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# Positive doubly periodic solutions of telegraph equations with delays

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**Abstract**

This paper deals with the existence of positive doubly periodic solutions for the nonlinear telegraph equation with delays $\mathcal{L}u = f(t,x,u(t-\tau_1,x),\dots,u(t-\tau_n,x))$, $(t,x) \in \mathbb{R}^2$, where $\mathcal{L}u := u_{tt} - u_{xx} + cu_t + a(t,x)u$ is a linear telegraph operator acting on function $u : \mathbb{R}^2 \to \mathbb{R}$, $c > 0$ is a constant, $a \in C(\mathbb{R}^2,(0,\infty))$ is $2\pi$-periodic in $t$ and $x$, $f \in C(\mathbb{R}^2 \times [0,\infty)^n,[0,\infty))$ is $2\pi$-periodic in $t$ and $x$, and $\tau_1,\dots,\tau_n \in [0,\infty)$ are constants. Some existence results of positive doubly $2\pi$-periodic weak solutions are obtained under that $f(t,x,\eta_1,\dots,\eta_n)$ satisfies some superlinear or sublinear growth conditions on $\eta_1,\dots,\eta_n$. The discussion is based on the fixed point index theory in cones.

**MSC:** 35B15; 47H10

**Keywords:** telegraph equation with delays; doubly periodic solution; cone; fixed point index of cone mapping

## 1 Introduction and main results

In this paper we are concerned with the existence of solutions for the nonlinear telegraph equation with time delays

$$\mathcal{L}u = f\bigl(t,x,u(t-\tau_1,x),\dots,u(t-\tau_n,x)\bigr), \quad (t,x) \in \mathbb{R}^2 \tag{1.1}$$

with doubly periodic boundary condition

$$u(t+2\pi,x) = u(t,x+2\pi) = u(t,x), \quad (t,x) \in \mathbb{R}^2, \tag{1.2}$$

where

$$\mathcal{L}u := u_{tt} - u_{xx} + cu_t + a(t,x)u \tag{1.3}$$

is a linear telegraph operator acting on function $u : \mathbb{R}^2 \to \mathbb{R}$, $c > 0$ is a constant, $a \in C(\mathbb{R}^2,(0,\infty))$ is $2\pi$-periodic in $t$ and $x$, $f \in C(\mathbb{R}^2 \times [0,\infty)^n,[0,\infty))$ is $2\pi$-periodic in $t$ and $x$, and $\tau_1,\dots,\tau_n \in [0,\infty)$ are constants.

As is well known, the telegraph equation describes a great deal of physical systems. For instance, the propagation of electromagnetic waves in an electrically conducting medium, the motion of a string or membrane with external damping, the motion of a viscoelastic fluid under the Maxwell body theory, the damped wave equation in a thermally conducting medium, *etc.* (see [1, 2]). In these models, the existence of time periodic solutions is an important problem which has attracted many authors' attention and concern, see [3–16] and the references therein. All of these works are on the telegraph equations without time delays. It has been widely argued and accepted [17, 18] that for various reasons, time delay should be taken into consideration in modeling. Obviously, the telegraph equation with time delay has more actual significance. For instance, in the control propagation of electromagnetic wave signals, the signal intensity $u(t,x)$ is subjected to a telegraph equation with time delay, in which the time delay expresses that the control act has delays. The purpose of this paper is to discuss existence of positive doubly periodic solutions for the nonlinear telegraph equation (1.1) with time delays.

Let $\mathbb{T}^2$ be the torus defined by

$$\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}). \tag{1.4}$$

Doubly $2\pi$-periodic functions will be identified to functions defined on $\mathbb{T}^2$. We use the following notations:

$$L^p(\mathbb{T}^2), \qquad C(\mathbb{T}^2), \qquad C^\alpha(\mathbb{T}^2), \qquad \mathcal{D}(\mathbb{T}^2) = C^\infty(\mathbb{T}^2), \qquad \ldots$$

to denote the spaces of doubly periodic functions with the indicated degree of regularity. $\mathcal{D}'(\mathbb{T}^2)$ denotes the space of distributions on $\mathbb{T}^2$.

We define a doubly periodic solution of Equation (1.1) as a function $u \in L^1(\mathbb{T}^2)$ that satisfies Equation (1.1) in the distribution sense, that is,

$$\begin{aligned}&\int_{\mathbb{T}^2} u\big(\phi_{tt} - \phi_{xx} - c\phi_t + a(t,x)\phi\big)\,\mathrm{d}t\,\mathrm{d}x \\ &\quad = \int_{\mathbb{T}^2} f\big(t,x,u(t-\tau_1,x),\ldots,u(t-\tau_n,x)\big)\phi\,\mathrm{d}t\,\mathrm{d}x, \quad \forall \phi \in \mathcal{D}(\mathbb{T}^2),\end{aligned} \tag{1.5}$$

with $f(t,x,u(t-\tau_1,x),\ldots,u(t-\tau_n,x)) \in L^1(\mathbb{T}^2)$.

We assume the following conditions throughout this paper:

(H1) $a \in C(\mathbb{T}^2)$ and $0 < a(t,x) \le \frac{c^2}{4}$ for $(t,x) \in \mathbb{R}^2$;

(H2) $f \in C(\mathbb{T}^2 \times [0,\infty)^n, [0,\infty))$,

and introduce the notations

$$\underline{a} = \min_{(t,x)\in\mathbb{T}^2} a(t,x), \qquad \overline{a} = \max_{(t,x)\in\mathbb{T}^2} a(t,x). \tag{1.6}$$

By assumption (H1), $0 < \underline{a} \le \overline{a} \le \frac{c^2}{4}$. Our main results are as follows.

**Theorem 1.1** *Assume that* (H1) *and* (H2) *hold. If $f$ satisfies the following conditions*:

(F1) *there exist positive constants $c_1,\ldots,c_n$ satisfying $c_1 + \cdots + c_n < \underline{a}$ and $\delta > 0$ such that*

$$f(t,x,\eta_1,\ldots,\eta_n) \le c_1\eta_1 + \cdots + c_n\eta_n$$

*for $(t,x) \in \mathbb{R}^2$ and $\eta_1,\ldots,\eta_n \in [0,\delta]$*;

(F2) *there exist positive constants* $d_1, \dots, d_n$ *satisfying* $d_1 + \cdots + d_n > \overline{a}$ *and* $H > 0$ *such that*

$$f(t, x, \eta_1, \dots, \eta_n) \geq d_1\eta_1 + \cdots + d_n\eta_n$$

*for* $(t, x) \in \mathbb{R}^2$ *and* $\eta_1, \dots, \eta_n \geq H$,

*then Equation* (1.1) *has at least one positive doubly periodic solution in* $C(\mathbb{T}^2)$.

**Theorem 1.2** *Assume that* (H1) *and* (H2) *hold. If* $f$ *satisfies the following conditions*:

(F3) *there exist positive constants* $d_1, \dots, d_n$ *satisfying* $d_1 + \cdots + d_n > \overline{a}$ *and* $\delta > 0$ *such that*

$$f(t, x, \eta_1, \dots, \eta_n) \geq d_1\eta_1 + \cdots + d_n\eta_n$$

*for* $(t, x) \in \mathbb{R}^2$ *and* $\eta_1, \dots, \eta_n \in [0, \delta]$;

(F4) *there exist positive constants* $c_1, \dots, c_n$ *satisfying* $c_1 + \cdots + c_n < \underline{a}$ *and* $H > 0$ *such that*

$$f(t, x, \eta_1, \dots, \eta_n) \leq c_1\eta_1 + \cdots + c_n\eta_n$$

*for* $(t, x) \in \mathbb{R}^2$ *and* $\eta_1, \dots, \eta_n \geq H$,

*then Equation* (1.1) *has at least one positive doubly periodic solution in* $C(\mathbb{T}^2)$.

In Theorem 1.1, conditions (F1) and (F2) allow the nonlinearity $f(t, x, \eta_1, \dots, \eta_n)$ to be superlinear growth on $\eta_1, \dots, \eta_n$. For instance,

$$f(t, x, \eta_1, \dots, \eta_n) = a_1(t, x)\eta_1^2 + \cdots + a_n(t, x)\eta_n^2$$

satisfies (F1) and (F2), where $a_1, \dots, a_n \in C(\mathbb{T}^2, [0, \infty))$.

In Theorem 1.2, conditions (F3) and (F4) allow the nonlinearity $f(t, x, \eta_1, \dots, \eta_n)$ to be sublinear growth on $\eta_1, \dots, \eta_n$. For instance,

$$f(t, x, \eta_1, \dots, \eta_n) = b_1(t, x)\sqrt{|\eta_1|} + \cdots + b_n(t, x)\sqrt{|\eta_n|}$$

satisfies (F3) and (F4), where $b_1, \dots, b_n \in C(\mathbb{T}^2, [0, \infty))$.

Conditions (F1) and (F2) in Theorem 1.1 and conditions (F3) and (F4) in Theorem 1.2 are optimal for the existence of positive double periodic solutions of Equation (1.1). This fact can been shown from the telegraph equation with linear time delays

$$u_{tt} - u_{xx} + cu_t + \frac{c^2}{4}u = c_1u(t - \tau_1, x) + \cdots + c_nu(t - \tau_n, x) + h(t, x), \tag{1.7}$$

where $c_1, \dots, c_n$ are positive constants, $h \in C(\mathbb{T}^2, (0, \infty))$ is a positive doubly periodic function. If $c_1, \dots, c_n$ satisfy

$$c_1 + c_2 + \cdots + c_n = \frac{c^2}{4}, \tag{1.8}$$

Equation (1.7) has no positive doubly $2\pi$-periodic solutions. In fact, if Equation (1.7) has a positive doubly $2\pi$-periodic solution $u$, by definition (1.5) of doubly periodic solution, choosing $\phi \equiv 1$ and using the periodicity of $u(t,x)$, noticing that

$$\int_{\mathbb{T}^2} u(t-\tau_k,x)\,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbb{T}^2} u(t,x)\,\mathrm{d}t\,\mathrm{d}x, \quad k = 1,2,\ldots,n, \tag{1.9}$$

we can obtain that $\int_{\mathbb{T}^2} h(t,x)\,\mathrm{d}t\,\mathrm{d}x = 0$, which contradicts the positivity of $h(t,x)$. Hence, Equation (1.7) has no positive doubly periodic solution. For $a(t) \equiv \frac{c^2}{4}$ and $f(t,x,\eta_1,\ldots,\eta_n) = c_1\eta_1 + \cdots + c_n\eta_n + h(t,x)$, if condition (1.8) holds, conditions (F1) and (F2) in Theorem 1.1 and conditions (F3) and (F4) in Theorem 1.2 are not satisfied. From this we see that the conditions in Theorems 1.1 and 1.2 are optimal.

The proofs of Theorems 1.1 and 1.2 are based on the fixed point index theory in cones, which will be given in Section 3. Some preliminaries to discuss Equation (1.1) are presented in Section 2.

## 2 Preliminaries

Let $c > 0$ be a constant and $a \in C(\mathbb{T}^2)$ satisfy (H1). We consider the doubly periodic problem of the linear telegraph equation

$$u_{tt} - u_{xx} + cu_t + a(t,x)u = h(t,x), \quad \text{in } \mathcal{D}'(\mathbb{T}^2), \tag{2.1}$$

where $h \in L^1(\mathbb{T}^2)$. A solution of (2.1) is a function $u \in L^1(\mathbb{T}^2)$ satisfying

$$\int_{\mathbb{T}^2} u\bigl(\phi_{tt} - \phi_{xx} + c\phi_t + a(t,x)\phi\bigr)\,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbb{T}^2} h\phi\,\mathrm{d}t\,\mathrm{d}x, \quad \forall \phi \in \mathcal{D}(\mathbb{T}^2). \tag{2.2}$$

For the special case where $a(t,x) \equiv \frac{c^2}{4}$, namely for the doubly periodic problem of the linear telegraph equation

$$u_{tt} - u_{xx} + cu_t + \frac{c^2}{4}u = h(t,x), \quad \text{in } \mathcal{D}'(\mathbb{T}^2), \tag{2.3}$$

the existence-uniqueness and regularity of solution have been discussed by Ortega and Robles-Perez in [15]. They obtained the Green function $G \in L^\infty(\mathbb{T}^2)$ of Equation (2.3) and proved that Equation (2.3) has a unique solution $u \in C(\mathbb{T}^2)$ which can be represented by the convolution product

$$u(t,x) = \int_{\mathbb{T}^2} G(t-s,x-y)h(s,y)\,\mathrm{d}s\,\mathrm{d}y. \tag{2.4}$$

See Equation (5.10) in [15]. The expression of the Green function $G(t,s)$ is given as follows.

Let $\mathcal{D} = \mathbb{R}^2 \setminus \mathcal{C}$, where $\mathcal{C}$ is the family of lines

$$x \pm t = 2k\pi, \quad k \in \mathbb{Z}.$$

Let $\mathcal{D}_{ij}$ denote the connected component of $\mathcal{D}$ with center at $(i\pi, j\pi)$, where $i + j$ is an old number. By periodicity, the value of $G$ on $\mathcal{D}_{10}$ and $\mathcal{D}_{01}$ completely determines the value of

$G$ on the whole set $\mathcal{D}$. In $\mathcal{D}_{10}$ and $\mathcal{D}_{01}$, $G(t,x)$ is explicitly given by

$$G(t,x)=\begin{cases}\gamma_{10}e^{-ct/2}, & (t,x)\in\mathcal{D}_{10},\\ \gamma_{01}e^{-ct/2}, & (t,x)\in\mathcal{D}_{01},\end{cases} \tag{2.5}$$

where

$$\gamma_{10}=\frac{1}{2}\frac{1+e^{-c\pi}}{(1-e^{-c\pi})^2},\qquad \gamma_{01}=\frac{e^{-c\pi}}{(1-e^{-c\pi})^2}.$$

See Lemma 5.2 in [15]. From (2.5), we have

$$\begin{aligned}&\underline{G}:=\operatorname{ess\,inf}G(t,x)=\frac{e^{-3c\pi/2}}{(1-e^{-c\pi})^2},\\ &\overline{G}:=\operatorname{ess\,sup}G(t,x)=\frac{1+e^{-c\pi}}{2(1-e^{-c\pi})^2}.\end{aligned} \tag{2.6}$$

For Equation (2.1), in [13] the present author using the above result and a perturbation method of positive operator has built the following existence-uniqueness and positive estimate result.

**Lemma 2.1** (Lemma 2 in [13]) *Assume that* (H1) *holds. For every* $h\in L^1(\mathbb{T}^2)$, *Equation* (3.1) *has a unique solution* $u:=Ph\in C(\mathbb{T}^2)$. *Moreover,* $P:L^1(\mathbb{T}^2)\to C(\mathbb{T}^2)$ *is a linear bounded operator with the following properties*:

(1) *The restriction of* $P$ *on* $C(\mathbb{T}^2)$, $P:C(\mathbb{T}^2)\to C(\mathbb{T}^2)$ *is a completely continuous operator.*

(2) *If* $h(t,x)\geq 0$, *a.e.* $(t,x)\in\mathbb{T}^2$, *Ph has the positivity estimate*

$$\underline{G}\|h\|_1\leq (Ph)(t,x)\leq\frac{\overline{G}}{\underline{G}\|a\|_1}\|h\|_1,\quad\forall (t,x)\in\mathbb{T}^2. \tag{2.7}$$

Let $E$ denote the Banach space $C(\mathbb{T}^2)$. We simply denote the norm in $E$ by $\|\cdot\|$, and in $L^p(\mathbb{T}^2)$ by $\|\cdot\|_p$. Notice that $E$ is an ordered Banach space with cone

$$K_0=\{u\in E\mid u(t,x)\geq 0,\forall (t,x)\in\mathbb{T}^2\}. \tag{2.8}$$

We define a mapping $A:K_0\to E$ by

$$Au=P\bigl(f\bigl(t,x,u(t-\tau_1,x),\ldots,u(t-\tau_n,x)\bigr)\bigr),\quad\forall u\in K_0. \tag{2.9}$$

By Lemma 2.1 and assumption (H2), $A:K_0\to E$ is completely continuous, and the doubly periodic solution of Equation (1.1) is equivalent to the fixed point of $A$. We will find the non-zero fixed point of $A$ by the fixed point index theory in cones. For this we choose a sub-cone of $K_0$ by

$$K=\{u\in E\mid u(t,x)\geq\sigma\|u\|,\forall (t,x)\in\mathbb{T}^2\}, \tag{2.10}$$

where

$$\sigma=\frac{\underline{G}^2\|a\|_1}{\overline{G}}=\frac{2e^{-3c\pi}\|a\|_1}{(1-e^{-c\pi})^2(1+e^{-c\pi})}\in(0,1) \tag{2.11}$$

is a positive constant as in [13].

**Lemma 2.2** *$A(K_0) \subset K$, and $A : K \to K$ is completely continuous.*

*Proof* Let $u \in K_0$. Set $h(t,x) = f(t,x,u(t-\tau_1,x),\ldots,u(t-\tau_n,x))$ for every $(t,x) \in \mathbb{T}^2$, then $h \in E$ and $Au = Ph$. From the latter inequality of (2.7) it follows that

$$\|Au\| = \|Ph\| \leq \frac{\overline{G}}{\underline{G}\|a\|_1}\|h\|_1. \tag{2.12}$$

Using this and the former inequality of (2.7), we have

$$\begin{aligned}(Au)(t,x) &= (Ph)(t,x) \geq \underline{G}\|h\|_1 \\ &\geq \frac{\underline{G}^2\|a\|_1}{\overline{G}}\|Au\| = \sigma\|Au\|, \quad (t,x) \in \mathbb{T}^2.\end{aligned}$$

This means that $Au \in K$. Thus $A(K_0) \subset K$. The complete continuity of $A : K \to K$ is obvious. □

By Lemma 2.2, the positive doubly periodic solution of Equation (1.1) is equivalent to the nontrivial fixed point of $A$. We will find the nontrivial fixed point of $A$ by using the fixed point index theory in cone $K$.

We recall some concepts and conclusions on the fixed point index in [19, 20]. Let $E$ be a Banach space and $K \subset E$ be a closed convex cone in $E$. Assume $\Omega$ is a bounded open subset of $E$ with boundary $\partial\Omega$, and $K \cap \Omega \neq \emptyset$. Let $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $Au \neq u$ for any $u \in K \cap \partial\Omega$, then the fixed point index $i(A, K \cap \Omega, K)$ has definition. One important fact is that if $i(A, K\cap\Omega, K) \neq 0$, then $A$ has a fixed point in $K \cap \Omega$, see Theorem 2.3.2 in [20]. The following two lemmas in [20] are needed in our argument.

**Lemma 2.3** (Lemma 2.3.1 in [20]) *Let $\Omega$ be a bounded open subset of $E$ with $\theta \in \Omega$, and $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $\lambda Au \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \lambda \leq 1$, then $i(A, K \cap \Omega, K) = 1$.*

**Lemma 2.4** (Corollary 2.3.1 in [20]) *Let $\Omega$ be a bounded open subset of $E$ and $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If there exists $e \in K \setminus \{\theta\}$ such that $u - Au \neq \mu e$ for every $u \in K \cap \partial\Omega$ and $\mu \geq 0$, then $i(A, K \cap \Omega, K) = 0$.*

In the next section, we will use Lemma 2.3 and Lemma 2.4 to prove Theorem 1.1 and Theorem 1.2.

## 3 Proofs of main results

*Proof of Theorem* 1.1 Choose the working space $E = C(\mathbb{T}^2)$. Let $K \subset C(\mathbb{T}^2)$ be the closed convex cone in $C(\mathbb{T}^2)$ defined by (2.10) and $A : K \to K$ be the completely continuous operator defined by (2.9). Then the positive doubly periodic solution of Equation (1.1) is equivalent to the nontrivial fixed point of $A$. Let $0 < r < R < +\infty$ and set

$$\Omega_1 = \{u \in E \mid \|u\| < r\}, \qquad \Omega_2 = \{u \in E \mid \|u\| < R\}. \tag{3.1}$$

We show that the operator $A$ has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when $r$ is small enough and $R$ large enough.

Let $r \in (0, \delta)$, where $\delta$ is the positive constant in condition (F1). We prove that $A$ satisfies the condition of Lemma 2.3 in $K \cap \partial\Omega_1$, namely $\lambda Au \neq u$ for every $u \in K \cap \partial\Omega_1$ and $0 < \lambda \leq 1$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $0 < \lambda_0 \leq 1$ such that $\lambda_0 Au_0 = u_0$, since $u_0 = P(\lambda_0 f(t, x, u_0(t - \tau_1, x), \dots, u_0(t - \tau_n, x)))$, by Lemma 2.1 and the definition of $P$, $u_0 \in C(\mathbb{T}^2)$ satisfies the differential equation

$$\mathcal{L}u_0(t,x) = \lambda_0 f\bigl(t, x, u_0(t - \tau_1, x), \dots, u_0(t - \tau_n, x)\bigr), \quad \text{in } \mathcal{D}'\bigl(\mathbb{T}^2\bigr). \tag{3.2}$$

By definition of the solution in the distribution sense, Equation (3.2) means that $u_0$ satisfies

$$\begin{aligned}&\int_{\mathbb{T}^2} u_0(t,x)\mathcal{L}\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x \\ &\quad = \int_{\mathbb{T}^2} \lambda_0 f\bigl(t, x, u_0(t - \tau_1, x), \dots, u_0(t - \tau_n, x)\bigr)\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x, \quad \forall \phi \in \mathcal{D}\bigl(\mathbb{T}^2\bigr).\end{aligned}$$

Choosing $\phi(t,x) \equiv 1 \in \mathcal{D}(\mathbb{T}^2)$, we obtain that

$$\int_{\mathbb{T}^2} a(t,x)u_0(t,x)\,\mathrm{d}t\,\mathrm{d}x = \lambda_0 \int_{\mathbb{T}^2} f\bigl(t, x, u_0(t - \tau_1, x), \dots, u_0(t - \tau_n, x)\bigr)\,\mathrm{d}t\,\mathrm{d}x. \tag{3.3}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of $K$ and $\Omega_1$, we have

$$0 \leq u_0(t - \tau_k, x) \leq \|u_0\| = r < \delta, \quad t, x \in \mathbb{R}, k = 1, \dots, n. \tag{3.4}$$

Hence from condition (F1) it follows that

$$f\bigl(t, x, u_0(t - \tau_1, x), \dots, u_0(t - \tau_n, x)\bigr) \leq c_1 u_0(t - \tau_1, x) + \cdots + c_n u_0(t - \tau_n, x)$$

for every $t, x \in \mathbb{R}$. By this inequality and (3.3), using the periodicity of $u_0$ and (1.9), we have

$$\begin{aligned}\int_{\mathbb{T}^2} a(t,x)u_0(t,x)\,\mathrm{d}t\,\mathrm{d}x &= \lambda_0 \int_{\mathbb{T}^2} f\bigl(t, x, u_0(t - \tau_1, x), \dots, u_0(t - \tau_n, x)\bigr)\,\mathrm{d}t\,\mathrm{d}x \\ &\leq \int_{\mathbb{T}^2} f\bigl(t, x, u_0(t - \tau_1, x), \dots, u_0(t - \tau_n, x)\bigr)\,\mathrm{d}t\,\mathrm{d}x \\ &\leq \int_{\mathbb{T}^2} \bigl(c_1 u_0(t - \tau_1, x) + \cdots + c_n u_0(t - \tau_n, x)\bigr)\,\mathrm{d}t\,\mathrm{d}x \\ &= (c_1 + \cdots + c_n) \int_{\mathbb{T}^2} u_0(t,x)\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

Consequently, we obtain that

$$\begin{aligned}\underline{a} \int_{\mathbb{T}^2} u_0(t,x)\,\mathrm{d}t\,\mathrm{d}x &\leq \int_{\mathbb{T}^2} a(t,x)u_0(t,x)\,\mathrm{d}t\,\mathrm{d}x \\ &\leq (c_1 + \cdots + c_n) \int_{\mathbb{T}^2} u_0(t,x)\,\mathrm{d}t\,\mathrm{d}x.\end{aligned} \tag{3.5}$$

By the definition of cone $K$, $\int_{\mathbb{T}^2} u_0(t,x)\,\mathrm{d}t\,\mathrm{d}x \geq \sigma\|u_0\| \cdot 4\pi^2 > 0$. From (3.5) it follows that $\underline{a} \leq c_1 + \cdots + c_n$, which contradicts the assumption in condition (F1). Hence $A$ satisfies the condition of Lemma 2.3 in $K \cap \partial\Omega_1$. By Lemma 2.3 we have

$$i(A, K \cap \Omega_1, K) = 1. \tag{3.6}$$

On the other hand, choose $R > \max\{H/\sigma, \delta\}$, where $H$ is the positive constant in condition (F2), and let $e(t,x) \equiv 1$. Clearly, $e \in K \setminus \{\theta\}$. We show that $A$ satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_2$, namely $u - Au \neq \mu e$ for every $u \in K \cap \partial\Omega_2$ and $\mu \geq 0$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $\mu_1 \geq 0$ such that $u_1 - Au_1 = \mu_1 e$, since $u_1 - \mu_1 e = Au_1 = P(f(t,x,u_1(t-\tau_1,x),\ldots,u_1(t-\tau_n,x)))$, by the definition of $P$ and Lemma 2.1, $u_1 - \mu_1 e \in C(\mathbb{T}^2)$ satisfies the differential equation

$$\mathcal{L}(u_1 - \mu_1 e) = f\bigl(t,x,u_1(t-\tau_1),\ldots,u_1(t-\tau_n)\bigr), \quad \text{in } \mathcal{D}'\bigl(\mathbb{T}^2\bigr). \tag{3.7}$$

In the definition of the solution of Equation (3.7) in the distribution sense, choosing $\phi(t,x) \equiv 1 \in \mathcal{D}(\mathbb{T}^2)$, we have

$$\int_{\mathbb{T}^2} a(t,x)\bigl(u_1(t,x) - \mu_1\bigr)\,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbb{T}^2} f\bigl(t,x,u_1(t-\tau_1,x),\ldots,u_1(t-\tau_n,x)\bigr)\,\mathrm{d}t\,\mathrm{d}x. \tag{3.8}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of $K$, we have

$$u_1(t-\tau_k,x) \geq \sigma\|u_1\| = \sigma R > H, \quad t,x \in \mathbb{R}, k = 1,\ldots,n. \tag{3.9}$$

From this and condition (F2), it follows that

$$f\bigl(t,x,u_1(t-\tau_1),\ldots,u_1(t-\tau_n)\bigr) \geq d_1 u_1(t-\tau_1,x) + \cdots + d_n u_n(t-\tau_n,x)$$

for every $t,x \in \mathbb{R}$. By this inequality and (3.8), using the periodicity of $u_1(t,x)$ and (1.9), we have

$$\begin{aligned}
\int_{\mathbb{T}^2} a(t,x)u_1(t,x)\,\mathrm{d}t\,\mathrm{d}x &\geq \int_{\mathbb{T}^2} a(t,x)\bigl(u_1(t,x) - \mu_1\bigr)\,\mathrm{d}t\,\mathrm{d}x \\
&\geq \int_{\mathbb{T}^2} f\bigl(t,x,u_1(t-\tau_1,x),\ldots,u_1(t-\tau_n,x)\bigr)\,\mathrm{d}t\,\mathrm{d}x \\
&\geq \int_{\mathbb{T}^2} \bigl(d_1 u_1(t-\tau_1,x) + \cdots + d_n u_1(t-\tau_n,x)\bigr)\,\mathrm{d}t\,\mathrm{d}x \\
&= (d_1 + \cdots + d_n)\int_{\mathbb{T}^2} u_1(t,x)\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

Hence, we get that

$$\begin{aligned}
\overline{a}\int_{\mathbb{T}^2} u_1(t,x)\,\mathrm{d}t\,\mathrm{d}x &\geq \int_{\mathbb{T}^2} a(t,x)u_1(t,x)\,\mathrm{d}t\,\mathrm{d}x \\
&\geq (d_1 + \cdots + d_n)\int_{\mathbb{T}^2} u_1(t,x)\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned} \tag{3.10}$$

Since $\int_{\mathbb{T}^2} u_1(t,x)\,\mathrm{d}t\,\mathrm{d}x \geq \sigma\|u_1\| \cdot 4\pi^2 > 0$, from this inequality it follows that $\overline{a} \geq d_1 + \cdots + d_n$, which contradicts the assumption in condition (F2). This means that $A$ satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_2$. By Lemma 2.4,

$$i(A, K \cap \Omega_2, K) = 0. \tag{3.11}$$

Now, using the additivity of fixed point index in cone $K$, by (3.6) and (3.11) we have that

$$i\big(A, K \cap (\Omega_2 \setminus \overline{\Omega}_1), K\big) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = -1.$$

Hence, $A$ has a fixed-point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive doubly periodic solution of Equation (1.1). □

*Proof of Theorem* 1.2 Let $\Omega_1, \Omega_2 \subset C(\mathbb{T}^2)$ be defined by (3.1). We prove that the operator $A$ defined by (2.9) has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when $r$ is small enough and $R$ large enough.

Let $r \in (0, \delta)$, where $\delta$ is the positive constant in condition (F3), and choose $e(t,x) \equiv 1$. We prove that $A$ satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_1$, namely $u - Au \neq \mu e$ for every $u \in K \cap \partial\Omega_1$ and $\mu \geq 0$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $\mu_0 \geq 0$ such that $u_0 - Au_0 = \mu_0 e$, since $u_0 - \mu_0 e = Au_0 = P(f(t, x, u_0(t-\tau_1, x), \ldots, u_0(t-\tau_n, x)))$, by the definition of $P$ and Lemma 2.1, $u_0 - \mu_0 e \in C(\mathbb{T}^2)$ satisfies the differential equation

$$\mathcal{L}(u_0 - \mu_0 e) = f\big(t, x, u_0(t-\tau_1), \ldots, u_0(t-\tau_n)\big), \quad \text{in } \mathcal{D}'\big(\mathbb{T}^2\big). \tag{3.12}$$

In the definition of the solution of Equation (3.12) in the distribution sense, choosing $\phi(t,x) \equiv 1 \in \mathcal{D}(\mathbb{T}^2)$, we get that

$$\int_{\mathbb{T}^2} a(t,x)\big(u_0(t,x) - \mu_0\big)\,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbb{T}^2} f\big(t, x, u_0(t-\tau_1, x), \ldots, u_0(t-\tau_n, x)\big)\,\mathrm{d}t\,\mathrm{d}x. \tag{3.13}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of $K$ and $\Omega_1$, $u_0$ satisfies (3.4). From (3.4) and condition (F3) we see that

$$f\big(t, x, u_0(t-\tau_1, x), \ldots, u_0(t-\tau_n, x)\big) \geq d_1 u_0(t-\tau_1, x) + \cdots + d_n u_0(t-\tau_n, x)$$

for every $t, x \in \mathbb{R}$. By this inequality and (3.13), we have

$$\begin{aligned}
\int_{\mathbb{T}^2} a(t,x) u_0(t,x)\,\mathrm{d}t\,\mathrm{d}x &\geq \int_{\mathbb{T}^2} a(t,x)\big(u_0(t,x) - \mu_0\big)\,\mathrm{d}t\,\mathrm{d}x \\
&\geq \int_{\mathbb{T}^2} f\big(t, x, u_0(t-\tau_1, x), \ldots, u_0(t-\tau_n, x)\big)\,\mathrm{d}t\,\mathrm{d}x \\
&\geq \int_{\mathbb{T}^2} \big(d_1 u_0(t-\tau_1, x) + \cdots + d_n u_0(t-\tau_n, x)\big)\,\mathrm{d}t\,\mathrm{d}x \\
&= (d_1 + \cdots + d_n) \int_{\mathbb{T}^2} u_0(t,x)\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

From this it follows that

$$\begin{aligned}
\overline{a} \int_{\mathbb{T}^2} u_0(t,x)\,\mathrm{d}t\,\mathrm{d}x &\geq \int_{\mathbb{T}^2} a(t,x) u_0(t,x)\,\mathrm{d}t\,\mathrm{d}x \\
&\geq (d_1 + \cdots + d_n) \int_{\mathbb{T}^2} u_0(t,x)\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned} \tag{3.14}$$

Since $\int_{\mathbb{T}^2} u_0(t,x)\,\mathrm{d}t\,\mathrm{d}x \geq \sigma \|u_0\| \cdot 4\pi^2 > 0$, from inequality (3.14) it follows that $\overline{a} \geq d_1 + \cdots + d_n$, which contradicts the assumption in (F3). Hence $A$ satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_1$. By Lemma 2.4 we have

$$i(A, K \cap \Omega_1, K) = 0. \tag{3.15}$$

Choosing $R > \max\{H/\sigma, \delta\}$, we show that $A$ satisfies the condition of Lemma 2.3 in $K \cap \partial\Omega_2$, namely $\lambda Au \neq u$ for every $u \in K \cap \partial\Omega_2$ and $0 < \lambda \leq 1$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $0 < \lambda_1 \leq 1$ such that $\lambda_1 A u_1 = u_1$, since $u_1 = P(\lambda_1 f(t,x,u_1(t-\tau_1,x),\ldots,u_1(t-\tau_n,x)))$, by the definition of $P$ and Lemma 2.1, $u_1 \in C(\mathbb{T}^2)$ satisfies the differential equation

$$\mathcal{L}u_1(t,x) = \lambda_1 f\big(t,x,u_1(t-\tau_1,x),\ldots,u_1(t-\tau_n,x)\big), \quad \text{in } \mathcal{D}'\big(\mathbb{T}^2\big). \tag{3.16}$$

In the definition of the solution of Equation (3.16) in the distribution sense, choosing $\phi(t,x) \equiv 1 \in \mathcal{D}(\mathbb{T}^2)$, we have

$$\int_{\mathbb{T}^2} a(t,x)u_1(t,x)\,\mathrm{d}t\,\mathrm{d}x = \lambda_1 \int_{\mathbb{T}^2} f\big(t,x,u_1(t-\tau_1,x),\ldots,u_1(t-\tau_n,x)\big)\,\mathrm{d}t\,\mathrm{d}x. \tag{3.17}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of $K$, $u_1$ satisfies (3.9). From (3.9) and condition (F4) it follows that

$$f\big(t,x,u_1(t-\tau_1,x),\ldots,u_1(t-\tau_n,x)\big) \leq c_1 u_1(t-\tau_1,x) + \cdots + c_n u_1(t-\tau_n,x)$$

for every $t, x \in \mathbb{R}$. By this inequality and (3.17), we have

$$\begin{aligned}
\int_{\mathbb{T}^2} a(t,x)u_1(t,x)\,\mathrm{d}t\,\mathrm{d}x &= \lambda_1 \int_{\mathbb{T}^2} f\big(t,x,u_1(t-\tau_1,x),\ldots,u_1(t-\tau_n,x)\big)\,\mathrm{d}t\,\mathrm{d}x \\
&\leq \int_{\mathbb{T}^2} f\big(t,x,u_1(t-\tau_1,x),\ldots,u_1(t-\tau_n,x)\big)\,\mathrm{d}t\,\mathrm{d}x \\
&\leq \int_{\mathbb{T}^2} \big(c_1 u_1(t-\tau_1,x) + \cdots + c_n u_1(t-\tau_n,x)\big)\,\mathrm{d}t\,\mathrm{d}x \\
&= (c_1 + \cdots + c_n) \int_{\mathbb{T}^2} u_1(t,x)\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

This means that

$$\begin{aligned}
\underline{a} \int_{\mathbb{T}^2} u_1(t,x)\,\mathrm{d}t\,\mathrm{d}x &\leq \int_{\mathbb{T}^2} a(t,x)u_1(t,x)\,\mathrm{d}t\,\mathrm{d}x \\
&\leq (c_1 + \cdots + c_n) \int_{\mathbb{T}^2} u_1(t,x)\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned} \tag{3.18}$$

Since $\int_{\mathbb{T}^2} u_1(t,x)\,\mathrm{d}t\,\mathrm{d}x \geq \sigma \|u_0\| \cdot 4\pi^2 > 0$, from inequality (3.18) it follows that $\underline{a} \leq c_1 + \cdots + c_n$, which contradicts the assumption in condition (F4). Hence $A$ satisfies the condition of Lemma 2.3 in $K \cap \partial\Omega_1$. By Lemma 2.3 we have

$$i(A, K \cap \Omega_2, K) = 1. \tag{3.19}$$

Now, from (3.15) and (3.19) it follows that

$$i\big(A, K \cap (\Omega_2 \setminus \overline{\Omega}_1), K\big) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = 1.$$

Hence $A$ has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive doubly periodic solution of Equation (1.1). □

**Example 3.1** Consider the existence of doubly $2\pi$-periodic solution of the following telegraph equation with time delay:

$$u_{tt} - u_{xx} + 2u_t + u = b_1(t,x)u^2(t,x) + b_2(t,x)u^2(t-\pi,x), \tag{3.20}$$

where $b_i \in C(\mathbb{T}^2)$ and $b_i(t,x) > 0$ for $(t,x) \in \mathbb{T}^2$, $i = 1, 2$. Corresponding to the general equation (1.1),

$$\begin{aligned} &c = 2, \qquad a(t,x) \equiv 1, \qquad n = 2, \qquad \tau_1 = 0, \qquad \tau_2 = \pi, \\ &f(t,x,\eta_1,\eta_2) = b_1(t,x)\eta_1^2 + b_2(t,x)\eta_2^2. \end{aligned} \tag{3.21}$$

We easily see that assumptions (H1) and (H2) hold. From (3.21) we can directly verify that $f$ satisfies conditions (F1) and (F2). By Theorem 1.1, Equation (3.20) has at least one positive doubly $2\pi$-periodic weak solution (in the distribution sense).

**Example 3.2** Consider the following nonlinear telegraph equation with time delays

$$u_{tt} - u_{xx} + 4u_t + \big(2 + \sin(t+x)\big)u = u^{2/3}(t-\pi/2,x) + u^{1/3}(t-\pi,x). \tag{3.22}$$

Corresponding to Equation (1.1),

$$\begin{aligned} &c = 4, \qquad a(t,x) = 2 + \sin(t+x), \qquad n = 2, \qquad \tau_1 = \pi/2, \qquad \tau_2 = \pi, \\ &f(t,x,\eta_1,\eta_2) = \eta_1^{2/3} + \eta_2^{1/3}. \end{aligned}$$

For these, we can directly verify that the conditions of Theorem 1.2 are satisfied. By Theorem 1.2, Equation (3.22) has a positive doubly $2\pi$-periodic weak solution (in the distribution sense).

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
YL carried out the main part of this article. All authors read and approved the final version of the manuscript.

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