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Blow-up of solution for an integro-differential equation with arbitrary positive initial energy

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Abstract

In this paper, we consider the integro-differential equation $u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + u_t = f(u)$, $(x,t) \in \Omega \times (0,T)$, with initial and Dirichlet boundary conditions. Under suitable assumptions on the functions g and the initial data, a blow-up result with arbitrary positive initial energy is established.

MSC: 35L05; 35L55; 35L70

Keywords: blow-up; arbitrary positive initial energy; integro-differential equation

1 Introduction

In this paper we study the following nonlinear integro-differential equations:

$$\begin{cases} u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + u_t = f(u), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & x \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, M is a positive C^1-function like $M(s) = m_0 + bs^\gamma$, $m_0 > 0$, $b \geq 0$, $\gamma \geq 1$, and $s \geq 0$, g represents the kernel of the memory term and f is a nonlinear function like $f(u) = |u|^{p-2}u$, $p > 2$, they will be specified later.

Before going further, (1.1) without the viscoelastic term, that is, $g \equiv 0$, for the case that $M \equiv 1$, (1.1) becomes a nonlinear wave equation which has been extensively studied and several results concerning existence and nonexistence have been established [1–5]. When M is not a constant function, a special case of (1.1) is Kirchhoff equation which has been introduced in order to describe the nonlinear vibrations of an elastic string. Kirchhoff [6] was the first one to study the oscillations of stretched strings and plates. In this case the existence and nonexistence of solutions have been discussed by many authors; see [7–11] and the references cited therein.

For (1.1) with $g \not\equiv 0$, in the case that $M \equiv 1$, (1.1) becomes a semilinear viscoelastic equation which has been extensively studied and many results concerning global existence and blow-up in finite time have been proved. See in this regard [12–16]. For instance, Messaoudi [13] studied (1.1) with damping term $a|u_t|^{m-2}u_t$ and $f(u) = b|u|^{p-2}u$ and proved a blow-up result for solutions with negative initial energy if $p > m \geq 2$ and a global result for $2 \leq p \leq m$. This result has later been improved by the same author in [14] to accommodate certain solutions with positive initial energy. In [15], Song and Zhong considered (1.1) with strong damping $-\Delta u_t$ and $f(u) = |u|^{p-2}u$ and proved a blow-up result for solutions with positive initial energy by using the ideas of the 'potential well' theory introduced by Payne and Sattinger [5].

For $g \neq 0$ and M is not a constant function, (1.1) is a model to describe the motion of deformable solids as hereditary effect is incorporated. It may also be used to describe the dynamics of an extensible string with fading memory. This equation states that the dynamic equilibrium of a body depends not only on the present state of deformation, but also on the previous history of the deformation [17]. Also, (1.1) is applied to the theory of the heat conduction with memory; see [18, 19]. Therefore, the dynamics of (1.1) is of great importance and interest as they have wide applications in natural sciences.

This type of problem have been considered by many authors and several results concerning existence, nonexistence, and asymptotic behavior have been established. Equation (1.1) was first studied by Torrejón and Young [20], who proved the existence of weakly asymptotic stable solution for a large analytical datum. Later, Munoz Rivera [17] showed the existence of global solutions for small datum and the total energy decays to zero exponentially under some restrictions. In [11], Wu and Tsai studied (1.1) for a strong damping $-\Delta u_t$ and proved the global existence, decay result, and blow-up properties. Recently, they [21] discussed the local existence and blow-up of solutions with positive initial energy for nonlinear damping under some conditions.

In this paper, we consider problem (1.1) and will establish a blow-up result for (1.1) with arbitrary positive initial energy under suitable assumptions on the functions g and the initial data. This result extends earlier ones [11, 21], in which only some a positive initial energy is considered. The main tool in proving blow-up result is the 'concavity method' where the basic idea of the method is to construct a positive defined functional $F(t)$ of the solution by the energy inequality and show that $F^{-\alpha}(t)$ is a concave function of t.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and state the main result. In Section 3, we prove the blow-up result by a concavity method. Section 4 is devoted to a simple discussion of the main result.

2 Preliminaries

First, let us introduce some notation used throughout this paper. We denote by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \leq q \leq \infty$ and by $\|\nabla\cdot\|_2$ the Dirichlet norm in $H_0^1(\Omega)$, which is equivalent to the $H^1(\Omega)$ norm. Moreover, we set

$$(\varphi, \psi) = \int_\Omega \varphi(x)\psi(x)\,dx$$

as the usual $L^2(\Omega)$ inner product.

We next state some assumptions on f, M, and g:

(A1) $f(0) = 0$ and there are two positive constants c_1 and δ such that

$$\left|f(s) - f(s')\right| \leq c_1\left|s - s'\right|\left(|s|^{p-2} + \left|s'\right|^{p-2}\right)$$

and

$$sf(s) \geq (2 + 4\delta)F(s)$$

for $s, s' \in \mathcal{R}$, $2 < p \leq \frac{2(n-1)}{n-2}$ if $n > 2$ and $2 < p < \infty$ if $n \leq 2$, where $F(s) = \int_0^s f(\tau)\,d\tau$.

(A2) M is a positive C^1-function like $M(s) = m_0 + bs^\gamma$, $m_0 > 0$, $b \geq 0$, $\gamma \geq 1$, and $s \geq 0$, and it satisfies

$$(2\delta + 1)\overline{M}(s) - \bigl(M(s) + 2\delta m_0\bigr)s \geq 0, \quad \forall s \geq 0,$$

where $\overline{M}(s) = \int_0^s M(\tau)\,d\tau$ and δ is the constant appeared in (A1).

(A3) Assume $g(t) : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $C^1(\mathbb{R}_+)$ and satisfy

$$g(t) \geq 0, \qquad g'(t) \leq 0 \quad \text{for } t \geq 0$$

and

$$l = \int_0^\infty g(s)\,ds < \frac{2\delta m_0}{1 + 2\delta}.$$

(A4) The function $e^{\frac{t}{2}} g(t)$ is of positive type in the following sense:

$$\int_0^t \upsilon(s) \int_0^s e^{\frac{s-\tau}{2}} g(s - \tau)\,d\tau\,ds \geq 0,$$

$\forall \upsilon \in C^1([0,\infty))$ and $\forall t > 0$.

Remark 2.1 It is clear that $f(u) = |u|^{p-2}u$, $p \geq 2\gamma + 2$, and $m(s) = m_0 + bs^\gamma$, where $m_0 > 0$, $b \geq 0$, $\gamma \geq 1$ satisfy the assumptions (A1) and (A2) with $\alpha/2 \leq \delta \leq (p-2)/4$. It is also obvious that $g(t) = \epsilon e^{-t}$ with $0 < \epsilon < m_0$ satisfies the assumptions (A3) and (A4).

Now we are ready to state the local existence of problem (1.1), whose proof can be found in [21].

Theorem 2.1 *Assume that* (A1)-(A3) *hold, and that* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in L^2(\Omega)$, *then there exists a unique solution* u *of* (1.1) *satisfying*

$$u \in C\bigl([0,T); H_0^1(\Omega) \cap H^2(\Omega)\bigr) \quad \textit{and} \quad u_t \in C\bigl([0,T); L^2(\Omega)\bigr) \cap L^2\bigl([0,T); H_0^1(\Omega)\bigr).$$

Moreover, at least one of the following statements holds true:

(i) $T = \infty$,

(ii) $\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 \to \infty$ *as* $t \to T^-$.

We introduce the energy functional $E(t)$ associated to our equation,

$$\begin{aligned} E(t) = {} & \frac{1}{2}\bigl\|u_t(t)\bigr\|_2^2 + \frac{1}{2}\overline{M}\bigl(\bigl\|\nabla u(t)\bigr\|_2^2\bigr) - \frac{1}{2}\int_0^t g(\tau)\,d\tau\,\bigl\|\nabla u(t)\bigr\|_2^2 \\ & + \frac{1}{2}(g \circ \nabla u)(t) - \int_\Omega F(u)\,dx, \end{aligned} \tag{2.1}$$

where

$$(g \circ \nabla w)(t) = \int_0^t g(t-\tau)\big\|\nabla w(t) - \nabla w(\tau)\big\|_2^2\,d\tau.$$

As in [21], we see that

$$\frac{d}{dt}E(t) = -\int_\Omega \big|u_t(t)\big|^2\,dx + \frac{1}{2}\big(g' \circ \nabla u\big)(t) - \frac{1}{2}g(t)\big\|\nabla u(t)\big\|_2^2 \le 0,$$

which implies

$$E(t) \le E(0) - \int_0^t \int_\Omega \big|u_t(\tau)\big|^2\,dx\,d\tau. \tag{2.2}$$

Let

$$I(u) = M\big(\big\|\nabla u(t)\big\|_2^2\big)\big\|\nabla u(t)\big\|_2^2 - \int_\Omega f(u)u\,dx$$

for $u \in H_0^1(\Omega)$. We finally state our main blow-up result for problem (1.1).

Theorem 2.2 *Assume* (A1)-(A4) *hold. If* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in L^2(\Omega)$, *satisfy the following conditions*:

$$E(0) > 0, \qquad \int_\Omega u_0(x)u_1(x)\,dx > 0 \tag{2.3}$$

and

$$I(u_0) < 0, \qquad \|u_0\|_2^2 > \frac{(4\delta + 2)E(0)}{(2\delta m_0 - (2\delta + 1)l)\min\{\lambda, 1\}}, \tag{2.4}$$

where δ *is the constant appeared in* (A1) *and* λ *is the constant of the Poincaré inequality on* Ω, *then the corresponding solution* $u(t)$ *of problem* (1.1) *blows up in a finite time* $T^* > 0$.

3 Proof of Theorem 2.2

In this section, we deal with the blow-up solutions of equation (1.1). Before we prove our blow-up result, we need the following lemmas.

Lemma 3.1 (see [16], Lemma 2.1) *Assume that* $g(t)$ *satisfies the assumptions* (A3) *and* (A2), *and* $\Lambda(t)$ *is a function that is twice continuously differentiable, satisfying*

$$\begin{cases} \Lambda''(t) + \Lambda'(t) > \int_0^t g(t-\tau)\int_\Omega \nabla u(\tau,x)\nabla u(t,x)\,dx\,d\tau, \\ \Lambda(0) > 0, \qquad \Lambda'(0) > 0 \end{cases}$$

for every $t \in [0, T_0)$, *where* $u(t)$ *is the corresponding solution of* (1.1) *with* u_0 *and* u_1. *Then the function* $\Lambda(t)$ *is strictly increasing on* $[0, T_0)$.

Lemma 3.2 *Assume* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in L^2(\Omega)$, *satisfy*

$$\int_\Omega u_0(x)u_1(x)\,dx > 0.$$

If the local solution $u(t)$ *of* (1.1) *satisfies*

$$I(u) < 0,$$

then $\|u(t)\|_2^2$ *is strictly increasing on* $[0, T)$.

Proof Since $u(t)$ is the local solution of (1.1), by a simple computation we have

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\int_\Omega \left|u(t,x)\right|^2 dx &= \int_\Omega \left(\left|u_t(t)\right|^2 + uu_{tt}\right) dx \\ &= \int_\Omega \left|u_t(t,x)\right|^2 dx - \int_\Omega uu_t\, dx - M\left(\left\|\nabla u(t)\right\|_2^2\right)\left\|\nabla u(t)\right\|_2^2 \\ &\quad + \int_\Omega f(u)u\, dx + \int_0^t g(t-\tau)\int_\Omega \nabla u(t)\cdot\nabla u(\tau)\, dx\, d\tau \\ &> -\int_\Omega uu_t\, dx + \int_0^t g(t-\tau)\int_\Omega \nabla u(t)\cdot\nabla u(\tau)\, dx\, d\tau,\end{aligned}$$

where the last inequality uses $I(u) < 0$, which implies

$$\frac{d^2}{dt^2}\int_\Omega \left|u(t,x)\right|^2 dx + \frac{d}{dt}\int_\Omega \left|u(t,x)\right|^2 dx > \int_0^t g(t-\tau)\int_\Omega \nabla u(t)\cdot\nabla u(\tau)\, dx\, d\tau.$$

Therefore, this lemma comes from Lemma 3.1. □

Proof of Theorem 2.2 We next prove Theorem 2.2 in two steps. First, by a contradiction argument we claim that

$$I\left(u(t)\right) < 0 \tag{3.1}$$

and

$$\left\|u(t)\right\|_2^2 > \frac{(4\delta+2)E(0)}{(2\delta m_0 - (2\delta+1)l)\min\{\lambda, 1\}} \tag{3.2}$$

for every $t \in [0, T)$. If this was not the case, then there would exist a time t_1 such that

$$t_1 = \min\left\{t \in (0, T) : I\left(u(t)\right) = 0\right\} > 0. \tag{3.3}$$

By the continuity of the solution $u(t)$ as a function of t, we see that $I(u(t)) < 0$ when $t \in (0, t_1)$ and $I(u(t_1)) = 0$. Thus by Lemma 3.1 we have

$$\left\|u(t)\right\|_2^2 > \|u_0\|_2^2 > \frac{(4\delta+2)E(0)}{(2\delta m_0 - (2\delta+1)l)\min\{\lambda, 1\}}$$

for every $t \in [0, t_1)$. In addition, it is obvious that $\|u(t)\|_2^2$ is continuous on $[0, t_1]$. Thus the following inequality is obtained:

$$\left\|u(t_1)\right\|_2^2 > \frac{(4\delta+2)E(0)}{(2\delta m_0 - (2\delta+1)l)\min\{\lambda, 1\}}. \tag{3.4}$$

On the other hand, it follows from the definition of $E(t)$ and (2.2) that

$$\begin{aligned}&\frac{1}{2}\overline{M}\big(\|\nabla u(t_1)\|_2^2\big)-\frac{1}{2}\int_0^{t_1}g(\tau)\,d\tau\,\|\nabla u(t_1)\|_2^2\\&\quad+\frac{1}{2}(g\circ\nabla u)(t_1)-\int_\Omega F\big(u(t_1)\big)\,dx\le E(0).\end{aligned}\tag{3.5}$$

Using the assumptions (A1) and (A2), we have

$$\begin{aligned}&M\big(\|\nabla u(t_1)\|_2^2\big)\|\nabla u(t_1)\|_2^2+2\delta m_0\|\nabla u(t_1)\|_2^2-(2\delta+1)l\|\nabla u(t_1)\|_2^2\\&\quad-\int_\Omega f\big(u(t_1)\big)u(t_1)\,dx\le(4\delta+2)E(0).\end{aligned}$$

Noting the fact that $I(u(t_1))=0$, we then have

$$\big(2\delta m_0-(2\delta+1)l\big)\|\nabla u(t_1)\|_2^2\le(4\delta+2)E(0).$$

Thus, by the Poincaré inequality, we have

$$\|u(t_1)\|_2^2\le\frac{(4\delta+2)E(0)}{(2\delta m_0-(2\delta+1)l)\min\{\lambda,1\}}.\tag{3.6}$$

Obviously, there is a contradiction between (3.4) and (3.6). Thus, we have proved that (3.1) is true for every $t\in[0,T)$. Furthermore, by Lemma 3.2 we see that (3.2) is also valid on $t\in[0,T)$.

Secondly, we prove that the solution of problem (1.1) blows up in a finite time. Assume by contradiction that the solution u is global. Then, for sufficiently large $T>0$, we consider $H(t):[0,T]\to\mathbb{R}_+$ defined by

$$H(t)=\|u(t)\|_2^2+\int_0^t\|u(\tau)\|_2^2\,d\tau+(T-t)\|u_0\|_2^2+\alpha(t_0+t)^2,$$

where t_0 and α are positive constants, which will be determined in the sequel. A direct computation yields

$$\begin{aligned}H'(t)&=2\int_\Omega u(t)u_t(t)\,dx+\|u(t)\|_2^2-\|u_0\|_2^2+2\alpha(t_0+t)\\&=2\int_\Omega u(t)u_t(t)\,dx+2\int_0^t\big(u(\tau),u_t(\tau)\big)\,d\tau+2\alpha(t_0+t)\end{aligned}$$

and

$$\begin{aligned}H''(t)&=2\big(u_{tt},u(t)\big)+2\|u_t(t)\|_2^2+2\big(u(t),u_t(t)\big)+2\alpha\\&=2\|u_t(t)\|_2^2-2M\big(\|\nabla u\|_2^2\big)\|\nabla u\|_2^2+2\int_0^t g(t-\tau)\int_\Omega\nabla u(\tau)\nabla u(t)\,dx\,d\tau\\&\quad+2\int_\Omega f(u)u\,dx+2\alpha.\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&H(t)H''(t) - (\delta+1)H'(t)^2 \\
&\quad = 2H(t)\biggl(\|u_t(t)\|_2^2 - M\bigl(\|\nabla u\|_2^2\bigr)\|\nabla u\|_2^2 + \int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\nabla u(t)\,dx\,d\tau \\
&\qquad + \int_\Omega f(u)u\,dx + \alpha\biggr) - 4(\delta+1)\biggl(\int_\Omega u(t)u_t(t)\,dx + \int_0^t \bigl(u(\tau),u_t(\tau)\bigr)\,d\tau + \alpha(t_0+t)\biggr)^2 \\
&\quad = 2H(t)\biggl(\|u_t(t)\|_2^2 - M\bigl(\|\nabla u\|_2^2\bigr)\|\nabla u\|_2^2 + \int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\nabla u(t)\,dx\,d\tau \\
&\qquad + \int_\Omega f(u)u\,dx + \alpha\biggr) + 4(\delta+1)\bigl(G(t) - \bigl(H(t) - (T-t)\|u_0\|_2^2\bigr)\Psi(t)\bigr),
\end{aligned}
\tag{3.7}
$$

where $\Psi(t), G(t) : [0,T] \to \mathbb{R}_+$ are the functions defined by

$$
\Psi(t) = \|u_t(t)\|_2^2 + \int_0^t \|u_t(\tau)\|_2^2\,d\tau + \alpha
$$

and

$$
\begin{aligned}
G(t) = {} & \Psi(t)\biggl(\|u(t)\|_2^2 + \int_0^t \|u(\tau)\|_2^2\,d\tau + \alpha(t_0+t)^2\biggr) \\
& - \biggl(\int_\Omega u(t)u_t(t)\,dx + \int_0^t (u,u_t)\,d\tau + \alpha(t_0+t)\biggr)^2.
\end{aligned}
$$

Using the Schwarz inequality, we have

$$
\begin{aligned}
&\biggl(\int_\Omega uu_t\,dx\biggr)^2 \le \|u(t)\|_2^2 \|u_t(t)\|_2^2, \\
&\biggl(\int_0^t (u,u_t)\,d\tau\biggr)^2 \le \int_0^t \|u(\tau)\|_2^2\,d\tau \int_0^t \|u_t(\tau)\|_2^2\,d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega u(t)u_t(t)\,dx \int_0^t \bigl(u(\tau),u_t(\tau)\bigr)\,d\tau \\
&\quad \le \|u(t)\|_2 \biggl(\int_0^t \|u_t(\tau)\|_2^2\,d\tau\biggr)^{\frac{1}{2}} \|u_t(t)\|_2 \biggl(\int_0^t \|u(\tau)\|_2^2\,d\tau\biggr)^{\frac{1}{2}} \\
&\quad \le \frac{1}{2}\|u(t)\|_2^2 \int_0^t \|u_t(\tau)\|_2^2\,d\tau + \frac{1}{2}\|u_t(t)\|_2^2 \int_0^t \|u(\tau)\|_2^2\,d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\alpha(t+t_0)\int_\Omega u(t)u_t(t)\,dx \\
&\quad \le \sqrt{\alpha}\sqrt{\alpha}(t+s_0)\|u(t)\|_2 \|u_t(t)\|_2 \\
&\quad \le \frac{1}{2}\alpha\|u(t)\|_2^2 + \frac{1}{2}\alpha(t+t_0)^2\|u_t(t)\|_2^2.
\end{aligned}
$$

Similarly, we have

$$\alpha(t+t_0)\int_0^t \big(u(\tau),u_t(\tau)\big)\,d\tau \le \frac{1}{2}\alpha\int_0^t \big\|u(\tau)\big\|_2^2\,d\tau + \frac{1}{2}\alpha(t+t_0)^2\int_0^t \big\|u_t(\tau)\big\|_2^2\,d\tau.$$

The previous inequalities entail $G(t) \ge 0$ for every $[0,T]$. Using (3.7), we get

$$H(t)H''(t) - (\delta+1)H'(t)^2 \ge H(t)L(t), \tag{3.8}$$

where

$$\begin{aligned}
L(t) &= -(4\delta+2)\big\|u_t(t)\big\|_2^2 - 2M\big(\|\nabla u\|_2^2\big)\|\nabla u\|_2^2 \\
&\quad + 2\int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\nabla u(t)\,dx\,d\tau \\
&\quad + 2\int_\Omega f(u)u\,dx - 4(\delta+1)\int_0^t \big\|u_t(\tau)\big\|_2^2\,d\tau - (4\delta+2)\alpha \\
&= -(4\delta+2)\big\|u_t(t)\big\|_2^2 - 2\left(M\big(\|\nabla u\|_2^2\big) - \int_0^t g(\tau)\,d\tau\right)\|\nabla u\|_2^2 \\
&\quad + 2\int_\Omega f(u)u\,dx - (4\delta+2)\alpha - 4(\delta+1)\int_0^t \big\|u_t(\tau)\big\|_2^2\,d\tau \\
&\quad + 2\int_0^t g(t-\tau)\int_\Omega \nabla u(t)\big(\nabla u(\tau) - \nabla u(t)\big)\,dx\,d\tau.
\end{aligned} \tag{3.9}$$

Using Young's inequality, we have

$$\begin{aligned}
&\int_0^t g(t-\tau)\int_\Omega \nabla u(t)\nabla\big(u(\tau)-u(t)\big)\,dx\,d\tau \\
&\quad \ge -(2\delta+1)(g\circ\nabla u)(t) - \frac{1}{(8\delta+4)}\int_0^t g(\tau)\,d\tau\,\big\|\nabla u(t)\big\|_2^2.
\end{aligned} \tag{3.10}$$

Inserting (3.10) into (3.9), we have

$$\begin{aligned}
L(t) &\ge -(4\delta+2)\big\|u_t(t)\big\|_2^2 - (4\delta+2)(g\circ\nabla u)(t) \\
&\quad - 2\left(M\big(\|\nabla u\|_2^2\big) - \int_0^t g(\tau)\,d\tau\right)\|\nabla u\|_2^2 \\
&\quad + 2\int_\Omega f(u)u\,dx - 4(\delta+1)\int_0^t \big\|u_t(\tau)\big\|_2^2\,d\tau \\
&\quad - \frac{1}{4\delta+2}\int_0^t g(\tau)\,d\tau\,\|\nabla u\|_2^2 - (4\delta+2)\alpha \\
&\ge -(8\delta+4)E(t) + 2\big((2\delta+1)\overline{M}\big(\|\nabla u\|_2^2\big) - M\big(\|\nabla u\|_2^2\big)\|\nabla u\|_2^2\big) \\
&\quad - 4(\delta+1)\int_0^t \big\|u_t(\tau)\big\|_2^2\,d\tau - 2\int_\Omega \big[(4\delta+2)F(u) - f(u)u\big]\,dx \\
&\quad - \left(4\delta + \frac{1}{4\delta+2}\right)\int_0^t g(\tau)\,d\tau\,\|\nabla u\|_2^2 - (4\delta+2)\alpha \\
&\ge -(8\delta+4)E(0) + 2\big((2\delta+1)\overline{M}\big(\|\nabla u\|_2^2\big) - M\big(\|\nabla u\|_2^2\big)\|\nabla u\|_2^2\big)
\end{aligned}$$

$$
\begin{aligned}
&+ 4\delta \int_0^t \|u_t(\tau)\|_2^2 \, d\tau - 2\int_\Omega \big[(4\delta+2)F(u) - f(u)u\big] \, dx \\
&- \left(4\delta + \frac{1}{4\delta+2}\right) \int_0^t g(\tau) \, d\tau \, \|\nabla u\|_2^2 - (4\delta+2)\alpha.
\end{aligned}
$$

Using the assumptions (A1) and (A2), we have

$$
\begin{aligned}
L(t) &\geq -(8\delta+4)E(0) + \left(4\delta m_0 - \left(4\delta + \frac{1}{4\delta+2}\right) \int_0^t g(\tau) \, d\tau\right) \|\nabla u\|_2^2 - (4\delta+2)\alpha \\
&\geq -(8\delta+4)E(0) + \big(4\delta m_0 - (4\delta+1)l\big)\lambda \big\|u(t)\big\|_2^2 - (4\delta+2)\alpha \\
&\geq -(8\delta+4)E(0) + \big(4\delta m_0 - (4\delta+1)l\big)\lambda \|u_0\|_2^2 - (4\delta+2)\alpha,
\end{aligned}
$$

where the last inequality follows from Lemma 3.2 and the Poincaré inequality. From (2.4), we have

$$
(8\delta+4)E(0) < \big(4\delta m_0 - (4\delta+2)l\big)\lambda \|u_0\|_2^2 < \big(4\delta m_0 - (4\delta+1)l\big)\lambda \|u_0\|_2^2.
$$

Thus, we can let α satisfy

$$
(4\delta+2)\alpha < \big(4\delta m_0 - (4\delta+2)l\big)\lambda \|u_0\|_2^2 - (8\delta+4)E(0),
$$

which implies that there exists $\theta > 0$ (independent of T) such that

$$
L(t) \geq \theta \quad \text{for } t \in [0, T]. \tag{3.11}
$$

By (3.8) and (3.11), it follows that

$$
H(t)H''(t) - (\delta+1)H'(t)^2 > 0.
$$

Moreover, we let t_0 satisfy

$$
\alpha t_0 + \int_\Omega u_0 u_1 \, dx > 0,
$$

which means $H'(0) > 0$. Thus by $H''(t) > 0$ we see that $H(t)$ and $H'(t)$ are strictly increasing on $[0, T]$.

Setting $y(t) = H(t)^{-\delta}$, then we have

$$
y'(t) = -\delta H(t)^{-(\delta+1)} H'(t) < 0
$$

and

$$
y''(t) = -\delta H^{-\delta-2}\big(H(t)H''(t) - (\delta+1)H'(t)^2\big) < 0
$$

for all $t \in [0, T]$, which implies that $y(t)$ reaches 0 in finite time, say as $t \to T^*$. Since T^* is independent of the initial choice of T, we may assume that $T^* < T$. This tells us that

$$
\lim_{t \to T^*} H(t) = \infty. \qquad \square
$$

4 Conclusions

In this paper, we consider the integro-differential equation

$$u_{tt} - M\big(\|\nabla u\|_2^2\big)\Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + u_t = f(u), \quad (x,t) \in \Omega \times (0,T),$$

with initial and Dirichlet boundary conditions which arises in the dynamics of an extensible string with fading memory. Under suitable assumptions on the relax function g and the initial data, we establish a blow-up result with arbitrary positive initial energy. The main tool in proving the blow-up result is the 'concavity method' where the basic idea of the method is to construct a positive defined functional $F(t)$ of the solution by the energy inequality and show that $F^{-\alpha}(t)$ is a concave function of t.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The article is a joint work of the two authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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