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3-D flow of a compressible viscous micropolar fluid with spherical symmetry: a global existence theorem

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Abstract

We consider the nonstationary 3-D flow of a compressible viscous heat-conducting micropolar fluid in the domain to be the subset of $\mathbf{R}^3$ bounded with two concentric spheres that present the solid thermo-insulated walls. In the thermodynamical sense the fluid is perfect and polytropic. We assume that the initial density and temperature are bounded from below with a positive constant and that the initial data are sufficiently smooth spherically symmetric functions. The starting problem is transformed into the Lagrangian description on the spatial domain $]0,1[$. In this work we prove that our problem has a generalized solution for any time interval $[0,T]$, $T \in \mathbf{R}^+$. The proof is based on the local existence theorem and the extension principle.

Keywords: micropolar fluid; spherical symmetry; generalized solution; global existence

1 Introduction

The model of micropolar fluids, introduced by Eringen (*e.g.* in [1]), has received considerable attention in the last two decades. The model has many potential applications (see [2], p.13) and has become an important area of interest for mathematicians and engineers (see *e.g.* [3–7]). From a mathematical point of view, the micropolar fluid model is considered in two directions-one explores the incompressible and the other the compressible flows. The incompressible flow has been very well analyzed (*e.g.* see [8]), but there are still many open problems. Lately, the incompressible flow of magneto-micropolar fluids is increasingly being explored (*e.g.* see [9]). The compressible flow of the micropolar fluid has begun to be intensively studied in the last few years (*e.g.* see [10–12]).

In this paper we consider the model for the compressible flow of the isotropic, viscous and heat-conducting micropolar fluid which is in the thermodynamical sense perfect and polytropic. The described model in the one-dimensional case was first described by Mujaković in [13]. This model was analyzed in relation to existence, regularity and stabilization for different kinds of problems with homogeneous and non-homogeneous boundary conditions (*e.g.* see [14–16]). A significant number of results related to this one-dimensional model has been systematized in the fifth and sixth chapter of [17], but researches concerning the three-dimensional model for this kind of fluid are still at the beginning. Till now the described model of the compressible micropolar fluid in the three-dimensional case has been considered just in [18] or [19] by Dražić and Mujaković in the spherically symmetric case.

Here, as in [18] and [19], we analyze the motion of the described fluid on the domain

$$\left\{\mathbf{x} = (x_1, x_2, x_3) : \mathbf{x} \in \mathbf{R}^3, a < |\mathbf{x}| < b, |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}\right\}. \tag{1}$$

We assume that initial functions are spherically symmetric and smooth enough on (1). Based on this assumption, we analyze the spherically symmetric solution to the governing system. Because of spherical symmetry, the starting three-dimensional problem becomes one-dimensional with the spatial domain $]a, b[$ in the Eulerian description or on the domain $]0, 1[$ in the Lagrangian description.

Using the Faedo-Galerkin method in [18] it is proved that the corresponding problem with homogeneous boundary conditions for velocity, microrotation and heat flux has a generalized solution locally in time, *i.e.* on the domain $]0, 1[\times]0, T_0[$, where $T_0 > 0$ is sufficiently small. In [19] the uniqueness of the generalized solution for the same problem is proved. This work is a natural continuation of the research presented in these two papers, where we prove that the problem has a generalized solution globally in time, *i.e.* on the domain $]0, 1[\times]0, T[$, for any finite $T > 0$. The proof is based on the local existence theorem and the extension principle.

To be able to apply the extension principle we first derive a set of *a priori* bounds with constants dependent only on initial data and the constant $T > 0$ (boundary of time domain). Let us note that our time domain is arbitrary, but finite. The solution on the described time domain $]0, T[$, we call global as in [20] and [16]. Such a solution is analyzed for a much simpler problem in [21], Chapter 2 as well, but under the name 'solution in the whole'.

The results from this paper are a generalization of the results from [20] where the one-dimensional variant of the problem, which we analyze here, is presented. We use here some ideas from [20], as well as from the book [21] and [22] where a similar problem was considered for the classical compressible fluid (problem without microrotation).

The paper is organized as follows. In Section 2 we formally present the results from [18] and [19] which are important for this work and formulate the main result of this paper. In Section 3 we give the proof of our result-the global existence theorem. We first briefly explain the extension principle on which the proof is based. After that, we derive the set of *a priori* bounds which are needed to employ the extension principle and at the end we give the formal proof based on the obtained lemmas.

2 Statement of the problem and the main result

Our model is based on the local forms of the conservation laws for mass, moment, angular (momentum) moment and energy, as was stated in [18] ((1)-(4)), as well as on the constitutive equations for compressible viscous and heat-conducting micropolar fluid with assumptions that the fluid is perfect and polytropic ((5)-(7), (10)-(11) in [18]). These equations relate mass density ρ, velocity $\mathbf{v}$, microrotation $\boldsymbol{\omega}$, and temperature θ.

In [18] we introduce the spherically symmetric initial conditions

$$\rho_0(\mathbf{x}) = \rho_0(r), \qquad \mathbf{v}_0(\mathbf{x}) = \frac{\mathbf{x}}{r} v_0(r), \qquad \boldsymbol{\omega}_0(\mathbf{x}) = \frac{\mathbf{x}}{r}\omega_0(r), \qquad \theta_0(\mathbf{x}) = \theta_0(r), \tag{2}$$

where ρ_0, v_0, ω_0, and θ_0 are known real functions defined on $]a,b[$,[a] $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$, $r = |\mathbf{x}|$, and we assume that ρ, $\mathbf{v}$, $\boldsymbol{\omega}$, and θ are spherically symmetric too:

$$\begin{aligned} v_i(\mathbf{x},t) &= \frac{x_i}{r} v(r,t), \qquad \omega_i(\mathbf{x},t) = \frac{x_i}{r}\omega(r,t), \quad i = 1,2,3, \\ \rho(\mathbf{x},t) &= \rho(r,t), \qquad \theta(\mathbf{x},t) = \theta(r,t). \end{aligned} \tag{3}$$

Using the assumptions (3), the spatial domain (1) becomes a one-dimensional domain $]a,b[$. The governing system in the Eulerian description is given by formulas (16)-(21) in [18]. As it was stated in the book [21], p.40. when the global estimates are deduced, it is convenient to use Lagrangian description. The transition from the Eulerian to the Lagrangian description was done in [18], pp.4-5, but for the reader's convenience we will briefly describe it here. The Eulerian coordinates (r,t) are connected to the Lagrangian coordinates (ξ,t) by the relation

$$r(\xi,t) = r_0(\xi) + \int_0^t \tilde{v}(\xi,t)\,d\tau, \qquad r_0(\xi) = r(\xi,0) = \xi, \tag{4}$$

where $\tilde{v}(\xi,t)$ is defined by $\tilde{v}(\xi,t) = v(r(\xi,t),t)$. Using the same procedure as in [21] we then introduce (see [18], (21)-(25)) the new function η by

$$\eta(\xi) = \int_a^\xi s^2 \rho_0(s)\,ds, \tag{5}$$

define the new constant L by

$$\eta(b) = \int_a^b s^2 \rho_0(s)\,ds = L \tag{6}$$

and introduce the new coordinate $x = L^{-1}\eta(\xi)$. With this new coordinate the spatial domain becomes $]0,1[$ and we get the following initial-boundary problem:

$$\frac{\partial \rho}{\partial t} = -\frac{1}{L}\rho^2 \frac{\partial}{\partial x}\left(r^2 v\right), \tag{7}$$

$$\frac{\partial v}{\partial t} = -\frac{R}{L} r^2 \frac{\partial}{\partial x}(\rho\theta) + \frac{\lambda + 2\mu}{L^2} r^2 \frac{\partial}{\partial x}\left(\rho \frac{\partial}{\partial x}\left(r^2 v\right)\right), \tag{8}$$

$$\rho \frac{\partial \omega}{\partial t} = -\frac{4\mu_r}{j_I}\omega + \frac{c_0 + 2c_d}{j_I L^2} r^2 \rho \frac{\partial}{\partial x}\left(\rho \frac{\partial}{\partial x}\left(r^2\omega\right)\right), \tag{9}$$

$$\begin{aligned} \rho \frac{\partial \theta}{\partial t} = {} & \frac{k}{c_v L^2}\rho \frac{\partial}{\partial x}\left(r^4 \rho \frac{\partial \theta}{\partial x}\right) - \frac{R}{c_v L}\rho^2\theta \frac{\partial}{\partial x}\left(r^2 v\right) + \frac{\lambda + 2\mu}{c_v L^2}\left[\rho \frac{\partial}{\partial x}\left(r^2 v\right)\right]^2 \\ & - \frac{4\mu}{c_v L}\rho \frac{\partial}{\partial x}\left(r v^2\right) + \frac{c_0 + 2c_d}{c_v L^2}\left[\rho \frac{\partial}{\partial x}\left(r^2\omega\right)\right]^2 - \frac{4c_d}{c_v L}\rho \frac{\partial}{\partial x}\left(r\omega^2\right) + \frac{4\mu_r}{c_v}\omega^2, \end{aligned} \tag{10}$$

$$\rho(x,0) = \rho_0(x), \qquad v(x,0) = v_0(x), \qquad \omega(x,0) = \omega_0(x), \qquad \theta(x,0) = \theta_0(x), \tag{11}$$

$$v(0,t) = v(1,t) = 0, \qquad \omega(0,t) = \omega(1,t) = 0, \qquad \frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(1,t) = 0, \tag{12}$$

considered on the domain $Q_T =]0,1[\times]0,T[$, where $T > 0$ is arbitrary. With the boundary conditions (12) we describe the acting of the solid thermo-insulated walls.

The positive constant j_I is microinertia density. λ and μ are coefficients of viscosity and μ_r, c_0, and c_d are the coefficients of microviscosity. Because of the Clausius-Duhamel inequalities they must have the properties:

$$\mu, \mu_r, c_d \geq 0, \qquad \lambda + 2\mu \geq 0, \qquad c_0 + 2c_d \geq 0. \tag{13}$$

By the constant k ($k \geq 0$) we denote the heat-conduction coefficient, the positive constant R is the specific gas constant, and the positive constant c_v denotes the specific heat for a constant volume.

If (7) is considered in the Eulerian description (see (16) in [18]) and by using (5) we can conclude as in [23], p.169 that

$$\frac{\partial r(x,t)}{\partial x} = \frac{L}{\rho(x,t)r^2(x,t)}. \tag{14}$$

Taking into account (14) and (4) we have

$$r_0(x) = \left(a^3 + 3L\int_0^x \frac{1}{\rho_0(y)}\,dy\right)^{\frac{1}{3}}, \quad x \in \left]0,1\right[\tag{15}$$

($a > 0$ is the radius of the smaller boundary sphere), and

$$r(x,t) = r_0(x) + \int_0^t v(x,\tau)\,d\tau, \quad (x,t) \in Q_T. \tag{16}$$

In this work we consider the properties of the so-called generalized solution to the problem (7)-(12) which is introduced in [18], p.6 as follows.

Definition 2.1 A generalized solution of the problem (7)-(12) in the domain Q_T is a function

$$(x,t) \mapsto (\rho, v, \omega, \theta)(x,t), \quad (x,t) \in Q_T, \tag{17}$$

where

$$\rho \in \mathrm{L}^\infty\big(0,T;\mathrm{H}^1(\left]0,1\right[)\big) \cap \mathrm{H}^1(Q_T), \quad \inf_{Q_T} \rho > 0, \tag{18}$$

$$v, \omega, \theta \in \mathrm{L}^\infty\big(0,T;\mathrm{H}^1(\left]0,1\right[)\big) \cap \mathrm{H}^1(Q_T) \cap \mathrm{L}^2\big(0,T;\mathrm{H}^2(\left]0,1\right[)\big), \tag{19}$$

that satisfies (7)-(10) a.e. in Q_T and conditions (11)-(12) in the sense of traces.

As was stated in [18], Remark 2.1, p.7, from the embedding and interpolation theorems one can conclude that our generalized solution could be treated as a strong solution.

We assume that the initial data (11) have the following smoothness properties:

$$\rho_0, \theta_0 \in H^1(\left]0,1\right[), \qquad v_0, \omega_0 \in H_0^1(\left]0,1\right[), \tag{20}$$

and that there exists a constant $m \in \mathbf{R}^+$ such that

$$\rho_0(x) \geq m, \qquad \theta_0(x) \geq m \quad \text{for } x \in \left]0,1\right[. \tag{21}$$

Because of the embedding $H^m(]0,1[) \hookrightarrow C^k([0,1])$, for $m-k > \frac{1}{2}$ from (20) and (21) we find that there exists $M \in \mathbf{R}^+$, such that

$$\rho_0(x), \left|v_0(x)\right|, \left|\omega_0(x)\right|, \theta_0(x) \leq M, \quad x \in [0,1]. \tag{22}$$

In this work we use the following result which is proved in [18] and [19].

Theorem 2.1 *Let the initial functions ρ_0, v_0, ω_0, and θ_0 satisfy conditions* (21) *and* (20). *Then there exists small enough $T_0 \in \mathbf{R}^+$ such that the problem* (7)-(12) *has at most one generalized solution $(\rho, v, \omega, \theta)$ in $Q_{T_0} =]0,1[\times]0,T_0[$, having the property*

$$\theta > 0 \quad in\ \overline{Q}_{T_0}. \tag{23}$$

For the function r we have

$$r \in L^\infty\bigl(0,T_0;H^2\bigl(]0,1[\bigr)\bigr) \cap H^2(Q_{T_0}) \cap C(\overline{Q}_{T_0}), \tag{24}$$

$$\frac{a}{2} \leq r \leq 2M \quad in\ \overline{Q}_{T_0}, \tag{25}$$

where the constant a is from (15) *and the constant M from* (22).

Using Theorem 2.1 and extension principle in this work we shall prove the following result.

Theorem 2.2 *Let the initial functions ρ_0, v_0, ω_0, and θ_0 satisfy conditions* (21) *and* (20). *Then for any $T \in \mathbf{R}^+$ there exists a generalized solution of the problem* (7)-(12) *on the domain Q_T with the property*

$$\theta > 0 \quad in\ \overline{Q}_T. \tag{26}$$

3 The proof of Theorem 2.2

The proof of Theorem 2.2 is based on the extension principle. The idea of this principle is explained for example in [24], in the proof of Theorem 2.4, p.233. For the reader's convenience let us explain it briefly. Theorem 2.1 ensures the existence of a unique solution of our problem on the time domain $]t_0, t_0+T_0[$ with initial functions $\rho(\cdot,t_0)$, $v(\cdot,t_0)$, $\omega(\cdot,t_0)$, and $\theta(\cdot,t_0)$. So, we have the existence on the time domain $]0,T_1[$, where $T_1 = t_0 + T_0$. After we repeat the procedure for k steps, we will have the existence on the time domain $]0,T_k[$, so we can continue this procedure as long as through *a priori* estimates we can ensure that $\rho(\cdot,t_0)$, $v(\cdot,t_0)$, $\omega(\cdot,t_0)$, and $\theta(\cdot,t_0)$ satisfy the conditions for initial functions for any $t_0 \in]0,T_k[$. We make this principle more formal in the following proposition, as was done for example in [16].

Proposition 3.1 *Let $T \in \mathbf{R}^+$ and let the function*

$$(x,t) \mapsto (\rho, v, \omega, \theta)(x,t), \quad (x,t) \in Q_{T'} \tag{27}$$

be the generalized solution of the problem (7)-(12) *on the domain $Q_{T'}$, for any $T' < T$ with the property $\theta > 0$ in $\overline{Q}_{T'}$. Then* (27) *is the generalized solution of the same problem on the domain Q_T with the property $\theta > 0$ in $\overline{Q}_T$.*

As it is explained in the book [21], p.40, to be able to use the Proposition 3.1 it is crucial to find a set of global *a priori* estimates in which the constants are independent of the length of time domain from the local existence theorem. The constants can depend on initial data and the constant T from the Proposition 3.1 only. We shall note these constants by C or C_i where $i = 1, 2, \dots$ and in different places they can take over different values.

3.1 Lower bounds for density and temperature

Following the procedure from the book [21], Chapter 2 we first shall derive some properties of the functions ρ and θ. To be precise we have to show that these two functions are bounded from below which is the hardest part of this paper. We will also show the upper boundedness for the function ρ and derive some important properties for the function θ. Apart from the book [21], in this part of the work we also use the ideas from articles [20] and [22]. In the cases when we will use the results from other papers we will omit the proofs or details of proofs, but we will refer to them appropriately.

In almost all lemmas hereafter we use the lower boundedness of the function r:

Lemma 3.1 (Lemma 3.1 in [19], p.4) *The function r defined by* (16) *satisfies the estimate*

$$r(x,t) \geq a, \quad (x,t) \in Q_T, \tag{28}$$

where $a > 0$ is the radius of the smaller boundary sphere of the starting domain.

3.1.1 The 'energy' estimate

We first introduce the function

$$U(x,t) = \frac{v^2}{2} + j_I \frac{\omega^2}{2} + R\psi\left(\frac{1}{\rho}\right) + c_v \psi(\theta), \tag{29}$$

where

$$\psi(x) = x - \ln x - 1 \tag{30}$$

is a non-negative and convex function. Let us note that the function U is the generalization of the energy function. The estimate of the energy function is crucial for obtaining the estimates and properties of the functions ρ, v, ω, and θ in the following sections.

Lemma 3.2 *There exists a constant $C \in \mathbf{R}^+$ such that*

$$\begin{aligned}&\int_0^1 U(x,t)\,dx + \int_0^t \int_0^1 \left[\frac{k}{L^2}\frac{r^4\rho}{\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\lambda + \frac{2}{3}\mu\right)\frac{\rho}{\theta}\left(\frac{\partial}{\partial x}(r^2 v)\right)^2 \right.\\ &\quad \left. \times \left(c_0 + \frac{2}{3}c_d\right)\frac{\rho}{\theta}\left(\frac{\partial}{\partial x}(r^2\omega)\right)^2\right] dx\,d\tau \leq C.\end{aligned} \tag{31}$$

Proof Multiplying (7), (8), (9), and (10), respectively, by $R(-\frac{1}{\rho^2} + \frac{1}{\rho})$, v, $j_I\omega\rho^{-1}$, and $c_v(1 - \frac{1}{\theta})\rho^{-1}$, after addition and integration over $[0,1]$ we obtain

$$\begin{aligned}&\int_0^1 \frac{\partial}{\partial t} U\,dx + \frac{\lambda + 2\mu}{L^2}\int_0^1 \frac{\rho}{\theta}\left[\frac{\partial}{\partial x}(r^2 v)\right]^2 dx\\ &\quad + \frac{c_0 + 2c_d}{L^2}\int_0^1 \frac{\rho}{\theta}\left[\frac{\partial}{\partial x}(r^2\omega)\right]^2 dx\end{aligned}$$

$$
\begin{aligned}
&+ \frac{k}{L^2} \int_0^1 \frac{r^4 \rho}{\theta^2} \left[\frac{\partial \theta}{\partial x} \right]^2 dx + 4\mu_r \int_0^1 \frac{\omega^2}{\rho \theta} dx \\
&= \frac{4\mu}{L} \int_0^1 \frac{1}{\theta} \frac{\partial}{\partial x} \left(r v^2 \right) dx + \frac{4c_d}{L} \int_0^1 \frac{1}{\theta} \frac{\partial}{\partial x} \left(r \omega^2 \right) dx.
\end{aligned}
\tag{32}
$$

Integrating (32) over $[0, t]$, using (14) as well as the equality

$$
\frac{\partial}{\partial x} \left(r v^2 \right) = \frac{2}{r} v \frac{\partial}{\partial x} \left(r^2 v \right) - \frac{3 L v^2}{\rho r^2},
\tag{33}
$$

which is also valid for the function ω, after some calculations we get

$$
\begin{aligned}
&\int_0^1 U(x,t)\, dx + \left(\lambda + \frac{2}{3} \mu \right) \frac{1}{L^2} \int_0^t \int_0^1 \frac{\rho}{\theta} \left[\frac{\partial}{\partial x} \left(r^2 v \right) \right]^2 dx\, d\tau \\
&+ \left(c_0 + \frac{2}{3} c_d \right) \frac{1}{L^2} \int_0^t \int_0^1 \frac{\rho}{\theta} \left[\frac{\partial}{\partial x} \left(r^2 \omega \right) \right]^2 dx\, d\tau \\
&+ \frac{k}{L^2} \int_0^t \int_0^1 \frac{r^4 \rho}{\theta^2} \left[\frac{\partial \theta}{\partial x} \right]^2 dx\, d\tau \leq \int_0^1 U(x,0)\, dx.
\end{aligned}
\tag{34}
$$

Taking into account (20) we easily conclude the following estimate:

$$
\int_0^1 U(x,0)\, dx \leq C \left(1 + \left\| (\rho_0, v_0, \omega_0, \theta_0) \right\|^2_{L^2(]0,1[)^4} \right) \leq C,
\tag{35}
$$

which together with (34) immediately gives (31). □

Lemma 3.3 *Let α_1 and α_2 be two positive solutions of the equation*

$$
\psi(x) = C c_v^{-1},
\tag{36}
$$

where C is the same constant as in (31), *and ψ is the function defined by* (30). *Then, for any $t \in]0, T[$ we have*

$$
\alpha_1 \leq \int_0^1 \theta(x,t)\, dx \leq \alpha_2
\tag{37}
$$

and there exists a function $a : [0, T] \to [0, 1]$ such that

$$
\alpha_1 \leq \theta\left(a(t), t \right) \leq \alpha_2.
\tag{38}
$$

Proof From (31) we immediately get

$$
\int_0^1 (\theta - \ln \theta - 1)(x,t)\, dx \leq C c_v^{-1}.
\tag{39}
$$

As the function ψ is convex, we are able to utilize the Jensen inequality and conclude that

$$
\int_0^1 \theta(x,t)\, dx - \ln \int_0^1 \theta(x,t)\, dx - 1 \leq C c_v^{-1}.
\tag{40}
$$

From (40) we easily get (37) and (38). □

3.1.2 Some auxiliary constructions

The aim of this section is to derive a useful representation of the function ρ, which is known in literature as the representation of the Kazhikov type. We will also list all important properties of the functions connected to this representation.

Lemma 3.4 *Let A be the constant defined by*

$$A = \int_0^1 \frac{1}{\rho_0(x)}\,dx. \tag{41}$$

For the function ρ and for any $t \in]0, T[$ we have

$$\int_0^1 \frac{1}{\rho(x,t)}\,dx = A. \tag{42}$$

Also, there exists a function g, $0 \leq g(t) \leq 1$ such that

$$\rho\bigl(g(t), t\bigr) = A^{-1}, \quad t \in [0, T]. \tag{43}$$

Proof In the same way as in Lemma 2.1 in [21], p.43, from (7) we obtain (42) and (43). □

In the next lemma we use the same procedure as in [21], p.44, as well as some ideas from [22], p.349 in order to make the aforementioned representation of the function ρ.

Lemma 3.5 *For the function ρ on Q_T we have*

$$\rho(x,t) = \frac{\rho_0(x)\cdot Y(t)\cdot B(x,t)}{1 + \frac{R}{\lambda+2\mu}\rho_0(x)\int_0^t \theta(x,\tau)\cdot Y(\tau)\cdot B(x,\tau)\,d\tau}, \tag{44}$$

where

$$Y(t) = \frac{1}{A\rho_0(g(t))}\exp\left\{\frac{R}{\lambda+2\mu}\int_0^t \rho\bigl(g(t),\tau\bigr)\theta\bigl(g(t),\tau\bigr)\,d\tau\right\} \tag{45}$$

and

$$B(x,t) = \exp\left\{-\frac{L}{\lambda+2\mu}\int_{g(t)}^x \int_0^t r^{-2}(y,\tau)\frac{\partial v(y,\tau)}{\partial t}\,d\tau\,dy\right\}. \tag{46}$$

(*The constant A and the function g are from Lemma* 3.4.)

Proof Let us write (7) in the form

$$\frac{1}{L}\rho\frac{\partial}{\partial x}\bigl(r^2 v\bigr) = -\frac{\partial}{\partial t}\ln\rho \tag{47}$$

and insert it in (8). After we integrate the obtained equality over $[0,t]$, $t \in]0, T[$, we get

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{\lambda+2\mu}{L}\ln\rho + \frac{R}{L}\int_0^t \rho(x,\tau)\theta(x,\tau)\,d\tau\right) \\
&\quad = \frac{\lambda+2\mu}{L}\frac{\partial}{\partial x}\ln\rho_0(x) - \int_0^t r^{-2}(x,\tau)\frac{\partial v(x,\tau)}{\partial t}\,d\tau.
\end{aligned} \tag{48}$$

Now, we integrate (48) over $[g(t),x]$, $x \in]0,1[$ for fixed t and get

$$\begin{aligned} &\frac{\lambda+2\mu}{L}\ln\rho + \frac{R}{L}\int_0^t \rho(x,\tau)\theta(x,\tau)\,d\tau \\ &\quad= \frac{\lambda+2\mu}{L}\ln\rho\bigl(g(t),t\bigr) + \frac{R}{L}\int_0^t \rho\bigl(g(t),\tau\bigr)\theta\bigl(g(t),\tau\bigr)\,d\tau \\ &\qquad+ \frac{\lambda+2\mu}{L}\ln\frac{\rho_0(x)}{\rho_0(g(t))} - \int_{g(t)}^x\int_0^t r^{-2}(y,\tau)\frac{\partial v(y,\tau)}{\partial t}\,d\tau\,dy. \end{aligned} \tag{49}$$

Taking into account (43), (45), and (46) we easily get (44). □

Lemma 3.6 *There exists $C \in \mathbf{R}^+$ such that for $(x,t) \in Q_T$ we have*

$$\left|\int_{g(t)}^x\int_0^t r^{-2}(y,\tau)\frac{\partial v(y,\tau)}{\partial t}\,d\tau\,dy\right| \le C, \tag{50}$$

where the function g is defined by (43).

Proof In the same way as in [22], p.349, (3.35), with the help of (28) and (31) we obtain (50). □

Lemma 3.7 *The function B defined by* (46) *has the properties*:

$$C^{-1} \le B(x,t) \le C, \tag{51}$$

$$\frac{\partial B(x,t)}{\partial x} = B(x,t)\varphi(x,t), \tag{52}$$

where

$$\varphi(x,t) = \frac{-L}{\lambda+2\mu}\int_0^t r^{-2}(y,\tau)\frac{\partial v(y,\tau)}{\partial t}\,d\tau, \tag{53}$$

for $(x,t) \in Q_T$ and $C \in \mathbf{R}^+$.

Proof Using (50) and (53) from (46) we immediately get (51) and (52). □

Lemma 3.8 *There exist constants $C_1, C_2 \in \mathbf{R}^+$ such that for any $t \in]0,T[$ we have*

$$C_1 \le Y(t) \le C_2. \tag{54}$$

Proof In the same way as in [21], Lemma 2.2, p.45, using (37), (42), (51), and the Gronwall inequality from (44) we get (54). □

Now we introduce the notations analogous to the one in [21], p.46 for the maximal and minimal values of the functions ρ and θ for fixed t:

$$\begin{aligned} m_\rho(t) &= \min_{x\in[0,1]}\rho(x,t), & M_\rho(t) &= \max_{[0,1]}\rho(x,t), \\ m_\theta(t) &= \min_{x\in[0,1]}\theta(x,t), & M_\theta(t) &= \max_{x\in[0,1]}\theta(x,t). \end{aligned} \tag{55}$$

The following relationships of the functions from (55) are crucial for deriving the bounds of the function ρ.

Lemma 3.9 *There exist positive constants C_1 and C_2 such that*

$$M_\rho(t) \le C_1\left(1+\int_0^t m_\theta(\tau)\,d\tau\right)^{-1} \tag{56}$$

and

$$m_\rho(t) \ge C_2\left(1+\int_0^t M_\theta(\tau)\,d\tau\right)^{-1}. \tag{57}$$

Proof In the same way as in Lemma 2.3 in [21], p.46, using (51) and (54) from (44) we immediately get (56) and (57). □

To derive the further properties of the function θ we will need the following result.

Lemma 3.10 (Lemma 2.4 in [21], p.47) *For any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$, such that for any $t \in]0,T[$ we have*

$$M_\theta^2(t) \le \varepsilon I_1(t) + C_\varepsilon\bigl(1+I_2(t)\bigr), \tag{58}$$

where

$$I_1(t) = \int_0^1 r^4\rho\left(\frac{\partial\theta}{\partial x}\right)^2 dx, \qquad I_2(t) = \int_0^t I_1(\tau)\,d\tau. \tag{59}$$

Let us mention that we slightly adapted the form of inequality (58) comparing to the one in [21], as well as the form of the function I_1, but the proof remains the same.

3.1.3 Lower bound for the function θ

In the proof of the following lemma we used the adapted approach from [21], Lemma 3.1, p.48, as well as some ideas from Lemma 2.3, p.202 in [20] and Lemma 3.12, p.356 in [22].

Lemma 3.11 *There exists a constant $C \in \mathbf{R}^+$ such that for any $t \in]0,T[$ we have*

$$m_\theta(t) \ge C. \tag{60}$$

Proof Multiplying (10) by $-\theta^{-2}\rho^{-1}$ we get

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{\theta}\right) = {}& \frac{k}{c_v L^2}\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial}{\partial x}\left(\frac{1}{\theta}\right)\right) - \frac{2k}{c_v L^2}\frac{\rho r^4}{\theta^3}\left(\frac{\partial\theta}{\partial x}\right)^2 + \frac{R}{c_v L}\frac{\rho}{\theta}\frac{\partial}{\partial x}\left(r^2 v\right)\\ &- \frac{1}{\theta^2}\left\{\frac{\lambda+2\mu}{c_v L^2}\rho\left[\frac{\partial}{\partial x}\left(r^2 v\right)\right]^2 - \frac{4\mu}{c_v L}\frac{\partial}{\partial x}\left(r v^2\right)\right\}\\ &- \frac{1}{\theta^2}\left\{\frac{c_2+2c_d}{c_v L^2}\rho\left[\frac{\partial}{\partial x}\left(r^2\omega\right)\right]^2 - \frac{4c_d}{c_v L}\frac{\partial}{\partial x}\left(r\omega^2\right)\right\} - \frac{4\mu_r}{c_v}\frac{\omega^2}{\rho\theta^2},\end{aligned} \tag{61}$$

which implies the inequality

$$\frac{\partial}{\partial t}\left(\frac{1}{\theta}\right) \leq \frac{k}{c_v L^2}\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial}{\partial x}\left(\frac{1}{\theta}\right)\right) + \frac{R^2}{4c_v(\lambda + \frac{2}{3}\mu)}\rho - \frac{\rho}{\theta^2}\left[\frac{1}{L}\sqrt{\frac{\lambda + \frac{2}{3}\mu}{c_v}}\frac{\partial}{\partial x}\left(r^2 v\right) - \frac{R}{2\sqrt{c_v}\sqrt{\lambda + \frac{2}{3}\mu}}\theta\right]^2, \tag{62}$$

i.e.

$$\frac{\partial}{\partial t}\left(\frac{1}{\theta}\right) \leq \frac{k}{c_v L^2}\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial}{\partial x}\left(\frac{1}{\theta}\right)\right) + \frac{R^2}{4c_v(\lambda + \frac{2}{3}\mu)}\rho. \tag{63}$$

After multiplying (63) by $p\theta^{-p+1}$, $p \geq 2$, we have

$$\frac{\partial}{\partial t}\left(\frac{1}{\theta^p}\right) \leq \frac{kp}{c_v L^2}\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial}{\partial x}\left(\frac{1}{\theta}\right)\right)\left(\frac{1}{\theta}\right)^{p-1} + \frac{R^2 p}{4c_v(\lambda + \frac{2}{3}\mu)}\rho\left(\frac{1}{\theta}\right)^{p-1} \tag{64}$$

which, after integration over $]0,1[$, gives

$$\frac{d}{dt}\left\|\frac{1}{\theta(t)}\right\|^p_{L^p(]0,1[)} \leq \frac{R^2 p}{4c_v(\lambda + \frac{2}{3}\mu)}\int_0^1 \rho\left(\frac{1}{\theta}\right)^{p-1} dx. \tag{65}$$

After applying the Hölder inequality to the right-hand side of (65) we get

$$\frac{d}{dt}\left\|\frac{1}{\theta(t)}\right\|^p_{L^p(]0,1[)} \leq \frac{R^2 p}{4c_v(\lambda + \frac{2}{3}\mu)}\left\|\rho(t)\right\|_{L^p(]0,1[)}\left\|\frac{1}{\theta(t)}\right\|^{p-1}_{L^p(]0,1[)}, \tag{66}$$

hence we have

$$\frac{d}{dt}\left\|\frac{1}{\theta(t)}\right\|_{L^p(]0,1[)} \leq \frac{R^2}{4c_v(\lambda + \frac{2}{3}\mu)}\left\|\rho(t)\right\|_{L^p(]0,1[)}. \tag{67}$$

Now we integrate (67) over $[0,t]$, $t \in]0,T[$ and obtain

$$\left\|\frac{1}{\theta(t)}\right\|_{L^p(]0,1[)} \leq \left\|\frac{1}{\theta(0)}\right\|_{L^p(]0,1[)} + \frac{R^2}{4c_v(\lambda + \frac{2}{3}\mu)}\int_0^t \left\|\rho(\tau)\right\|_{L^p(]0,1[)} d\tau, \tag{68}$$

which implies the assertion of the lemma, analogously to the proof of Lemma 3.1 in [21], p.49. □

With the help of (60) and (56) we immediately get the following property of the function ρ.

Corollary 3.1 *There exists a constant $C \in \mathbf{R}^+$ such that for any $t \in]0,T[$ we have*

$$M_\rho \leq C. \tag{69}$$

3.1.4 Lower bound for the function ρ

In obtaining the lower bound for the density we were not able to use the method proposed in [21], p.50 which is used in [20], p.203, for the one-dimensional model, so we adapted here the idea from [22], Lemma 3.6, p.348.

Lemma 3.12 *There exists a constant $C \in \mathbf{R}^{+}$ such that for any $t \in]0,T[$ we have*

$$m_\rho(t) \geq C. \tag{70}$$

Proof By using the Cauchy-Schwarz inequality as well as (38), (37), and (28) we get

$$\begin{aligned}\left|\sqrt{\theta(x,t)} - \sqrt{\theta\big(a(t),t\big)}\right| &\leq C \int_{a(t)}^{x} \frac{1}{\sqrt{\theta}} \left|\frac{\partial\theta}{\partial x}\right| dy \\ &\leq C \left(\int_0^1 \frac{r^4\rho}{\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 \frac{\theta}{r^4\rho}\,dx\right)^{\frac{1}{2}} \\ &\leq C \left(\int_0^1 \frac{r^4\rho}{\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 dx\right)^{\frac{1}{2}} \frac{1}{\sqrt{m_\rho}}.\end{aligned} \tag{71}$$

Taking into account estimate (38), from (71) we obtain

$$\theta(x,t) \leq C\left(1 + \left(\int_0^1 \frac{r^4\rho}{\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 dx\right)\frac{1}{m_\rho}\right), \tag{72}$$

which we insert into (57) and get

$$\frac{1}{m_\rho(t)} \leq C\left(1 + \int_0^t \int_0^1 \frac{r^4\rho}{\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 dx \frac{1}{m_\rho(\tau)}\,d\tau\right). \tag{73}$$

After we apply the Gronwall inequality to (73) and use estimate (31) we immediately get (70). □

3.2 *A priori* estimates for derivatives

To be able to derive the estimates of derivatives for functions ρ, v, ω, and θ we will apply the energy method. Therefore, we will make the estimate of the function

$$\Phi = \frac{1}{2}v^2 + \frac{1}{2}j_I\omega^2 + c_v\theta, \tag{74}$$

adapting the procedure used in the proof of Lemma 2.4 in [20], p.203.

Lemma 3.13 *There exists a constant $C \in \mathbf{R}^{+}$ such that for any $t \in]0,T[$ we have*

$$\int_0^1 \left(\Phi^2 + v^4 + \omega^4\right) dx + I_2 \leq C, \tag{75}$$

where the function I_2 is defined by (59).

Proof First we multiply (8), (9), and (10), respectively, by v, $j_I\omega\rho^{-1}$ and $c_v\rho^{-1}$, and integrate them over $]0,1[$. After addition of the obtained equalities, making use of boundary conditions and (14), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\Phi^2(x,t)\,dx = {}&-\int_0^1\biggl(\frac{\lambda+2\mu}{L^2}\rho r^4\frac{\partial\Phi}{\partial x}+\frac{2\lambda}{L}rv^2-\frac{R}{L}\rho\theta r^2v+\frac{2c_o}{L}r\omega^2\\&+\left(\frac{c_0+2c_d}{L^2}-j_I\frac{\lambda+2\mu}{L^2}\right)\rho r^4\omega\frac{\partial\omega}{\partial x}\\&+\left(\frac{k}{L^2}-c_v\frac{\lambda+2\mu}{L^2}\right)r^4\rho\frac{\partial\theta}{\partial x}\biggr)\frac{\partial\Phi}{\partial x}\,dx,\end{aligned}\tag{76}$$

which, using the Young inequality, implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\Phi^2(x,t)\,dx\\&\quad+\int_0^1\rho r^4\left[\frac{\lambda+2\mu}{L^2}\left(\frac{\partial\Phi}{\partial x}\right)^2dx+\left(\frac{k}{L^2}-c_v\frac{\lambda+2\mu}{L^2}\right)\frac{\partial\theta}{\partial x}\frac{\partial\Phi}{\partial x}\,dx\right]\\&\quad\le C_1\frac{\varepsilon^{-1}}{2}\int_0^1\left(\left(\rho r^2\right)^{-1}v^4+\rho\theta^2v^2+\left(\rho r^2\right)^{-1}\omega^4+\rho r^4\omega^2\left(\frac{\partial\omega}{\partial x}\right)^2\right)dx\\&\qquad+2C_1\varepsilon\int_0^1\rho r^4\left(\frac{\partial\Phi}{\partial x}\right)^2,\end{aligned}\tag{77}$$

where $\varepsilon>0$ is arbitrary.

To simplify (77), using elementary algebraic operations, we derive the following inequality:

$$\begin{aligned}&(A-B)(a+b+c)^2+(C-A)c(a+b+c)\\&\quad\ge(C-3B)c^2-\left(2B+\frac{(A-2B+C)^2}{4B}\right)a^2-\frac{1}{2B}\left[(A-B)^2+\frac{(A-2B+C)^2}{2}\right]b^2,\end{aligned}\tag{78}$$

where $A,B,C,a,b,c\in\mathbf{R}$, and $A>B>0$. In (78) we insert $a=v\frac{\partial v}{\partial x}$, $b=j_I\omega\frac{\partial\omega}{\partial x}$, $c=c_v\frac{\partial\theta}{\partial x}$, $A=\frac{\lambda+2\mu}{L^2}$, $B=2C_1\varepsilon$, $C=\frac{k}{c_vL^2}$, and choose ε such that $A-B>0$ and $C-3B>0$. For simplicity reasons we denote $D=C-3B$ and

$$\begin{aligned}C_2=\max\biggl\{&2B+\frac{(A-2B+C)^2}{4B},\\&\frac{1}{2B}\left[(A-B)^2+\frac{(A-2B+C)^2}{2}\right]+C_1\frac{\varepsilon^{-2}}{2}\cdot C_1\frac{\varepsilon^{-2}}{2}\biggr\}.\end{aligned}\tag{79}$$

We get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\Phi^2(x,t)\,dx+D\int_0^1\rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2dx\\&\quad\le C_2\int_0^1\biggl(\left(\rho r^2\right)^{-1}v^4+\rho\theta^2v^2+\left(\rho r^2\right)^{-1}\omega^4\\&\qquad+\rho r^4v^2\left(\frac{\partial v}{\partial x}\right)^2+\rho r^4\omega^2\left(\frac{\partial\omega}{\partial x}\right)^2\biggr)dx.\end{aligned}\tag{80}$$

Using (69), (70), and (28), from (80) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \Phi^2(x,t)\,dx + D\int_0^1 \rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2 dx \\ \leq C_3\int_0^1\left(v^4+\theta^2 v^2+\omega^4+r^4 v^2\left(\frac{\partial v}{\partial x}\right)^2+r^4\omega^2\left(\frac{\partial\omega}{\partial x}\right)^2\right)dx. \tag{81}$$

To be able to bound the terms v^4 and ω^4 on the right-hand side of (81) we multiply (8) and (9), respectively, by v^3 and ω^3, integrate over $]0,1[$ and utilize the Young inequality. After some calculations and making use of (69), (70), and (28), we get

$$\frac{1}{4}\int_0^1 \frac{\partial v^4}{\partial t}\,dx + C_4\int_0^1 r^4 v^2\left(\frac{\partial v}{\partial x}\right)^2 dx \leq C_5\int_0^1\left(v^4+\theta^2 v^2\right)dx, \tag{82}$$

$$\frac{1}{4}\int_0^1 \frac{\partial \omega^4}{\partial t}\,dx + C_6\int_0^1 r^4 \omega^2\left(\frac{\partial \omega}{\partial x}\right)^2 dx \leq C_7\int_0^1 \omega^4\,dx. \tag{83}$$

Now, we multiply (82) by $C_3C_4^{-1}$ and (83) by $C_3C_6^{-1}$. After the addition of the obtained inequalities with (81), we find

$$\frac{d}{dt}\left[\int_0^1\left(\Phi^2(x,t)+\frac{C_3C_4^{-1}}{2}v^4+\frac{C_3C_6^{-1}}{2}\omega^4\right)dx + 2D\int_0^t\int_0^1 \rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2 dx\,d\tau\right] \\ \leq 2C_3\int_0^1\left(\left(1+C_5C_4^{-1}\right)\left(v^4+\theta^2v^2\right)+\left(1+C_7C_6^{-1}\right)\omega^4\right)dx. \tag{84}$$

To finish the proof we need the following inequality which is the direct consequence of (58):

$$\int_0^1 \theta^2 v^2\,dx \leq C_8\epsilon\int_0^1 r^4\rho\left(\frac{\partial\theta}{\partial x}\right)^2 dx + C_\epsilon\left(1+\int_0^t\int_0^1 \rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2 dx\,d\tau\right) \tag{85}$$

which we insert into (84) and use the suitable ϵ. Hence we have

$$\frac{d}{dt}\int_0^1\left(\Phi^2(x,t)+\frac{C_3C_4^{-1}}{2}v^4+\frac{C_3C_6^{-1}}{2}\omega^4+D\int_0^t \rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2 d\tau\right)dx \\ \leq C_9\int_0^1\left(\Phi^2(x,t)+\frac{C_3C_4^{-1}}{2}v^4+\frac{C_3C_6^{-1}}{2}\omega^4+D\int_0^t \rho r^4\left(\frac{\partial\theta}{\partial x}\right)^2 d\tau\right)dx+1. \tag{86}$$

Using the Gronwall inequality, from (86) we immediately get (75). □

Equations (75) and (58) imply an important property of the function M_θ, which is given in the next corollary.

Corollary 3.2 *There exists a constant $C\in\mathbf{R}^+$ such that we have*

$$\|M_\theta\|_{L^2(]0,T[)} \leq C. \tag{87}$$

Let us notice that (70), (75) and (28) imply

$$\frac{\partial\theta}{\partial x}\in L^2(Q_T). \tag{88}$$

We will derive the estimates for the first spatial derivatives of other functions (ρ, v, and ω) in the following two lemmas.

Lemma 3.14 *There exists a constant $C \in \mathbf{R}^+$ such that for any $t \in]0,T[$ we have*

$$\left\| \frac{\partial \rho}{\partial x}(t) \right\| \leq C. \tag{89}$$

Proof Taking into account (52) and (53), for the derivative of (44) we get

$$\frac{\partial \rho}{\partial x} = \rho\varphi - \rho^2 Y^{-1} B^{-1} \left[\frac{d}{dx}\left(\frac{1}{\rho_0}\right) + \frac{RL}{\lambda + 2\mu} \int_0^t BY\left(\frac{\partial \theta}{\partial x} + \theta\varphi\right) d\tau \right]. \tag{90}$$

With the help of (51), (54), and (69), after integration over $]0,1[$, (90) implies

$$\left\| \frac{\partial \rho}{\partial x} \right\|^2 \leq C\left(\|\varphi\|^2 + \int_0^1 \frac{1}{\rho_0^4}(\rho_0')^2\,dx + \int_0^t \int_0^1 \left(\frac{\partial \theta}{\partial x}\right)^2 dx\,d\tau + \int_0^t M_\theta^2 \|\varphi\|^2\,d\tau \right). \tag{91}$$

Using (28), integration by parts and properties of the initial data, from (53) we obtain

$$\|\varphi(t)\|^2 \leq C\left(1 + \|v\|^2 + \int_0^t \int_0^1 v^4\,dx\,d\tau\right). \tag{92}$$

Inserting (92) into (91), using the properties of the initial data as well as (31), (75), (87), and (88), we immediately get (89). □

Lemma 3.15 *There exists a constant $C \in \mathbf{R}^+$ such that for any $t \in]0,T[$ we have*

$$\|v(t)\|^2 + \int_0^t \left\| \frac{\partial v}{\partial x}(\tau) \right\|^2 \leq C, \tag{93}$$

$$\|\omega(t)\|^2 + \int_0^t \left\| \frac{\partial \omega}{\partial x}(\tau) \right\|^2 \leq C. \tag{94}$$

Proof After we multiply (8) by v and integrate over $]0,1[$, we get

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \frac{\lambda + 2\mu}{L^2} \int_0^1 \rho\left(\frac{\partial}{\partial x}(r^2 v)\right)^2 dx = \frac{R}{L}\int_0^1 \rho\theta \frac{\partial}{\partial x}(r^2 v)\,dx. \tag{95}$$

Using the Young inequality, (69) and (70) from (95) we obtain

$$\frac{d}{dt}\|v(t)\|^2 + \int_0^1 \left(\frac{\partial}{\partial x}(r^2 v)\right)^2 dx \leq CM_\theta^2. \tag{96}$$

To simplify the left hand side of (96) we use the inequality

$$\left[\frac{\partial}{\partial x}(r^2 v)\right]^2 \geq \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 - Cv^2, \tag{97}$$

which can easily be derived using (14), (28), and (69). After inserting (97) into (96), integrating over $]0,t[$ and using (31) and (87) we immediately get (93).

Now we prove (94). Let us first notice that (97) is also valid for the function ω. After we multiply (9) by $\rho^{-1}\omega$ and integrate over $]0,1[$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|^2+\frac{c_0+2c_d}{j_I L^2}\int_0^1\rho\left(\frac{\partial}{\partial x}\left(r^2\omega\right)\right)^2dx=\frac{4\mu_r}{j_I}\int_0^1\frac{\omega^2}{\rho}\,dx, \tag{98}$$

from which, by using (70) and the same procedure as before, we immediately arrive at (94). □

Lemma 3.15 enables us to derive the upper boundedness of the function r, which is crucial for the estimates of the second spatial derivatives.

Corollary 3.3 *There exists a constant $C\in\mathbf{R}^+$ such that for any $(x,t)\in\overline{Q}_T$ we have*

$$r(x,t)\leq C. \tag{99}$$

Proof From (16) and (15), using the Gagliardo-Ladyzhenskaya inequality as well as the Young inequality together with (31), we get

$$r(x,t)\leq C\left(1+\int_0^t\left\|\frac{\partial v}{\partial x}\right\|^2\right)dx \tag{100}$$

from which, using (93) we obtain the assertion of the lemma. □

Lemma 3.16 *There exists a constant $C\in\mathbf{R}^+$ such that for any $t\in\,]0,T[$ we have*

$$\left\|\frac{\partial v}{\partial x}(t)\right\|^2+\int_0^t\left\|\frac{\partial^2 v}{\partial x^2}(\tau)\right\|^2d\tau\leq C, \tag{101}$$

$$\left\|\frac{\partial \omega}{\partial x}(t)\right\|^2+\int_0^t\left\|\frac{\partial^2 \omega}{\partial x^2}(\tau)\right\|^2d\tau\leq C, \tag{102}$$

$$\left\|\frac{\partial \theta}{\partial x}(t)\right\|^2+\int_0^t\left\|\frac{\partial^2 \theta}{\partial x^2}(\tau)\right\|^2d\tau\leq C. \tag{103}$$

Proof Multiplying (8) by $\frac{\partial^2 v}{\partial x^2}$ and integrating over $]0,1[$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial v}{\partial x}(t)\right\|^2+\frac{\lambda+2\mu}{L^2}\int_0^1 r^4\rho\left(\frac{\partial^2 v}{\partial x^2}\right)^2dx\\
&\quad=\frac{R}{L}\int_0^1 r^2\theta\frac{\partial^2 v}{\partial x^2}\frac{\partial\rho}{\partial x}\,dx+\frac{R}{L}\int_0^1 r^2\rho\frac{\partial^2 v}{\partial x^2}\frac{\partial\theta}{\partial x}\,dx\\
&\qquad+2(\lambda+2\mu)\int_0^1\frac{v}{r^2\rho}\frac{\partial^2 v}{\partial x^2}\,dx-\frac{\lambda+2\mu}{L^2}\int_0^1 r^4\frac{\partial\rho}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx\\
&\qquad-\frac{4(\lambda+2\mu)}{L}\int_0^1 r\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx.
\end{aligned} \tag{104}$$

Using the Hölder, Gagliardo-Ladyzhenskaya, and Young inequalities as well as (89) and (99), we obtain the estimates of the integrals on the right-hand side of (104) as follows:

$$\left|\frac{R}{L}\int_0^1 r^2\theta\frac{\partial^2 v}{\partial x^2}\frac{\partial\rho}{\partial x}\,dx\right|\leq CM_\theta\left\|\frac{\partial^2 v}{\partial x^2}\right\|\left\|\frac{\partial\rho}{\partial x}\right\|\leq\varepsilon\left\|\frac{\partial^2 v}{\partial x^2}\right\|^2+CM_\theta^2, \tag{105}$$

$$\left|\frac{R}{L}\int_0^1 r^2\rho\frac{\partial^2 v}{\partial x^2}\frac{\partial\theta}{\partial x}\,dx\right| \le \varepsilon\left\|\frac{\partial^2 v}{\partial x^2}\right\|^2 + C\left\|\frac{\partial\theta}{\partial x}\right\|^2, \tag{106}$$

$$\left|2(\lambda+2\mu)\int_0^1 \frac{v}{r^2\rho}\frac{\partial^2 v}{\partial x^2}\,dx\right| \le \varepsilon\left\|\frac{\partial^2 v}{\partial x^2}\right\|^2 + C\|v\|^2, \tag{107}$$

$$\begin{aligned}\left|-\frac{\lambda+2\mu}{L^2}\int_0^1 r^4\frac{\partial\rho}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx\right| &\le C\left\|\frac{\partial v}{\partial x}\right\|^{\frac{1}{2}}\left\|\frac{\partial^2 v}{\partial x^2}\right\|^{\frac{1}{2}}\int_0^1\left|\frac{\partial\rho}{\partial x}\frac{\partial^2 v}{\partial x^2}\right|dx \\ &\le C\left\|\frac{\partial v}{\partial x}\right\|^{\frac{1}{2}}\left\|\frac{\partial^2 v}{\partial x^2}\right\|^{\frac{3}{2}}\left\|\frac{\partial\rho}{\partial x}\right\| \\ &\le \varepsilon\left\|\frac{\partial^2 v}{\partial x^2}\right\|^2 + C\left\|\frac{\partial v}{\partial x}\right\|^2,\end{aligned} \tag{108}$$

$$\left|-\frac{4(\lambda+2\mu)}{L}\int_0^1 r\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx\right| \le \varepsilon\left\|\frac{\partial^2 v}{\partial x^2}\right\|^2 + C\left\|\frac{\partial v}{\partial x}\right\|^2. \tag{109}$$

Inserting (105)-(109) into (104), and by using the small enough $\varepsilon > 0$ we get

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial v}{\partial x}(t)\right\|^2 + C_1\left\|\frac{\partial^2 v}{\partial x^2}\right\|^2 \le C_2\left(1 + M_\theta^2 + \|v\|^2 + \left\|\frac{\partial\theta}{\partial x}\right\|^2 + \left\|\frac{\partial v}{\partial x}\right\|^2\right), \tag{110}$$

from which by integration over $]0,t[$ and by using the properties of the initial data, (88) and (93) we get the assertion (101).

Now we prove (102). By multiplying (8) by $\rho^{-1}\frac{\partial^2\omega}{\partial x^2}$ and integrating over $]0,1[$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial\omega}{\partial x}(t)\right\|^2 + \frac{c_0+2c_d}{j_I L^2}\int_0^1 r^4\rho\left(\frac{\partial^2\omega}{\partial x^2}\right)^2 dx \\ &\quad= \frac{4\mu_r}{j_I}\int_0^1\frac{\omega}{\rho}\frac{\partial^2\omega}{\partial x^2}\,dx + 2\frac{c_0+2c_d}{j_I}\int_0^1\frac{\omega}{r^2\rho}\frac{\partial^2\omega}{\partial x^2}\,dx \\ &\qquad - \frac{c_0+2c_d}{j_I L^2}\int_0^1 r^4\frac{\partial\rho}{\partial x}\frac{\partial\omega}{\partial x}\frac{\partial^2\omega}{\partial x^2}\,dx - \frac{4(c_0+2c_d)}{j_I L}\int_0^1 r\frac{\partial\omega}{\partial x}\frac{\partial^2\omega}{\partial x^2}\,dx.\end{aligned} \tag{111}$$

In the same way as before, we get the estimates

$$\left|\frac{4\mu_r}{j_I}\int_0^1\frac{\omega}{\rho}\frac{\partial^2\omega}{\partial x^2}\,dx\right| \le C\left\|\frac{\partial^2\omega}{\partial x^2}\right\|\|\omega\| \le \varepsilon\left\|\frac{\partial^2\omega}{\partial x^2}\right\|^2 + C\|\omega\|^2, \tag{112}$$

$$\left|2\frac{c_0+2c_d}{j_I}\int_0^1\frac{\omega}{r^2\rho}\frac{\partial^2\omega}{\partial x^2}\,dx\right| \le \varepsilon\left\|\frac{\partial^2\omega}{\partial x^2}\right\|^2 + C\|\omega\|^2, \tag{113}$$

$$\left|-\frac{c_0+2c_d}{j_I L^2}\int_0^1 r^4\frac{\partial\rho}{\partial x}\frac{\partial\omega}{\partial x}\frac{\partial^2\omega}{\partial x^2}\,dx\right| \le \varepsilon\left\|\frac{\partial^2\omega}{\partial x^2}\right\|^2 + C\left\|\frac{\partial\omega}{\partial x}\right\|^2\left\|\frac{\partial\rho}{\partial x}\right\|^4, \tag{114}$$

$$\left|-\frac{4(c_0+2c_d)}{j_I L}\int_0^1 r\frac{\partial\omega}{\partial x}\frac{\partial^2\omega}{\partial x^2}\,dx\right| \le \varepsilon\left\|\frac{\partial^2\omega}{\partial x^2}\right\|^2 + C\|\omega\|^2. \tag{115}$$

By inserting (112)-(115) into (111), for small enough $\varepsilon > 0$, we get

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial\omega}{\partial x}(t)\right\|^2 + C_1\left\|\frac{\partial^2\omega}{\partial x^2}\right\|^2 \le C_2\left(1 + \|\omega\|^2 + \left\|\frac{\partial\omega}{\partial x}\right\|^2\right), \tag{116}$$

from which, by integration over $]0,t[$, using the properties of the initial data and (94), we obtain (102).

To prove (103) we multiply (10) by $\rho^{-1}\frac{\partial^2\theta}{\partial x^2}$. After integrating over $]0,1[$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial\theta}{\partial x}(t)\right\|^2+\frac{k}{c_vL^2}\int_0^1 r^4\rho\left(\frac{\partial^2\theta}{\partial x^2}\right)^2dx\\
&\quad=-\frac{4k}{c_vL}\int_0^1 r\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}\,dx\\
&\qquad-\frac{k}{c_vL^2}\int_0^1 r^4\frac{\partial\rho}{\partial x}\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}\,dx\frac{2R}{c_v}\int_0^1\frac{\theta v}{r}\frac{\partial^2\theta}{\partial x^2}\,dx+\frac{R}{c_vL}\int_0^1 r^2\rho\theta\frac{\partial v}{\partial x}\frac{\partial^2\theta}{\partial x^2}\,dx\\
&\qquad-\frac{4(\lambda+\mu)}{c_v}\int_0^1\frac{v^2}{r^2\rho}\frac{\partial^2\theta}{\partial x^2}\,dx-\frac{4\lambda}{c_vL}\int_0^1 rv\frac{\partial v}{\partial x}\frac{\partial^2\theta}{\partial x^2}\,dx\\
&\qquad-\frac{\lambda+2\mu}{c_vL^2}\int_0^1 r^4\rho\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2\theta}{\partial x^2}\,dx-\frac{4(c_0+c_d)}{c_v}\int_0^1\frac{\omega^2}{r^2\rho}\frac{\partial^2\theta}{\partial x^2}\,dx\\
&\qquad-\frac{4c_0}{c_vL}\int_0^1 r\omega\frac{\partial\omega}{\partial x}\frac{\partial^2\theta}{\partial x^2}\,dx-\frac{c_0+2c_d}{c_vL^2}\int_0^1 r^4\rho\left(\frac{\partial\omega}{\partial x}\right)^2\frac{\partial^2\theta}{\partial x^2}\,dx\\
&\qquad-\frac{4\mu_r}{c_v}\int_0^1\frac{\omega^2}{\rho}\frac{\partial^2\theta}{\partial x^2}.
\end{aligned}\tag{117}$$

Analogously to before we conclude the following:

$$\left|-\frac{4k}{c_vL}\int_0^1 r\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}\,dx\right|\le C\left\|\frac{\partial\theta}{\partial x}\right\|\left\|\frac{\partial^2\theta}{\partial x^2}\right\|\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+C\left\|\frac{\partial\theta}{\partial x}\right\|^2,\tag{118}$$

$$\left|-\frac{k}{c_vL^2}\int_0^1 r^4\frac{\partial\rho}{\partial x}\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}\,dx\right|\le C\left\|\frac{\partial\theta}{\partial x}\right\|^{\frac{1}{2}}\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^{\frac{3}{2}}\left\|\frac{\partial\rho}{\partial x}\right\|\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+C\left\|\frac{\partial\theta}{\partial x}\right\|^2,\tag{119}$$

$$\left|\frac{2R}{c_v}\int_0^1\frac{\theta v}{r}\frac{\partial^2\theta}{\partial x^2}\,dx\right|\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+CM_\theta^2\|v\|^2\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+CM_\theta^2,\tag{120}$$

$$\left|\frac{R}{c_vL}\int_0^1 r^2\rho\theta\frac{\partial v}{\partial x}\frac{\partial^2\theta}{\partial x^2}\,dx\right|\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+CM_\theta^2\left\|\frac{\partial v}{\partial x}\right\|^2\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+CM_\theta^2,\tag{121}$$

$$\left|-\frac{4(\lambda+\mu)}{c_v}\int_0^1\frac{v^2}{r^2\rho}\frac{\partial^2\theta}{\partial x^2}\,dx\right|\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+C\left\|v^2\right\|^2\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+C,\tag{122}$$

$$\left|-\frac{4\lambda}{c_vL}\int_0^1 rv\frac{\partial v}{\partial x}\frac{\partial^2\theta}{\partial x^2}\,dx\right|\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+C\left(\|v\|^2+\left\|\frac{\partial v}{\partial x}\right\|^6\right)\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+C,\tag{123}$$

$$\begin{aligned}
\left|-\frac{\lambda+2\mu}{c_vL^2}\int_0^1 r^4\rho\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2\theta}{\partial x^2}\,dx\right|&\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+C\left(\left\|\frac{\partial^2 v}{\partial x^2}\right\|^2+\left\|\frac{\partial v}{\partial x}\right\|^6\right)\\
&\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+C\left(1+\left\|\frac{\partial^2 v}{\partial x^2}\right\|^2\right),
\end{aligned}\tag{124}$$

$$\left|-\frac{4(c_0+c_d)}{c_v}\int_0^1\frac{\omega^2}{r^2\rho}\frac{\partial^2\theta}{\partial x^2}\,dx\right|\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+C,\tag{125}$$

$$\left|-\frac{4c_0}{c_vL}\int_0^1 r\omega\frac{\partial\omega}{\partial x}\frac{\partial^2\theta}{\partial x^2}\,dx\right|\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+C,\tag{126}$$

$$\left|-\frac{c_0+2c_d}{c_vL^2}\int_0^1 r^4\rho\left(\frac{\partial\omega}{\partial x}\right)^2\frac{\partial^2\theta}{\partial x^2}\,dx\right|\le\varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+C\left(1+\left\|\frac{\partial^2\omega}{\partial x^2}\right\|^2\right),\tag{127}$$

$$\left|-\frac{4\mu_r}{c_v}\int_0^1 \frac{\omega^2}{\rho}\frac{\partial^2\theta}{\partial x^2}\right| \le \varepsilon\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2 + C. \tag{128}$$

By inserting (118)-(128) into (117) and taking the sufficiently small $\varepsilon > 0$, we get

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial\theta}{\partial x}(t)\right\|^2 + C_1\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2 \le C_2\left(1 + M_\theta^2 + \left\|\frac{\partial\theta}{\partial x}\right\|^2 + \left\|\frac{\partial^2 v}{\partial x^2}\right\|^2 + \left\|\frac{\partial^2\omega}{\partial x^2}\right\|^2\right), \tag{129}$$

which, after integration over $]0,t[$, and making use of (87), (88), (101), and (102) as well as the properties of initial data, gives the assertion (103). □

In the next lemma we estimate the time derivatives of the functions ρ, v, ω, and θ.

Lemma 3.17 *There exists a constant $C \in \mathbf{R}^+$ such that for any $t \in]0,T[$ we have*

$$\int_0^t \left\|\frac{\partial\rho}{\partial t}(\tau)\right\|^2 d\tau \le C, \tag{130}$$

$$\int_0^t \left\|\frac{\partial v}{\partial t}(\tau)\right\|^2 d\tau \le C, \tag{131}$$

$$\int_0^t \left\|\frac{\partial\omega}{\partial t}(\tau)\right\|^2 d\tau \le C, \tag{132}$$

$$\int_0^t \left\|\frac{\partial\theta}{\partial t}(\tau)\right\|^2 d\tau \le C. \tag{133}$$

Proof From (7), by integrating over $]0,1[$, using (28), (99), and (69) we get

$$\left\|\frac{\partial\rho}{\partial t}\right\|^2 = \frac{1}{L^2}\int_0^1 \rho^4\left[\frac{\partial}{\partial x}\left(r^2 v\right)\right]^2 dx \le C\left(\|v\|^2 + \left\|\frac{\partial v}{\partial x}\right\|^2\right). \tag{134}$$

Using (93) and (101) from (134) we can easily conclude (130).

In a similar way, from (8), using estimates (28), (99), (69), (70),(69), (70), and (89), we get the inequality

$$\begin{aligned}\left\|\frac{\partial v}{\partial t}\right\|^2 &\le C\int_0^1\left(r^4\rho^2\left(\frac{\partial\theta}{\partial x}\right)^2 + r^4\theta^2\left(\frac{\partial\rho}{\partial x}\right)^2 + \frac{v^2}{r^4\rho^2} + r^8\left(\frac{\partial\rho}{\partial x}\right)^2\left(\frac{\partial v}{\partial x}\right)^2\right.\\ &\qquad\left. + r^2\left(\frac{\partial v}{\partial x}\right)^2 + r^8\rho^2\left(\frac{\partial^2 v}{\partial x^2}\right)^2\right)dx\\ &\le C\left(1 + M_\theta^2 + \left\|\frac{\partial\theta}{\partial x}\right\|^2 + \|v\|^2 + \left\|\frac{\partial v}{\partial x}\right\|^2 + \left\|\frac{\partial^2 v}{\partial x^2}\right\|^2\right).\end{aligned} \tag{135}$$

Using (87), (88), (93), and (101) from (135) we can easily conclude (131).

To prove (132), we multiply (9) by ρ^{-1} and using the same procedure as in (135), we get

$$\left\|\frac{\partial\omega}{\partial t}\right\|^2 \le C\left(1 + \|\omega\|^2 + \left\|\frac{\partial\omega}{\partial x}\right\|^2 + \left\|\frac{\partial^2\omega}{\partial x^2}\right\|^2\right) \tag{136}$$

which, using (94) and (102) leads to (132).

Analogously, from (10) we derive

$$\left\|\frac{\partial\theta}{\partial t}\right\|^2 \le C\left(1+\int_0^1\left(\left(\frac{\partial^2\theta}{\partial x^2}\right)^2+\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\rho}{\partial x}\right)^2\left(\frac{\partial\theta}{\partial x}\right)^2+\theta^2 v^2 + \theta^2\left(\frac{\partial v}{\partial x}\right)^2+v^2\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^4+\omega^2\left(\frac{\partial\omega}{\partial x}\right)^2+\left(\frac{\partial\omega}{\partial x}\right)^4\right)dx\right). \tag{137}$$

Using (89), (93), (101), (102), and (103) as well as the Gagliardo-Ladyzhenskaya and the Young inequalities, from (137) we get

$$\begin{aligned}\left\|\frac{\partial\theta}{\partial t}\right\|^2 &\le C\left(1+M_\theta^2+\left\|\frac{\partial\theta}{\partial x}\right\|^2+\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+\left\|\frac{\partial\theta}{\partial x}\right\|\left\|\frac{\partial^2\theta}{\partial x^2}\right\|\left\|\frac{\partial\rho}{\partial x}\right\|^2\right.\\ &\quad\left.+\left\|\frac{\partial v}{\partial x}\right\|^3\|v\|+\left\|\frac{\partial\omega}{\partial x}\right\|^3\|\omega\|+\left\|\frac{\partial v}{\partial x}\right\|^3\left\|\frac{\partial^2 v}{\partial x^2}\right\|+\left\|\frac{\partial\omega}{\partial x}\right\|^3\left\|\frac{\partial^2\omega}{\partial x^2}\right\|\right)\\ &\le C\left(1+M_\theta^2+\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2+\left\|\frac{\partial^2 v}{\partial x^2}\right\|+\left\|\frac{\partial^2\omega}{\partial x^2}\right\|\right),\end{aligned} \tag{138}$$

from which, using (87), (101), (102), and (103), we obtain the assertion (133). □

3.3 Final proof of Theorem 2.2

Corollary 3.1 and Lemma 3.14 gives the assertion

$$\rho \in \mathrm{L}^\infty\bigl(0,T;\mathrm{H}^1(]0,1[)\bigr). \tag{139}$$

From Lemmas 3.13, 3.15, and 3.16 we have

$$v,\omega,\theta \in \mathrm{L}^2\bigl(0,T;\mathrm{H}^2(]0,1[)\bigr)\cap \mathrm{L}^\infty\bigl(0,T;\mathrm{H}^1(]0,1[)\bigr). \tag{140}$$

Using inclusion (139) and (140) as well as the results from Lemma 3.17 we get

$$\rho,v,\omega,\theta \in \mathrm{H}^1(Q_T). \tag{141}$$

Now, using Lemmas 3.11 and 3.12 as well as inclusions (139), (140), and (141) in accordance with the Proposition 3.1, we have the statement of Theorem 2.2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The paper is the result of joint work of both authors who contributed equally to the final version of the paper. All authors have read and approved the final manuscript.

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Endnote
[a] *a* and *b* are the radii of boundary spheres from (1).

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