In a similar manner we obtain

$$\begin{aligned}
\left|h_1'(t) - h_2'(t)\right| &\le \|m\| \left\{ \frac{1}{\Gamma(\alpha)} + \frac{\xi^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{1}{2}C_1'\frac{1}{\Gamma(\alpha+1)} \right. \\
&\quad \left. + \Gamma(3-p)\left(1 + \frac{1}{12}C_1'\right)\frac{1}{\Gamma(\alpha-p+1)} + \frac{1}{2}C_1'\frac{1}{\Gamma(\alpha+2)} \right\} \|u-w\| \\
&= \|m\| \Lambda_2 \|u-w\|, \\
\left|h_1''(t) - h_2''(t)\right| &\le \|m\| \left\{ \frac{1}{\Gamma(\alpha-1)} + C_1''\frac{1}{\Gamma(\alpha+1)} + \Gamma(3-p)\left(1 + \frac{1}{12}C_1''\right)\frac{1}{\Gamma(\alpha-p+1)} \right. \\
&\quad \left. + \frac{1}{2}C_1''\frac{1}{\Gamma(\alpha+2)} \right\} \|u-w\| \\
&= \|m\| \Lambda_3 \|u-w\|
\end{aligned}$$

and

$$\begin{aligned}
\left|{}^{c}D^{p}h_1(t) - {}^{c}D^{p}h_2(t)\right| &\le \|m\| \left\{ \frac{1}{\Gamma(\alpha-p+1)} + D_1\frac{1}{\Gamma(\alpha+1)} \right. \\
&\quad \left. + \left(1 + \frac{\Gamma(3-p)}{12}D_1\right)\frac{1}{\Gamma(\alpha-p+1)} + \frac{1}{2}D_1\frac{1}{\Gamma(\alpha+2)} \right\} \|u-w\| \\
&= \|m\| \Lambda_4 \|u-w\|.
\end{aligned}$$

Hence,

$$\|h_1 - h_2\| \le \|m\| \sum_{i=1}^{4} \Lambda_i \|u-w\|.$$

Analogously, interchanging the roles of u and w, we obtain

$$H_d\big(\mathcal{N}(u), \mathcal{N}(w)\big) \le \|m\| \sum_{i=1}^{4} \Lambda_i \|u-w\|.$$

Since $\delta = \|m\| \sum_{i=1}^{4} \Lambda_i < 1$, $\mathcal{N}$ is a contraction, it follows by Lemma 1.4 that $\mathcal{N}$ has a fixed point u which is a solution of (1.1)-(1.2). This completes the proof. □

Now, we give an illustrative example.

Example 2.6 We consider the following fractional differential inclusion:

$$\begin{aligned}
{}^{c}D^{\frac{5}{2}}u(t) \in \Bigg[0, &\frac{\frac{t}{200}|\sin\frac{\pi}{2}t||u(t)|}{1+|u(t)|} + \frac{t|\sin u'(t)|^3}{200+200|\sin u'(t)|^3} + \frac{\frac{t}{200}|\cos u''(t)|^4}{1+|\cos u''(t)|^4} \\
&+ \frac{t|{}^{c}D^{\frac{5}{3}}u(t)|}{200|\cos\pi t|(1+|{}^{c}D^{\frac{5}{3}}u(t)|)}\Bigg],
\end{aligned} \tag{2.10}$$

via the boundary value conditions

$$u(0) + u(1) = 0, \qquad {}^{c}D^{\frac{5}{3}} u(1) = \frac{1}{1{,}000} \int_0^1 u(\tau)\, d\tau,$$
$$u'(0) + u''\left(\frac{1}{100}\right) = \frac{1}{1{,}000} \int_0^1 u(\tau)\, d\tau,$$

where $t \in J = [0,1]$. By the above inclusion problem, we have $\alpha = 5/2$, $p = 5/3$, $\xi = 1/100$, and $\gamma = \eta = 1/1{,}000$. Now, we define an operator $F : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to P_{\mathrm{cp}}(\mathbb{R})$ by

$$F(t, x_1, x_2, x_3, x_4) = \left[0, \frac{\frac{t}{200} |\sin \frac{\pi}{2} t||x_1|}{1 + |x_1|} + \frac{t|\sin x_2|^3}{200 + 200|\sin x_2|^3} + \frac{\frac{t}{200} |\cos x_3|^4}{1 + |\cos x_3|^4} + \frac{t|x_4|}{200|\cos \pi t|(1 + |x_4|)}\right].$$

Also, we define the function $m : J \to \mathbb{R}^+$ by $m(t) = t/200$. It is clear that m is continuous on J and $\|m\| = 1/200$. Finally, one can write

$$H_d\big(F(t, x_1, x_2, x_3, x_4), F(t, y_1, y_2, y_3, y_4)\big) \leq m(t)\big(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |x_4 - y_4|\big),$$

where $t \in J$ and $x_i, y_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$. On the other hand, there exist the following values:

$$\Lambda_1 = 2.13208, \qquad \Lambda_2 = 1.79407, \qquad \Lambda_3 = 2.057974, \qquad \Lambda_4 = 2.08023.$$

Then $\|m\|(\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4) = \frac{1}{200} \times 8.064354 = 0.04 < 1$. Consequently, all assumptions and conditions of Theorem 2.5 are satisfied. Hence, Theorem 2.5 implies that the fractional differential inclusion problem (2.10) has at least one solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

Author details
[1]Department of Mathematics, University of Ioannina, Ioannina, 451 10, Greece. [2]Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia. [3]Young Researchers and Elite Club, Tabriz Branch, Islamic Azad University, Tabriz, Iran. [4]Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand. [5]Centre of Excellence in Mathematics, CHE, Si Ayutthaya Rd., Bangkok, 10400, Thailand.

Acknowledgements
This research is partially supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

Received: 9 January 2015 Accepted: 19 May 2015 Published online: 10 June 2015

References

1. Agarwal, RP, Ahmad, B: Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions. J. Appl. Math. Comput. **62**, 1200-1214 (2011)
2. Agarwal, RP, Belmekki, M, Benchohra, M: A survey on semilinear differential equations and inclusions invovling Riemann-Liouville fractional derivative. Adv. Differ. Equ. **2009**, Article ID 981728 (2009)
3. Agarwal, RP, Benchohra, M, Hamani, S: A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl. Math. **109**, 973-1033 (2010)

4. Agarwal, RP, Ntouyas, SK, Ahmad, B, Alhothuali, M: Existence of solutions for integro-differential equations of fractional order with nonlocal three-point fractional boundary conditions. Adv. Differ. Equ. **2013**, 128 (2013)
5. Ahmad, B, Nieto, JJ: Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory. Topol. Methods Nonlinear Anal. **35**, 295-304 (2010)
6. Ahmad, B, Ntouyas, SK: Boundary value problem for fractional differential inclusions with four-point integral boundary conditions. Surv. Math. Appl. **6**, 175-193 (2011)
7. Ahmad, B, Ntouyas, SK, Alsedi, A: On fractional differential inclusions with anti-periodic type integral boundary conditions. Bound. Value Probl. **2013**, 82 (2013)
8. Bai, Z, Sun, W: Existence and multiplicity of positive solutions for singular fractional boundary value problems. Comput. Math. Appl. **63**, 1369-1381 (2012)
9. Baleanu, D, Agarwal, RP, Mohammadi, H, Rezapour, S: Some existence results for a nonlinear fractional differential equation on partially ordered Banach spaces. Bound. Value Probl. **2013**, 112 (2013)
10. Baleanu, D, Diethelm, K, Scalas, E, Trujillo, JJ: Fractional Calculus Models and Numerical Methods. World Scientific, Singapore (2012)
11. Baleanu, D, Mohammadi, H, Rezapour, S: The existence of solutions for a nonlinear mixed problem of singular fractional differential equations. Adv. Differ. Equ. **2013**, 359 (2013)
12. Baleanu, D, Mohammadi, H, Rezapour, S: Positive solutions of a boundary value problem for nonlinear fractional differential equations. Abstr. Appl. Anal. **2012**, Article ID 837437 (2012)
13. Baleanu, D, Mohammadi, H, Rezapour, S: Some existence results on nonlinear fractional differential equations. Philos. Trans. R. Soc. Lond. A **371**, 20120144 (2013)
14. Baleanu, D, Mohammadi, H, Rezapour, S: On a nonlinear fractional differential equation on partially ordered metric spaces. Adv. Differ. Equ. **2013**, 83 (2013)
15. Baleanu, D, Nazemi, Z, Rezapour, S: The existence of positive solutions for a new coupled system of multi-term singular fractional integro-differential boundary value problems. Abstr. Appl. Anal. **2013**, Article ID 368659 (2013)
16. Baleanu, D, Nazemi, Z, Rezapour, S: Existence and uniqueness of solutions for multi-term nonlinear fractional integro-differential equations. Adv. Differ. Equ. **2013**, 368 (2013)
17. Baleanu, D, Nazemi, Z, Rezapour, S: Attractivity for a k-dimensional system of fractional functional differential equations and global attractivity for a k-dimensional system of nonlinear fractional differential equations. J. Inequal. Appl. **2014**, 31 (2014)
18. Bragdi, M, Debbouche, A, Baleanu, D: Existence of solutions for fractional differential inclusions with separated boundary conditions in Banach space. Adv. Math. Phys. **2013**, Article ID 426061 (2013)
19. Campos, LM: On the solution of some simple fractional differential equations. Int. J. Math. Sci. **13**(3), 481-496 (1990)
20. Luchko, YF, Srivastava, HM: The exact solution of certain differential equations of fractional order by using operational calculus. Comput. Math. Appl. **29**, 73-85 (1995)
21. Ouahab, A: Some results for fractional boundary value problem of differential inclusions. Nonlinear Anal. **69**, 3877-3896 (2008)
22. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
23. Phung, PD, Truong, LX: On a fractional differential inclusion with integral boundary conditions in Banach space. Fract. Calc. Appl. Anal. **16**(3), 538-558 (2013)
24. Samko, G, Kilbas, A, Marichev, O: Fractional Integrals and Derivatives: Theory and Applications. Gordon & Breach, New York (1993)
25. Diethelm, K: Analysis of Fractional Differential Equations. Springer, Berlin (2010)
26. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies. Elsevier, Amsterdam (2006)
27. Miller, S, Ross, B: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
28. Oldham, KB, Spanier, J: The Fractional Calculus. Academic Press, San Diego (1974)
29. Deimling, K: Multi-Valued Differential Equations. de Gruyter, Berlin (1992)
30. Hu, S, Papageorgiou, N: Handbook of Multivalued Analysis, Volume I: Theory. Kluwer Academic, Dordrecht (1997)
31. Aubin, J, Cellna, A: Differential Inclusions: Set-Valued Maps and Viability Theory. Springer, Berlin (1984)
32. Lasota, A, Opial, Z: An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. **13**, 781-786 (1965)
33. Granas, A, Dugundji, J: Fixed Point Theory. Springer, New York (2003)
34. Covitz, H, Nadler, S: Multivalued contraction mappings in generalized metric spaces. Isr. J. Math. **8**, 5-11 (1970)
35. Su, X: Boundary value problem for a coupled system of nonlinear fractional differential equations. Appl. Math. Lett. **22**, 64-69 (2009)
36. Castaing, C, Valadier, M: Convex Analysis and Measurable Multifunctions. Lecture Notes in Mathematics, vol. 580. Springer, Berlin (1977)