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New existence and uniqueness results for an elastic beam equation with nonlinear boundary conditions

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Abstract

In this article, we study the existence and uniqueness of monotone positive solutions for an elastic beam equation with nonlinear boundary conditions, and some sufficient conditions which guarantee the existence of unique monotone positive solution are established. The methods employed are two fixed point theorems for mixed monotone operators with perturbation. Our results can not only guarantee the existence of unique monotone positive solution, but also be applied to construct an iterative scheme for approximating it. Two examples are given to illustrate our main results.

MSC: 34B18; 34B15

Keywords: existence and uniqueness; monotone positive solution; elastic beam equation; fixed point theorem for mixed monotone operators

1 Introduction

In this article, we are concerned with the existence and uniqueness of monotone positive solutions for the following nonlinear fourth-order two-point boundary value problem for elastic beam equation:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u^{(3)}(1) = g(u(1)), \end{cases} \tag{1.1}$$

where $f \in C([0,1] \times \mathbf{R} \times \mathbf{R})$ and $g \in C(\mathbf{R})$ are real functions. Here, monotone positive solutions mean increasing positive solutions. As we know, problem (1.1) models an elastic beam of length 1 subject to a nonlinear foundation given by the function f. The first boundary condition $u(0) = u'(0) = 0$ means that the left end of the beam is fixed. The second boundary condition $u''(1) = 0$, $u'''(1) = g(u(1))$ means that the right end of the beam is attached to a bearing device, given by the function g. Owing to the importance in engineering, physics and material mechanics, boundary value problems for elastic beam equations have attracted much attention. For a small sample of such work, we refer the reader to the works [1–20] and the references therein. In these papers, most of the authors have investigated the existence of solutions or positive solutions. On the other hand, the uniqueness of positive solutions for nonlinear fourth-order boundary value problems has been studied by some authors, see [16, 17, 19] for example. In [17], the authors utilized a fixed point theorem of generalized concave operators to study problem (1.1) and established the existence and uniqueness of monotone positive solutions. In [19], by using a fixed point theorem of cone expansion and a fixed point theorem of generalized concave operators, the authors considered the existence, nonexistence, and uniqueness of convex monotone positive solutions of an elastic beam equation with a parameter.

Motivated by the work [17, 21], we will discuss the existence and uniqueness of monotone positive solutions for problem (1.1) by using two fixed point theorems for mixed monotone operators with perturbation. As we know, there are still very few works to utilize fixed point theorems of mixed monotone operators to study fourth-order boundary value problems. So it is worthwhile to investigate problem (1.1) and the methods used here are relatively new to the literature. The main features of this article are as follows. First, we consider the monotone positive solutions for fourth-order boundary value problems. Second, comparing with [16, 17, 19], we establish the existence and uniqueness of monotone positive solutions via different methods. Third, our results can not only guarantee the existence of a unique monotone positive solution, but also be applied to construct an iterative scheme for approximating it. In addition, few papers can be found in the literature on the existence and uniqueness of monotone positive solutions for fourth-order boundary value problems. Hence we improve the results of [17] to some degree, and so it is important to study the existence and uniqueness of monotone positive solutions for problem (1.1).

2 Preliminaries

In the following, for completeness we list some basic concepts in ordered Banach spaces and two fixed point theorems for mixed monotone operators which will be used later. For the convenience of readers, we refer them to [21–23] for details.

Let $(E, \|\cdot\|)$ be a real Banach space which is partially ordered by a cone $P \subset E$, *i.e.*, $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. By θ we denote the zero element of E. A non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P$, $-x \in P \Rightarrow x = \theta$.

P is called normal if there is a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$; in this case N is the infimum of such constants, it is called the normality constant of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leq x \leq x_2\}$ is called the order interval between x_1 and x_2. We say that an operator $A : E \to E$ is increasing (decreasing) if $x \leq y$ implies $Ax \leq Ay$ ($Ax \geq Ay$).

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (*i.e.*, $h \geq \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$.

Definition 2.1 (See [21, 23]) $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, *i.e.*, $u_i, v_i \in P$, $i = 1, 2$, $u_1 \leq u_2$, $v_1 \geq v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$. Element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Definition 2.2 An operator $A : P \to P$ is said to be sub-homogeneous if it satisfies

$$A(tx) \geq tA(x), \quad \forall t \in (0, 1), x \in P.$$

Definition 2.3 Let α be a real number with $0 \leq \alpha < 1$. An operator $A : P \to P$ is said to be α-concave if it satisfies

$$A(tx) \geq t^{\alpha} A(x), \quad \forall t \in (0,1), x \in P.$$

To prove our results, we need the following fixed point theorems for mixed monotone operators, which were established in [21].

Lemma 2.4 (See Theorem 2.1 in [21]) *Let $h > \theta$ and $\alpha \in (0,1)$. $A : P \times P \to P$ is a mixed monotone operator and satisfies*

$$A(tx, t^{-1}y) \geq t^{\alpha} A(x,y), \quad \forall t \in (0,1), x, y \in P. \tag{2.1}$$

$B : P \to P$ is an increasing sub-homogeneous operator. Suppose that

(i) *there is $h_0 \in P_h$ such that $A(h_0, h_0) \in P_h$ and $Bh_0 \in P_h$;*
(ii) *there exists a constant $\delta_0 > 0$ such that $A(x,y) \geq \delta_0 Bx$, $\forall x, y \in P$.*

Then:

(1) *$A : P_h \times P_h \to P_h$ and $B : P_h \to P_h$;*
(2) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$rv_0 \leq u_0 < v_0, \qquad u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0;$$

(3) *the operator equation $A(x,x) + Bx = x$ has a unique solution x^* in P_h;*
(4) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Lemma 2.5 (See Theorem 2.4 in [21]) *Let $h > \theta$ and $\alpha \in (0,1)$. $A : P \times P \to P$ is a mixed monotone operator and satisfies*

$$A(tx, t^{-1}y) \geq tA(x,y), \quad \forall t \in (0,1), x, y \in P. \tag{2.2}$$

$B : P \to P$ is an increasing α-concave operator. Suppose that

(i) *there is $h_0 \in P_h$ such that $A(h_0, h_0) \in P_h$ and $Bh_0 \in P_h$;*
(ii) *there exists a constant $\delta_0 > 0$ such that $A(x,y) \leq \delta_0 Bx$, $\forall x, y \in P$.*

Then:

(1) *$A : P_h \times P_h \to P_h$ and $B : P_h \to P_h$;*
(2) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$rv_0 \leq u_0 < v_0, \qquad u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0;$$

(3) *the operator equation $A(x,x) + Bx = x$ has a unique solution x^* in P_h;*
(4) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

3 Main results

In this section, we use Lemmas 2.4, 2.5 to study problem (1.1) and present two new results on the existence and uniqueness of monotone positive solutions. The main results obtained here are relatively new in the literature.

In our considerations we shall consider the Banach space $E = C^1[0,1]$ equipped with the norm $\|u\| = \max\{\max_{0\le t\le 1}|u(t)|, \max_{0\le t\le 1}|u'(t)|\}$. In order to find monotone positive solutions, we consider the closed convex cone of nonnegative increasing functions $P = \{u \in E | u(t) \ge 0, u'(t) \ge 0, \forall t \in [0,1]\}$. Note that this induces an order relation $\dot{\le}$ in E by defining $u \dot{\le} v$ if and only if $v - u \in P$. Clearly, this cone is normal. That is, if $u \dot{\le} v$, then $u(t) \le v(t)$, $u'(t) \le v'(t)$, $t \in [0,1]$. Therefore, $\|u\| \le \|v\|$ and the normality constant is 1.

Let $G(t,s)$ be the Green function of the linear problem $u^{(4)}(t) = 0$ with the boundary conditions in (1.1); from [3] we know that

$$G(t,s) = \frac{1}{6}\begin{cases} s^2(3t-s), & 0 \le s \le t \le 1, \\ t^2(3s-t), & 0 \le t \le s \le 1. \end{cases} \tag{3.1}$$

Then problem (1.1) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f\bigl(s, u(s), u'(s)\bigr)\,ds - g\bigl(u(1)\bigr)\phi(t), \quad \text{where } \phi(t) = \frac{1}{2}t^2 - \frac{1}{6}t^3, t \in [0,1].$$

From [17], we give the following properties of the Green function $G(t,s)$ and $\phi(t)$.

Lemma 3.1 *For any $t, s \in [0,1]$, we have*

$$\frac{1}{3}s^2t^2 \le G(t,s) \le \frac{1}{2}st^2, \qquad \frac{1}{3}t^2 \le \phi(t) \le \frac{1}{2}t^2,$$
$$\frac{1}{2}s^2t \le \frac{\partial G(t,s)}{\partial t} \le st, \qquad \frac{1}{2}t \le \phi'(t) \le 2t.$$

Theorem 3.2 *Assume that*

(H_1) *$f(t,x,y) : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and $g : [0,+\infty) \to (-\infty, 0]$;*

(H_2) *$f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$ and $y \in [0,+\infty)$, decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$ and $x \in [0,+\infty)$, and $g(x)$ is decreasing in $x \in [0,+\infty)$;*

(H_3) *$g(\lambda x) \le \lambda g(x)$ for $\lambda \in (0,1)$, $x \in [0,+\infty)$, and there exists a constant $\alpha \in (0,1)$ such that $f(t, \lambda x, \lambda^{-1} y) \ge \lambda^\alpha f(t,x,y)$, $\forall t \in [0,1]$, $\lambda \in (0,1)$, $x, y \in [0,+\infty)$;*

(H_4) *there exists a constant $\sigma > 0$ such that $f(t,x,y) \ge \sigma \ge -g(x) > 0$, $t \in [0,1]$, $x, y \ge 0$.*

Then:

(1) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \dot{\le} u_0 \dot{<} v_0$ and*

$$u_0(t) \le \int_0^1 G(t,s) f\bigl(s, u_0(s), v_0'(s)\bigr)\,ds - g\bigl(u_0(1)\bigr)\phi(t), \quad t \in [0,1],$$

$$u_0'(t) \le \int_0^1 G_t(t,s) f\bigl(s, u_0(s), v_0'(s)\bigr)\,ds - g\bigl(u_0(1)\bigr)\phi'(t), \quad t \in [0,1],$$

$$v_0(t) \ge \int_0^1 G(t,s) f\bigl(s, v_0(s), u_0'(s)\bigr)\,ds - g\bigl(v_0(1)\bigr)\phi(t), \quad t \in [0,1],$$

$$v_0'(t) \geq \int_0^1 G_t(t,s) f\big(s, v_0(s), u_0'(s)\big)\, ds - g\big(v_0(1)\big)\phi'(t), \quad t \in [0,1],$$

where $h(t) = t^2$, $t \in [0,1]$ *and* $G(t,s)$ *is given as in* (3.1);

(2) *problem* (1.1) *has a unique monotone positive solution* u^* *in* P_h;

(3) *for any* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_n(t) = \int_0^1 G(t,s) f\big(s, x_{n-1}(s), y_{n-1}'(s)\big)\, ds - g\big(x_{n-1}(1)\big)\phi(t), \quad n = 1,2,\ldots,$$

$$y_n(t) = \int_0^1 G(t,s) f\big(s, y_{n-1}(s), x_{n-1}'(s)\big)\, ds - g\big(y_{n-1}(1)\big)\phi'(t), \quad n = 1,2,\ldots,$$

we have $\|x_n - u^*\| \to 0$ *and* $\|y_n - u^*\| \to 0$ *as* $n \to \infty$.

Proof Define two operators $A : P \times P \to E$ and $B : P \to E$ by

$$A(u,v)(t) = \int_0^1 G(t,s) f\big(s, u(s), v'(s)\big)\, ds, \qquad Bu(t) = -g\big(u(1)\big)\phi(t), \quad t \in [0,1].$$

Then

$$\big(A(u,v)\big)'(t) = \int_0^1 G_t(t,s) f\big(s, u(s), v'(s)\big)\, ds, \qquad (Bu)'(t) = -g\big(u(1)\big)\phi'(t), \quad t \in [0,1].$$

Evidently, u is the solution of problem (1.1) if and only if $u = A(u,u) + Bu$. For $u, v \in P$, we know that $u(t), v(t) \geq 0$, $u'(t), v'(t) \geq 0$, $t \in [0,1]$. From (H_1) and Lemma 3.1, we have $A(u,v)(t) \geq 0$, $Bu(t) \geq 0$, $(A(u,v))'(t) \geq 0$, $(Bu)'(t) \geq 0$, $t \in [0,1]$. Therefore, $A(u,v) \in P$, $Bu \in P$. That is, $A : P \times P \to P$ and $B : P \to P$. In the sequel we check that A, B satisfy all assumptions of Lemma 2.4.

Firstly, we prove that A is a mixed monotone operator. In fact, for $u_i, v_i \in P$, $i = 1,2$ with $u_1 \dot{\geq} u_2$, $v_1 \dot{\leq} v_2$, we know that $u_1(t) \geq u_2(t)$, $v_1(t) \leq v_2(t)$, $u_1'(t) \geq u_2'(t)$, $v_1'(t) \leq v_2'(t)$, $t \in [0,1]$ and by (H_2) and Lemma 3.1, we have

$$\begin{aligned} A(u_1,v_1)(t) &= \int_0^1 G(t,s) f\big(s, u_1(s), v_1'(s)\big)\, ds \\ &\geq \int_0^1 G(t,s) f\big(s, u_2(s), v_2'(s)\big)\, ds \\ &= A(u_2,v_2)(t), \\ \big(A(u_1,v_1)\big)'(t) &= \int_0^1 G_t(t,s) f\big(s, u_1(s), v_1'(s)\big)\, ds \\ &\geq \int_0^1 G_t(t,s) f\big(s, u_2(s), v_2'(s)\big)\, ds \\ &= \big(A(u_2,v_2)\big)'(t). \end{aligned}$$

That is, $A(u_1,v_1) \dot{\geq} A(u_2,v_2)$.

Further, we show B is increasing. For any $u, v \in P$ with $u \dot{\leq} v$, we know that $u(t) \leq v(t)$, $u'(t) \leq v'(t)$, $t \in [0,1]$. It follows from (H_1), (H_2) and Lemma 3.1 that $Bu(t) \leq Bv(t)$,

$(Bu)'(t) \leq (Bv)'(t)$, $t \in [0,1]$. That is, $Bu \dot{\leq} Bv$. Next we show that operator A satisfies the condition (2.1). For any $\lambda \in (0,1)$ and $u, v \in P$, by (H_3) we have

$$\begin{aligned}
A(\lambda u, \lambda^{-1} v)(t) &= \int_0^1 G(t,s) f\bigl(s, \lambda u(s), \lambda^{-1} v'(s)\bigr)\,ds \\
&\geq \lambda^\alpha \int_0^1 G(t,s) f\bigl(s, u(s), v'(s)\bigr)\,ds \\
&= \lambda^\alpha A(u,v)(t), \\
\bigl(A(\lambda u, \lambda^{-1} v)\bigr)'(t) &= \int_0^1 G_t(t,s) f\bigl(s, \lambda u(s), \lambda^{-1} v'(s)\bigr)\,ds \\
&\geq \lambda^\alpha \int_0^1 G_t(t,s) f\bigl(s, u(s), v'(s)\bigr)\,ds \\
&= \bigl(\lambda^\alpha A(u,v)\bigr)'(t).
\end{aligned}$$

That is, $A(\lambda u, \lambda^{-1} v) \dot{\geq} \lambda^\alpha A(u,v)$ for $\lambda \in (0,1)$, $u, v \in P$. So operator A satisfies (2.1). Also, for any $\lambda \in (0,1)$, $u \in P$, from (H_3) we know that

$$\begin{aligned}
&B(\lambda u)(t) = -g\bigl(\lambda u(1)\bigr)\phi(t) \geq \lambda\bigl(-g\bigl(u(1)\bigr)\phi(t)\bigr) = \lambda Bu(t), \\
&\bigl(B(\lambda u)\bigr)'(t) = -g\bigl(\lambda u(1)\bigr)\phi'(t) \geq \lambda\bigl(-g\bigl(u(1)\bigr)\phi'(t)\bigr) = (\lambda Bu)'(t),
\end{aligned}$$

that is, $B(\lambda u) \dot{\geq} \lambda Bu$ for $\lambda \in (0,1)$, $u \in P$. That is, operator B is sub-homogeneous. Now we show that $A(h,h) \in P_h$ and $Bh \in P_h$. On the one hand, from (H_1), (H_2) and Lemma 3.1, for any $t \in [0,1]$, we have

$$\begin{aligned}
A(h,h)(t) &= \int_0^1 G(t,s) f\bigl(s, h(s), h'(s)\bigr)\,ds \\
&= \int_0^1 G(t,s) f\bigl(s, s^2, 2s\bigr)\,ds \\
&\leq \int_0^1 \frac{1}{2} t^2 s f\bigl(s, s^2, 2s\bigr)\,ds \leq \frac{1}{2}\int_0^1 s f(s,1,0)\,ds \cdot h(t), \\
A(h,h)(t) &\geq \int_0^1 \frac{1}{3} t^2 s^2 f\bigl(s, s^2, 2s\bigr)\,ds \geq \frac{1}{3}\int_0^1 s^2 f(s,0,2)\,ds \cdot h(t).
\end{aligned}$$

On the other hand, also from (H_1), (H_2) and Lemma 3.1, for any $t \in [0,1]$, we obtain

$$\begin{aligned}
\bigl(A(h,h)\bigr)'(t) &= \int_0^1 G_t(t,s) f\bigl(s, s^2, 2s\bigr)\,ds \\
&\leq \int_0^1 st f(s,1,0)\,ds = \frac{1}{2}\int_0^1 s f(s,1,0)\,ds \cdot h'(t), \\
\bigl(A(h,h)\bigr)'(t) &\geq \int_0^1 \frac{1}{2} s^2 t f(s,0,2)\,ds = \frac{1}{4}\int_0^1 s^2 f(s,0,2)\,ds \cdot h'(t).
\end{aligned}$$

Let

$$c_1 = \frac{1}{4}\int_0^1 s^2 f(s,0,2)\,ds, \qquad c_2 = \frac{1}{2}\int_0^1 s f(s,1,0)\,ds.$$

From (H_2), (H_4), we have

$$c_2 \geq c_1 \geq \frac{1}{4}\int_0^1 s^2\sigma\,ds = \frac{1}{12}\sigma > 0,$$

and in consequence,

$$\begin{aligned} &c_1 h(t) \leq A(h,h)(t) \leq c_2 h(t),\\ &(c_1 h)'(t) = c_1 h'(t) \leq \bigl(A(h,h)\bigr)'(t) \leq c_2 h'(t) = (c_2 h)'(t), \quad t \in [0,1]. \end{aligned}$$

Thus, $c_1 h \dot{\leq} A(h,h) \dot{\leq} c_2 h$. That is, $A(h,h) \in P_h$. Similarly, from (H_1), (H_2) and Lemma 3.1, for any $t \in [0,1]$, we have

$$\begin{aligned} &-\frac{1}{3}g(1)h(t) = -g(1)\frac{1}{3}t^2 \leq Bh(t) = -g\bigl(h(1)\bigr)\phi(t) \leq -g(1)\frac{1}{2}t^2 = -\frac{1}{2}g(1)h(t),\\ &-\frac{1}{4}g(1)h'(t) = -g(1)\frac{1}{2}t \leq (Bh)'(t) = -g\bigl(h(1)\bigr)\phi'(t) \leq -g(1)2t = -g(1)h'(t). \end{aligned}$$

Let $c_3 = -\frac{1}{4}g(1)$, $c_4 = -g(1)$. Then, from (H_1), (H_4), we have $c_4 \geq c_3 > 0$ and thus

$$\begin{aligned} &c_3 h(t) \leq Bh(t) \leq c_4 h(t),\\ &(c_3 h)'(t) = c_3 h'(t) \leq (Bh)'(t) \leq c_4 h'(t) = (c_4 h)'(t), \quad t \in [0,1]. \end{aligned}$$

Therefore, $c_3 h \dot{\leq} Bh \dot{\leq} c_4 h$. That is, $Bh \in P_h$. Hence the condition (i) of Lemma 2.4 is satisfied.

In the following we show the condition (ii) of Lemma 2.4 is satisfied. For $u, v \in P$ and any $t \in [0,1]$, from (H_4) and Lemma 3.1,

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)f\bigl(s,u(s),v'(s)\bigr)\,ds\\ &\geq \int_0^1 \frac{1}{3}t^2 s^2 f\bigl(s,u(s),v'(s)\bigr)\,ds \geq \frac{1}{3}t^2\int_0^1 s^2\sigma\,ds\\ &= \frac{1}{9}\sigma t^2 \geq \frac{1}{9}t^2\bigl[-g\bigl(u(1)\bigr)\bigr] = \frac{2}{9}\bigl[-g\bigl(u(1)\bigr)\bigr]\frac{1}{2}t^2\\ &\geq \frac{2}{9}\bigl[-g\bigl(u(1)\bigr)\bigr]\phi(t) = \frac{2}{9}Bu(t),\\ \bigl(A(u,v)\bigr)'(t) &= \int_0^1 G_t(t,s)f\bigl(s,u(s),v'(s)\bigr)\,ds\\ &\geq \int_0^1 \frac{1}{2}ts^2 f\bigl(s,u(s),v'(s)\bigr)\,ds \geq \frac{1}{2}t\int_0^1 s^2\sigma\,ds\\ &= \frac{1}{6}\sigma t \geq \frac{1}{6}t\bigl[-g\bigl(u(1)\bigr)\bigr] = \frac{1}{12}\bigl[-g\bigl(u(1)\bigr)\bigr]2t\\ &\geq \frac{1}{12}\bigl[-g\bigl(u(1)\bigr)\bigr]\phi'(t) = \frac{1}{12}(Bu)'(t). \end{aligned}$$

Let $\delta_0 = \frac{1}{12}$. Then

$$A(u,v)(t) \geq \delta_0 Bu(t), \qquad \bigl(A(u,v)\bigr)'(t) \geq \delta_0 (Bu)'(t), \quad t \in [0,1].$$

Therefore, we get $A(u,v)\dot{\geq}\delta_0 Bu$ for $u,v \in P$. Finally, an application of Lemma 2.4 implies: there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \dot{\leq} u_0 \dot{<} v_0$, $u_0 \dot{\leq} A(u_0,v_0) + Bu_0 \dot{\leq} A(v_0,u_0) + Bv_0 \dot{\leq} v_0$; the operator equation $A(u,u) + Bu = u$ has a unique solution u^* in P_h; for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1,2,\ldots,$$

we have $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty$. That is,

$$\begin{aligned}
u_0(t) &\leq \int_0^1 G(t,s) f\bigl(s, u_0(s), v_0'(s)\bigr)\,ds - g\bigl(u_0(1)\bigr)\phi(t), \quad t \in [0,1],\\
u_0'(t) &\leq \int_0^1 G_t(t,s) f\bigl(s, u_0(s), v_0'(s)\bigr)\,ds - g\bigl(u_0(1)\bigr)\phi'(t), \quad t \in [0,1],\\
v_0(t) &\geq \int_0^1 G(t,s) f\bigl(s, v_0(s), u_0'(s)\bigr)\,ds - g\bigl(v_0(1)\bigr)\phi(t), \quad t \in [0,1],\\
v_0'(t) &\geq \int_0^1 G_t(t,s) f\bigl(s, v_0(s), u_0'(s)\bigr)\,ds - g\bigl(v_0(1)\bigr)\phi'(t), \quad t \in [0,1];
\end{aligned}$$

problem (1.1) has a unique positive solution u^* in P_h and $u^*(t)$ is monotone increasing; for any $x_0, y_0 \in P_h$, constructing successively the sequences

$$\begin{aligned}
x_n(t) &= \int_0^1 G(t,s) f\bigl(s, x_{n-1}(s), y_{n-1}'(s)\bigr)\,ds - g\bigl(x_{n-1}(1)\bigr)\phi(t), \quad n = 1,2,\ldots,\\
y_n(t) &= \int_0^1 G(t,s) f\bigl(s, y_{n-1}(s), x_{n-1}'(s)\bigr)\,ds - g\bigl(y_{n-1}(1)\bigr)\phi'(t), \quad n = 1,2,\ldots,
\end{aligned}$$

we have $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$. □

Theorem 3.3 *Assume* (H_1), (H_2) *and*

(H_5) *there exists a constant* $\alpha \in (0,1)$ *such that* $g(\lambda x) \leq \lambda^\alpha g(x)$, $\forall \lambda \in (0,1)$, $x \in [0,+\infty)$, *and* $f(t, \lambda x, \lambda^{-1} y) \geq \lambda f(t,x,y)$ *for* $\lambda \in (0,1)$, $t \in [0,1]$, $x,y \in [0,+\infty)$;

(H_6) $f(t,0,2) \not\equiv 0$ *for* $t \in [0,1]$ *and there exists a constant* $\sigma > 0$ *such that* $f(t,x,y) \leq \sigma \leq -g(x)$, $t \in [0,1]$, $x,y \geq 0$.

Then:

(1) *there exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that* $rv_0 \dot{\leq} u_0 \dot{<} v_0$ *and*

$$\begin{aligned}
u_0(t) &\leq \int_0^1 G(t,s) f\bigl(s, u_0(s), v_0'(s)\bigr)\,ds - g\bigl(u_0(1)\bigr)\phi(t), \quad t \in [0,1],\\
u_0'(t) &\leq \int_0^1 G_t(t,s) f\bigl(s, u_0(s), v_0'(s)\bigr)\,ds - g\bigl(u_0(1)\bigr)\phi'(t), \quad t \in [0,1],\\
v_0(t) &\geq \int_0^1 G(t,s) f\bigl(s, v_0(s), u_0'(s)\bigr)\,ds - g\bigl(v_0(1)\bigr)\phi(t), \quad t \in [0,1],\\
v_0'(t) &\geq \int_0^1 G_t(t,s) f\bigl(s, v_0(s), u_0'(s)\bigr)\,ds - g\bigl(v_0(1)\bigr)\phi'(t), \quad t \in [0,1],
\end{aligned}$$

where $h(t) = t^2$, $t \in [0,1]$ *and* $G(t,s)$ *is given as in* (3.1);

(2) *problem* (1.1) *has a unique monotone positive solution* u^* *in* P_h;
(3) *for any* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_n(t) = \int_0^1 G(t,s) f\bigl(s, x_{n-1}(s), y'_{n-1}(s)\bigr)\, ds - g\bigl(x_{n-1}(1)\bigr)\phi(t), \quad n = 1, 2, \ldots,$$
$$y_n(t) = \int_0^1 G(t,s) f\bigl(s, y_{n-1}(s), x'_{n-1}(s)\bigr)\, ds - g\bigl(y_{n-1}(1)\bigr)\phi'(t), \quad n = 1, 2, \ldots,$$

we have $\|x_n - u^*\| \to 0$ *and* $\|y_n - u^*\| \to 0$ *as* $n \to \infty$.

Sketch of the proof Consider two operators A, B defined in the proof of Theorem 3.2. Similarly, from (H_1), (H_2), we obtain that $A : P \times P \to P$ is a mixed monotone operator and $B : P \to P$ is increasing. From (H_5), we have

$$A\bigl(\lambda u, \lambda^{-1} v\bigr) \dot{\succeq} \lambda A(u,v); \qquad B(\lambda u) \dot{\succeq} \lambda^{\alpha} Bu, \quad \text{for } \lambda \in (0,1), u, v \in P.$$

Since $f(t,0,2) \not\equiv 0$, we get $\int_0^1 s^2 f(s,0,2)\, ds > 0$, and in sequence, $c_2 \geq c_1 > 0$, here c_1, c_2 are defined in the proof of Theorem 3.2. So we can easily prove that $A(h,h) \in P_h$. From (H_6), we know that $-g(1) > 0$, and from the proof of Theorem 3.2 we get $Bh \in P_h$. For $u, v \in P$ and any $t \in [0,1]$, from (H_6),

$$\begin{aligned}
A(u,v)(t) &= \int_0^1 G(t,s) f\bigl(s, u(s), v'(s)\bigr)\, ds \\
&\leq \int_0^1 \frac{1}{2} t^2 s f\bigl(s, u(s), v'(s)\bigr)\, ds \leq \frac{1}{2} t^2 \int_0^1 s\sigma\, ds \\
&= \frac{1}{4}\sigma t^2 \leq \frac{1}{4} t^2 \bigl[-g\bigl(u(1)\bigr)\bigr] = \frac{3}{4}\bigl[-g\bigl(u(1)\bigr)\bigr]\frac{1}{3} t^2 \\
&\leq \frac{3}{4}\bigl[-g\bigl(u(1)\bigr)\bigr]\phi(t) = \frac{3}{4} Bu(t), \\
\bigl(A(u,v)\bigr)'(t) &= \int_0^1 G_t(t,s) f\bigl(s, u(s), v'(s)\bigr)\, ds \\
&\leq \int_0^1 t s f\bigl(s, u(s), v'(s)\bigr)\, ds \leq t \int_0^1 s\sigma\, ds \\
&= \frac{1}{2}\sigma t \leq \frac{1}{2} t \bigl[-g\bigl(u(1)\bigr)\bigr] \leq \bigl[-g\bigl(u(1)\bigr)\bigr]\phi'(t) = (Bu)'(t).
\end{aligned}$$

Let $\delta_0 = 1$. Then

$$A(u,v)(t) \leq \delta_0 Bu(t), \qquad \bigl(A(u,v)\bigr)'(t) \leq \delta_0 (Bu)'(t), \quad t \in [0,1].$$

Therefore, we get $A(u,v) \dot{\leq} \delta_0 Bu$ for $u, v \in P$. Finally, an application of Lemma 2.5 implies: there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $r v_0 \dot{\preceq} u_0 \dot{<} v_0$, $u_0 \dot{\preceq} A(u_0, v_0) + Bu_0 \dot{\preceq} A(v_0, u_0) + Bv_0 \dot{\preceq} v_0$; the operator equation $A(u,u) + Bu = u$ has a unique solution u^* in P_h; for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots,$$

we have $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty$. That is,

$$u_0(t) \le \int_0^1 G(t,s)f\big(s,u_0(s),v_0'(s)\big)\,ds - g\big(u_0(1)\big)\phi(t), \quad t \in [0,1],$$
$$u_0'(t) \le \int_0^1 G_t(t,s)f\big(s,u_0(s),v_0'(s)\big)\,ds - g\big(u_0(1)\big)\phi'(t), \quad t \in [0,1],$$
$$v_0(t) \ge \int_0^1 G(t,s)f\big(s,v_0(s),u_0'(s)\big)\,ds - g\big(v_0(1)\big)\phi(t), \quad t \in [0,1],$$
$$v_0'(t) \ge \int_0^1 G_t(t,s)f\big(s,v_0(s),u_0'(s)\big)\,ds - g\big(v_0(1)\big)\phi'(t), \quad t \in [0,1];$$

problem (1.1) has a unique positive solution u^* in P_h and $u^*(t)$ is monotone increasing; for any $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n(t) = \int_0^1 G(t,s)f\big(s,x_{n-1}(s),y_{n-1}'(s)\big)\,ds - g\big(x_{n-1}(1)\big)\phi(t), \quad n = 1,2,\ldots,$$
$$y_n(t) = \int_0^1 G(t,s)f\big(s,y_{n-1}(s),x_{n-1}'(s)\big)\,ds - g\big(y_{n-1}(1)\big)\phi'(t), \quad n = 1,2,\ldots,$$

we have $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$.

Remark 3.4 Comparing Theorems 3.2-3.3 with the main results in [17], we provide some alternative approaches to study the same type of problems under different conditions. Our results can guarantee the existence of a unique monotone positive solution and the existence of upper-lower solutions, which are seldom seen in the literature.

4 Examples

To illustrate how our main results can be used in practice we present two examples.

Example 4.1 Consider the following fourth-order boundary value problem:

$$\begin{cases} u^{(4)}(t) = u^{\frac{1}{2}}(t) + [u'(t)+2]^{-\frac{1}{3}} + 2, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u^{(3)}(1) = -\frac{u(1)}{1+u(1)} - 1. \end{cases} \tag{4.1}$$

Obviously, problem (4.1) fits the framework of problem (1.1). In this example, let

$$f(t,x,y) = x^{\frac{1}{2}} + (y+1)^{-\frac{1}{3}} + 2, \qquad g(x) = -\frac{x}{1+x} - 1, \qquad \alpha = \frac{1}{2}, \qquad \sigma = 2.$$

Obviously, $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous and $g : [0,+\infty) \to (-\infty,0]$ is continuous. And $f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$ and $y \in [0,+\infty)$, decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$ and $x \in [0,+\infty)$, and $g(x)$ is decreasing in $x \in [0,+\infty)$. Besides, for $\lambda \in (0,1)$, $t \in [0,1]$, $x, y \in [0,+\infty)$, we have

$$g(\lambda x) = -\frac{\lambda x}{1+\lambda x} - 1 \le -\frac{\lambda x}{1+x} - \lambda = \lambda g(x),$$
$$f\big(t,\lambda x,\lambda^{-1}y\big) = \lambda^{\frac{1}{2}}x^{\frac{1}{2}} + \lambda^{\frac{1}{3}}(y+\lambda)^{-\frac{1}{3}} + 2 \ge \lambda^{\frac{1}{2}}\big[x^{\frac{1}{2}} + (y+1)^{-\frac{1}{3}} + 2\big] = \lambda^{\alpha}f(t,x,y).$$

Moreover,

$$f(t,x,y) = x^{\frac{1}{2}} + (y+1)^{-\frac{1}{3}} + 2 \geq \sigma = 2 \geq \frac{x}{1+x} + 1 = -g(x) > 0.$$

Hence all the conditions of Theorem 3.2 are satisfied. An application of Theorem 3.2 implies that problem (4.1) has a unique monotone positive solution in P_h, where P_h is the same set in Section 3 and $h(t) = t^2$, $t \in [0,1]$.

Example 4.2 Consider the following fourth-order boundary value problem:

$$\begin{cases} u^{(4)}(t) = \cos^2 t + \frac{u(t)}{1+u(t)} + \frac{1}{2+u'(t)}, & 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u^{(3)}(1) = -[u(1)]^{\frac{1}{3}} - 3. \end{cases} \tag{4.2}$$

Obviously, problem (4.2) fits the framework of problem (1.1). In this example, let

$$f(t,x,y) = \cos^2 t + \frac{x}{1+x} + \frac{1}{2+y}, \qquad g(x) = -x^{\frac{1}{3}} - 3, \qquad \alpha = \frac{1}{3}, \qquad \sigma = 3.$$

Obviously, $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous and $g : [0,+\infty) \to (-\infty,0]$ is continuous. And $f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$ and $y \in [0,+\infty)$, decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$ and $x \in [0,+\infty)$, and $g(x)$ is decreasing in $x \in [0,+\infty)$. Besides, for $\lambda \in (0,1)$, $t \in [0,1]$, $x, y \in [0,+\infty)$, we have

$$g(\lambda x) = -\lambda^{\frac{1}{3}} x^{\frac{1}{3}} - 3 \leq \lambda^{\frac{1}{3}} \left(-x^{\frac{1}{3}} - 3\right) = \lambda^{\alpha} g(x),$$
$$f\left(t, \lambda x, \lambda^{-1} y\right) = \cos^2 t + \frac{\lambda x}{1+\lambda x} + \frac{1}{2+\lambda^{-1} y} \geq \cos^2 t + \frac{\lambda x}{1+x} + \frac{\lambda}{2+y} \geq \lambda f(t,x,y).$$

Moreover,

$$f(t,x,y) = \cos^2 t + \frac{x}{1+x} + \frac{1}{2+y} \leq \sigma = 3 \leq x^{\frac{1}{3}} + 3 = -g(x).$$

Hence all the conditions of Theorem 3.3 are satisfied. An application of Theorem 3.3 implies that problem (4.2) has a unique monotone positive solution in P_h, where P_h is the same set in Section 3 and $h(t) = t^2$, $t \in [0,1]$.

Remark 4.3 Examples 4.1, 4.2 imply that there are many functions that satisfy the conditions of Theorems 3.2, 3.3. So the conditions of our results are easy to check.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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