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Existence of solutions to fourth-order differential equations with deviating arguments

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Abstract

In this paper, we consider fourth-order differential equations on a half-line with deviating arguments of the form $u^{(4)}(t) + q(t)f(t, [u(t)], [u'(t)], [u''(t)], u'''(t)) = 0$, $0 < t < +\infty$, with the boundary conditions $u(0) = A$, $u'(0) = B$, $u''(t) - au'''(t) = \theta(t)$, $-\tau \leq t \leq 0$; $u'''(+\infty) = C$. We present sufficient conditions for the existence of a solution between a pair of lower and upper solutions by using Schäuder's fixed point theorem. Also, we establish the existence of three solutions between two pairs of lower and upper solutions by using topological degree theory. An important feature of our existence criteria is that the obtained solutions may be unbounded. We illustrate the importance of our results through two simple examples.

MSC: 34B15; 34B40

Keywords: fourth-order; boundary value problem; half-line; upper solution; lower solution

1 Introduction

In recent years considerable attention has been focused on the existence of solutions to boundary value problems involving differential equations with deviating arguments (DEDA) [1–16]. While most of these works deal with problems on finite intervals and the literature is satisfactory, study of infinite interval problems has been just initiated in [2, 13–17]. This study compare to boundary value problems for second and higher order ordinary differential equations over infinite intervals (and their wide variety of applications to real world problems) [18–24] is far from complete, and needs attention. To fill some of this gap, in this paper we shall provide existence criteria for fourth-order differential equations with deviating arguments of the form

$$u^{(4)}(t) + q(t)f\bigl(t, \bigl[u(t)\bigr], \bigl[u'(t)\bigr], \bigl[u''(t)\bigr], u'''(t)\bigr) = 0, \quad 0 < t < +\infty, \tag{1.1}$$

where

$$\bigl[u(t)\bigr] = \bigl(u(t), u\bigl(t - \tau_{0,1}(t)\bigr), \ldots, u\bigl(t - \tau_{0,n}(t)\bigr)\bigr),$$
$$\bigl[u'(t)\bigr] = \bigl(u'(t), u'\bigl(t - \tau_{1,1}(t)\bigr), \ldots, u'\bigl(t - \tau_{1,n}(t)\bigr)\bigr),$$
$$\bigl[u''(t)\bigr] = \bigl(u''(t), u''\bigl(t - \tau_{2,1}(t)\bigr), \ldots, u''\bigl(t - \tau_{2,n}(t)\bigr)\bigr),$$

and $q:(0,+\infty)\to(0,+\infty)$, $f:[0,+\infty)\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\times\mathbb{R}\to\mathbb{R}$ are continuous, and $\tau_{j,i}$: $[0,+\infty)\to(0,+\infty)$ are continuous for all $j=0,1,2$, $i=1,2,\dots,n$. In what follows we shall always assume that $\lim_{t\to+\infty}(t-\tau_{j,i}(t))=+\infty$, $j=0,1,2$, $i=1,2,\dots,n$. We define the positive real number τ as

$$\tau=-\min_{0\le j\le 2,1\le i\le n}\min_{t\ge 0}\bigl(t-\tau_{j,i}(t)\bigr)$$

and seek the solutions of (1.1) which satisfy the boundary conditions

$$\begin{cases} u(0)=A, \qquad u'(0)=B, \qquad u''(t)-au'''(t)=\theta(t), \quad -\tau\le t\le 0;\\ u'''(+\infty)=C, \end{cases} \tag{1.2}$$

where $\theta\in\mathcal{C}[-\tau,0]$, $A,B\in\mathbb{R}$, $a,C\ge 0$, and $u'''(+\infty)=\lim_{t\to+\infty}u'''(t)$. For this, following as in several above works, and inspired by the contributions in [25–30], we shall employ the method of upper and lower solutions.

The plan of this paper is as follows: In Section 2, we state some definitions and lemmas which are needed to prove the main results. In Section 3, we show that in the presence of a pair of upper and lower solutions the problem (1.1)-(1.2) has at least one solution. Also in this section, we establish that in the presence of two pairs of upper and lower solutions the problem (1.1)-(1.2) has at least three solutions. Finally, in Section 4, we illustrate two examples which show the importance of our results.

2 Preliminaries

In this section we introduce some necessary definitions, lemmas, and preliminary results that will be used in main results which give the existence of solutions of the problem (1.1)-(1.2). First, we construct the Green's function for the linear boundary value problem

$$\begin{cases} u^{(4)}(t)+e(t)=0, & 0<t<+\infty;\\ u(0)=A, \qquad u'(0)=B, \qquad u''(t)-au'''(t)=\theta(t), & -\tau\le t\le 0;\\ u'''(+\infty)=C. \end{cases} \tag{2.1}$$

Lemma 2.1 *Let $e\in L^1[0,+\infty)$. Then the solution $u\in\mathcal{C}^3[-\tau,+\infty)\cap\mathcal{C}^4(0,+\infty)$ of the problem* (2.1) *can be expressed as*

$$u(t)=\begin{cases} \phi(t), & -\tau\le t\le 0;\\ A+Bt+(aC+\theta(0))\frac{t^2}{2}+C\frac{t^3}{3!}+\int_0^\infty G(t,s)e(s)\,ds, & 0\le t<+\infty, \end{cases} \tag{2.2}$$

where

$$G(t,s)=\begin{cases} \frac{a}{2}t^2+\frac{st^2}{2}-\frac{s^2t}{2}+\frac{s^3}{3!}, & 0\le s\le t<+\infty;\\ \frac{a}{2}t^2+\frac{t^3}{3!}, & 0\le t\le s<+\infty, \end{cases} \tag{2.3}$$

and

$$\phi(t)=A+Bt+\left(\theta(0)+aC+a\int_0^\infty e(s)\,ds\right)\bigl(-at-a^2+a^2e^{\frac{t}{a}}\bigr) +\int_t^0\bigl(s-a-t+ae^{\frac{t-s}{a}}\bigr)\theta(s)\,ds.$$

Proof Since $e \in L^1[0,+\infty)$, we can integrate (2.1) from t to $+\infty$, and use $u'''(+\infty) = C$, to get

$$u'''(t) = C + \int_t^{\infty} e(s)\,ds, \quad t \geq 0.$$

Integrating the above equation on $[0,t]$, and applying Fubini's theorem and using $u''(0) - au'''(0) = \theta(0)$, we obtain

$$u''(t) = aC + a\int_0^{\infty} e(s)\,ds + \theta(0) + Ct + \int_0^t se(s)\,ds + \int_t^{\infty} te(s)\,ds. \tag{2.4}$$

Again integrating (2.4) twice on $[0,t]$, and applying Fubini's theorem and using $u(0) = A$ and $u'(0) = B$, we find

$$\begin{aligned} u(t) &= A + Bt + \bigl(aC + \theta(0)\bigr)\frac{t^2}{2} + C\frac{t^3}{3!} + \int_0^t \left(\frac{a}{2}t^2 + \frac{st^2}{2} - \frac{s^2t}{2} + \frac{s^3}{3!}\right)e(s)\,ds \\ &\quad + \int_t^{\infty} \left(\frac{a}{2}t^2 + \frac{t^3}{3!}\right)e(s)\,ds, \end{aligned}$$

for $t \in [0,+\infty)$. Now we consider the following third order linear differential equation:

$$u''(t) - au'''(t) = \theta(t), \quad t \in [-\tau, 0].$$

If the above equation is rearranged, we have

$$u'''(t) - \frac{1}{a}u''(t) = -\frac{1}{a}\theta(t), \quad t \in [-\tau, 0],$$

and solving this linear equation on $[t,0]$, we find

$$u''(t) = u''(0)e^{\frac{t}{a}} + \frac{1}{a}\int_t^0 e^{\frac{t-s}{a}}\theta(s)\,ds. \tag{2.5}$$

Next, integrating (2.5) twice on $[t,0]$, and applying Fubini's theorem and using the following boundary conditions:

$$u(0) = A, \qquad u'(0) = B \quad \text{and} \quad u''(0) = \theta(0) + au'''(0) = \theta(0) + aC + a\int_0^{\infty} e(s)\,ds,$$

we obtain

$$\begin{aligned} u(t) &= A + Bt + \left(\theta(0) + aC + a\int_0^{\infty} e(s)\,ds\right)\bigl(-at - a^2 + a^2e^{\frac{t}{a}}\bigr) \\ &\quad + \int_t^0 \bigl(s - a - t + ae^{\frac{t-s}{a}}\bigr)\theta(s)\,ds \end{aligned}$$

for $t \in [-\tau, 0]$. This completes the proof of the lemma. □

Remark 2.2 $G(t,s)$ defined in (2.3) is the Green's function of the BVP

$$\begin{cases} -u^{(4)}(t) = 0, & 0 < t < +\infty; \\ u(0) = u'(0) = 0, \quad u''(0) = au'''(0), & u'''(+\infty) = 0. \end{cases}$$

Lemma 2.3 *The Green's function $G(t,s)$ has the following properties*:

(1) *$G(t,s)$ is twice continuously differentiable on* $[0,+\infty) \times [0,+\infty)$ *and*

$$\frac{\partial^3 G(t,s)}{\partial t^3}\bigg|_{t=s^+} - \frac{\partial^3 G(t,s)}{\partial t^3}\bigg|_{t=s^-} = -1;$$

(2) $\frac{\partial^i G(t,s)}{\partial t^i} \geq 0$, $\forall (t,s) \in [0,+\infty) \times [0,+\infty)$, *for* $i = 0,1,2,3$;

(3) $\sup_{t\in[0,+\infty)} \frac{G(t,s)}{1+t^3} \leq (\frac{a\sqrt[3]{4}+1}{6})$, $\sup_{t\in[0,+\infty)}(\frac{1}{1+t^2}\frac{\partial G(t,s)}{\partial t}) \leq (\frac{a+1}{2})$, $\sup_{t\in[0,+\infty)}(\frac{1}{1+t}\frac{\partial^2 G(t,s)}{\partial t^2}) \leq (a+1)$, $\sup_{t\in[0,+\infty)} \frac{\partial^3 G(t,s)}{\partial t^3} \leq 1$.

Proof (1) and (2) are obvious. Here we shall prove the first inequality of (3). We note that for all integers k and l

$$\sup_{t\in[0,+\infty)} \frac{t^k}{1+t^l} = \begin{cases} \frac{l-k}{l}(\frac{k}{l-k})^{\frac{k}{l}}, & k < l; \\ 1, & k = l; \\ +\infty, & k > l. \end{cases}$$

For $s \leq t$, we have

$$\begin{aligned} \sup_{t\in[0,+\infty)} \frac{G(t,s)}{1+t^3} &= \sup_{t\in[0,+\infty)} \left(\frac{\frac{a}{2}t^2 + \frac{st^2}{2} - \frac{s^2t}{2} + \frac{s^3}{6}}{1+t^3} \right) \leq \sup_{t\in[0,+\infty)} \left(\frac{\frac{at^2}{2}}{1+t^3} + \frac{\frac{t^3}{6}}{1+t^3} \right) \\ &\leq \frac{a}{2} \sup_{t\in[0,+\infty)} \frac{t^2}{1+t^3} + \frac{1}{6} \sup_{t\in[0,+\infty)} \frac{t^3}{1+t^3} \leq \frac{a\sqrt[3]{4}+1}{6} \end{aligned}$$

and for $s \geq t$

$$\begin{aligned} \sup_{t\in[0,+\infty)} \frac{G(t,s)}{1+t^3} &= \sup_{t\in[0,+\infty)} \left(\frac{\frac{a}{2}t^2 + \frac{t^3}{6}}{1+t^3} \right) \leq \sup_{t\in[0,+\infty)} \left(\frac{\frac{at^2}{2}}{1+t^3} + \frac{\frac{t^3}{6}}{1+t^3} \right) \\ &\leq \frac{a}{2} \sup_{t\in[0,+\infty)} \frac{t^2}{1+t^3} + \frac{1}{6} \sup_{t\in[0,+\infty)} \frac{t^3}{1+t^3} \leq \frac{a\sqrt[3]{4}+1}{6}. \end{aligned}$$

The other parts can be proved similarly. □

We consider the space X defined by

$$\begin{aligned} X = \Bigg\{ u \in C^3[-\tau,+\infty) : &\sup_{t\in[0,+\infty)} \frac{|u(t)|}{1+t^3} < +\infty, \sup_{t\in[0,+\infty)} \frac{|u'(t)|}{1+t^2} < +\infty, \\ &\sup_{t\in[0,+\infty)} \frac{|u''(t)|}{1+t} < +\infty, \lim_{t\to+\infty} u'''(t) \text{ exists} \Bigg\} \end{aligned}$$

with the norm

$$\|u\| = \max\big\{ \|u\|_0, \|u\|_1, \|u\|_2, \|u\|_\infty^0, \|u\|_\infty^1, \|u\|_\infty^2, \|u\|_\infty^3 \big\},$$

where

$$\|u\|_0 = \max_{t\in[-\tau,0]} \bigl|u(t)\bigr|, \qquad \|u\|_\infty^0 = \sup_{t\in[0,+\infty)} \frac{|u(t)|}{1+t^3},$$

$$\|u\|_1 = \max_{t\in[-\tau,0]} \bigl|u'(t)\bigr|, \qquad \|u\|_\infty^1 = \sup_{t\in[0,+\infty)} \frac{|u'(t)|}{1+t^2},$$

$$\|u\|_2 = \max_{t\in[-\tau,0]} \bigl|u''(t)\bigr|, \qquad \|u\|_\infty^2 = \sup_{t\in[0,+\infty)} \frac{|u''(t)|}{1+t}, \qquad \|u\|_\infty^3 = \sup_{t\in[-\tau,+\infty)} \bigl|u'''(t)\bigr|.$$

It is clear that $(X, \|\cdot\|)$ is a Banach space. Next we define the mapping $T : X \to \mathcal{C}^3[-\tau,+\infty) \cap \mathcal{C}^4(0,+\infty)$ by

$$Tu(t) = \begin{cases} \psi(t), & -\tau \le t \le 0; \\ l(t) + \int_0^\infty G(t,s)q(s)f(s,[u(s)],[u'(s)],[u''(s)],u'''(s))\,ds, & 0 \le t < +\infty, \end{cases} \tag{2.6}$$

where

$$\psi(t) = A + Bt + \left(\theta(0) + aC + a\int_0^\infty q(s)f\bigl(s,\bigl[u(s)\bigr],\bigl[u'(s)\bigr],\bigl[u''(s)\bigr],u'''(s)\bigr)\,ds\right) \times \bigl(-at - a^2 + a^2 e^{\frac{t}{a}}\bigr) + \int_t^0 \bigl(s - a - t + ae^{\frac{t-s}{a}}\bigr)\theta(s)\,ds$$

and

$$l(t) = A + Bt + \bigl(aC + \theta(0)\bigr)\frac{t^2}{2} + C\frac{t^3}{3!}. \tag{2.7}$$

Lemma 2.4 *The mapping $T : X \to \mathcal{C}^3[-\tau,+\infty) \cap \mathcal{C}^4(0,+\infty)$ in* (2.6) *has the following properties*:

(1) $Tu(0) = A$, $(Tu)'(0) = B$, $(Tu)''(t) - a(Tu)'''(t) = \theta(t)$ *for* $t \in [-\tau, 0]$,
(2) $Tu(t)$ *is three-times continuously differentiable on* $t \in [-\tau,+\infty)$,
(3) $(Tu)^{(4)}(t) = -q(t)f(t,[u(t)],[u'(t)],[u''(t)],u'''(t))$, $t \in (0,+\infty)$,
(4) *fixed points of* T *are solutions of BVP* (1.1)-(1.2).

When applying the Schäuder fixed point theorem to prove the existence result, it is necessary to show that the operator T_1 (defined later) is completely continuous. For this, we need the following modified version of the Arzela-Ascoli lemma (see [18, 20]).

Lemma 2.5 *$M \subset X$ is relatively compact if the following conditions hold*:

(1) *all functions belonging to M are uniformly bounded*,
(2) *all functions belonging to M are equi-continuous on any compact sub-interval of* $[-\tau,+\infty)$,
(3) *all functions from M are equi-convergent at infinity, that is, for any* $\epsilon > 0$, *there exists a* $T = T(\epsilon) > 0$ *such that, for all* $t \ge T$ *and any* $u \in M$,

$$\left|\frac{u(t)}{1+t^3} - \lim_{t\to+\infty}\frac{u(t)}{1+t^3}\right| < \epsilon, \qquad \left|\frac{u'(t)}{1+t^2} - \lim_{t\to+\infty}\frac{u'(t)}{1+t^2}\right| < \epsilon,$$

$$\left|\frac{u''(t)}{1+t} - \lim_{t\to+\infty}\frac{u''(t)}{1+t}\right| < \epsilon \quad \textit{and} \quad \bigl|u'''(t) - \lim_{t\to+\infty}u'''(t)\bigr| < \epsilon.$$

Definition 2.6 A function $\alpha \in X \cap \mathcal{C}^4(0,+\infty)$ is called a lower solution of (1.1)-(1.2) provided

$$\alpha^{(4)}(t) + q(t)f\bigl(t, [\alpha(t)], [\alpha'(t)], [\alpha''(t)], \alpha'''(t)\bigr) \geq 0, \quad 0 < t < +\infty; \tag{2.8}$$

$$\begin{aligned} &\alpha(0) \leq A, \qquad \alpha'(0) = B, \qquad \alpha''(t) - a\alpha'''(t) \leq \theta(t), \quad -\tau \leq t \leq 0; \\ &\alpha'''(+\infty) \leq C. \end{aligned} \tag{2.9}$$

Similarly, a function $\beta \in X \cap \mathcal{C}^4(0,+\infty)$ is called an upper solution of (1.1)-(1.2) provided

$$\beta^{(4)}(t) + q(t)f\bigl(t, [\beta(t)], [\beta'(t)], [\beta''(t)], \beta'''(t)\bigr) \leq 0, \quad 0 < t < +\infty; \tag{2.10}$$

$$\begin{aligned} &\beta(0) \geq A, \qquad \beta'(0) = B, \qquad \beta''(t) - a\beta'''(t) \geq \theta(t), \quad -\tau \leq t \leq 0; \\ &\beta'''(+\infty) \geq C. \end{aligned} \tag{2.11}$$

Also, we say $\alpha(\beta)$ is a strict lower solution (strict upper solution) for problem (1.1)-(1.2) if all the above inequalities are strict.

Remark 2.7 If

$$\alpha''(t) \leq \beta''(t), \quad \text{for every } t \in [-\tau, +\infty), \tag{2.12}$$

then on integrating (2.12) and using the boundary restrictions in Definition 2.6, we find that $\alpha'(t) \leq \beta'(t)$, $\alpha(t) \leq \beta(t)$ for all $t \in [0,+\infty)$ and $\beta'(t) \leq \alpha'(t)$, $\alpha(t) \leq \beta(t)$ for all $t \in [-\tau, 0)$.

Definition 2.8 Let α, β be lower and upper solutions for the problem (1.1)-(1.2) satisfying

$$\alpha''(t) \leq \beta''(t), \quad \text{for all } t \in [-\tau, +\infty).$$

A continuous function f is said to satisfy Nagumo's condition with respect to the pair of functions α, β if there exist positive functions φ and $h \in \mathcal{C}[0,+\infty)$ such that

$$\bigl|f(t, x_0, \ldots, x_n, y_0, \ldots, y_n, z_0, \ldots, z_n, w)\bigr| \leq \varphi(t)h\bigl(|w|\bigr)$$

for all $t \in [0,+\infty)$, and $(x_0, \ldots, x_n) \in [[\alpha(t)], [\beta(t)]]$, $y_i(t - \tau_{1,i}(t)) \in [\alpha'(t - \tau_{1,i}(t)), \beta'(t - \tau_{1,i}(t))]$ if $t - \tau_{1,i}(t) > 0$, $y_i(t - \tau_{1,i}(t)) \in [\beta'(t - \tau_{1,i}(t)), \alpha'(t - \tau_{1,i}(t))]$ if $t - \tau_{1,i}(t) \leq 0$, $0 \leq i \leq n$, $\tau_{1,0} = 0$, $(z_0, \ldots, z_n) \in [[\alpha''(t)], [\beta''(t)]]$, $w \in \mathbb{R}$, and

$$\int_0^\infty q(s)\varphi(s)\,ds < +\infty, \qquad \int_0^\infty \frac{s}{h(s)}\,ds = +\infty.$$

3 Main results

In this section we state and prove our existence results. We begin with the following lemma.

Lemma 3.1 *Suppose the following conditions hold.*

(H_1) *BVP* (1.1)-(1.2) *has a pair of lower and upper solutions* α, β *satisfying*

$$\alpha''(t) \le \beta''(t), \quad \text{for } t \in [-\tau, +\infty),$$

and f satisfies Nagumo's condition with respect to the pair of functions α, β.

(H_2) *There exists a constant* $\gamma > 1$ *such that*

$$\sup_{0 \le t < +\infty} (1+t)^\gamma q(t)\varphi(t) < +\infty,$$

where φ *is the function in Nagumo's condition of f*.

Then there exists a constant $R > 0$ (*depending on* α, β, h, *and* C) *such that every solution u of* (1.1)-(1.2) *with*

$$\begin{aligned} &\alpha(t) \le u(t) \le \beta(t), \qquad \alpha'(t) \le u'(t) \le \beta'(t), \\ &\alpha''(t) \le u''(t) \le \beta''(t) \quad \text{for all } t \in [0, +\infty) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\alpha(t) \le u(t) \le \beta(t), \qquad \beta'(t) \le u'(t) \le \alpha'(t), \\ &\alpha''(t) \le u''(t) \le \beta''(t) \quad \text{for all } t \in [-\tau, 0) \end{aligned} \tag{3.2}$$

satisfies $\|u\|_\infty^3 < R$.

Proof We can choose $R > \eta$ such that

$$\eta \ge \max\left\{ \sup_{t\in[0,+\infty)} \left|\beta'''(t)\right|, \sup_{t\in[0,+\infty)} \left|\alpha'''(t)\right|, \frac{\|\beta'' - \theta\|_0}{a}, \frac{\|\alpha'' - \theta\|_0}{a}, C \right\}$$

and

$$\int_\eta^R \frac{s}{h(s)}\,ds > M\left(\sup_{t\in[0,+\infty)} \frac{\beta''(t)}{(1+t)^\gamma} - \inf_{t\in[0,+\infty)} \frac{\alpha''(t)}{(1+t)^\gamma} + \frac{\gamma N}{\gamma - 1} \right),$$

where C is the nonhomogeneous boundary value, and

$$M = \sup_{t\in[0,+\infty)} (1+t)^\gamma q(t)\varphi(t), \qquad N = \max\left\{ \|\beta\|_\infty^2, \|\alpha\|_\infty^2 \right\}.$$

Let u be a solution of the differential equation (1.1) satisfying (3.1) and (3.2). If $|u'''(t)| < R$, for all $t \in [0, +\infty)$, there is nothing to prove. If this is not true, there exists a $t_0 \in [0, +\infty)$ such that $|u'''(t_0)| \ge R$. Since $\lim_{t\to+\infty} u'''(t) = C < R$, there exists a $T > 0$ such that

$$\left|u'''(t)\right| < R \quad \text{for all } t \ge T.$$

Let $t_1 = \inf\{t \le T : |u'''(s)| < R \text{ for all } s \in [t, +\infty)\}$. Then $|u'''(t_1)| = R$ and $|u'''(t)| < R$ for all $t > t_1$ and there exists a t_2 such that $|u'''(t)| \ge R$ for $t \in [t_2, t_1]$. So we have two cases to consider $u'''(t_1) = R$ and $u'''(t) \geq R$ for $t \in [t_2, t_1]$, or $u'''(t_1) = -R$ and $u'''(t) \leq -R$ for $t \in [t_2, t_1]$. We assume that $u'''(t_1) = R$ and $u'''(t) \geq R$ for $t \in [t_2, t_1]$, then we have

$$\begin{aligned}
\int_\eta^R \frac{s}{h(s)}\,ds &\leq \int_C^R \frac{s}{h(s)}\,ds \\
&= -\int_{t_1}^\infty \frac{u'''(s)u^{(4)}(s)}{h(u'''(s))}\,ds \\
&= -\int_{t_1}^\infty \frac{-q(s)f(s,[u(s)],[u'(s)],[u''(s)],u'''(s))u'''(s)}{h(u'''(s))}\,ds \\
&\leq \int_{t_1}^\infty q(s)\varphi(s)u'''(s)\,ds \\
&\leq M\int_{t_1}^\infty \frac{u'''(s)}{(1+s)^\gamma}\,ds \\
&= M\left(\int_{t_1}^\infty \left(\frac{u''(s)}{(1+s)^\gamma}\right)'ds + \int_{t_1}^\infty \frac{u''(s)}{1+s}\cdot\frac{\gamma}{(1+s)^\gamma}\,ds\right) \\
&\leq M\left(\sup_{t\in[0,+\infty)} \frac{\beta''(t)}{(1+t)^\gamma} - \inf_{t\in[0,+\infty)} \frac{\alpha''(t)}{(1+t)^\gamma} + \frac{\gamma N}{\gamma-1}\right) \\
&< \int_\eta^R \frac{s}{h(s)}\,ds,
\end{aligned}$$

which is a contradiction. In the case $u'''(t_1) = -R$ and $u'''(t) \leq -R$ for $t \in [t_2, t_1]$, we obtain a similar contradiction. Thus, $|u'''(t)| < R$ for all $t \in [0, +\infty)$. From the boundary condition (1.2) we also have

$$-R < -\eta \leq \frac{\alpha''(t)-\theta(t)}{a} \leq u'''(t) = \frac{u''(t)-\theta(t)}{a} \leq \frac{\beta''(t)-\theta(t)}{a} \leq \eta < R$$

for all $t \in [-\tau, 0]$. Therefore, $|u'''(t)| < R$ for $t \in [-\tau, 0)$. To sum up, we have $\|u\|_\infty^3 < R$. □

Theorem 3.2 *Suppose conditions* (H_1) *and* (H_2) *hold. Suppose further that*

(H_3) *For any fixed* $t \in [0, +\infty)$, $y_i, z_i, w \in \mathbb{R}$, $i = 0, \ldots, n$, *when*

$$\begin{aligned}
&\alpha\big(t-\tau_{0,i}(t)\big) \leq x_i \leq \beta\big(t-\tau_{0,i}(t)\big), \quad i = 0,1,\ldots,n, \\
&f\big(t, x_0, x_1, \ldots, \alpha\big(t-\tau_{0,i}(t)\big), \ldots, x_n, y_0, \ldots, y_n, z_0, \ldots, z_n, w\big) \\
&\quad \leq f(t, x_0, x_1, \ldots, x_i, \ldots, x_n, y_0, \ldots, y_n, z_0, \ldots, z_n, w) \\
&\quad \leq f\big(t, x_0, x_1, \ldots, \beta\big(t-\tau_{0,i}(t)\big), \ldots, x_n, y_0, \ldots, y_n, z_0, \ldots, z_n, w\big).
\end{aligned}$$

(H_4) *For any fixed* $t \in [0, +\infty)$, $x_i, z_i, w \in \mathbb{R}$, $i = 0, \ldots, n$ *when*

$$\alpha'\big(t-\tau_{1,i}(t)\big) \leq y_i \leq \beta'\big(t-\tau_{1,i}(t)\big), \quad t-\tau_{1,i}(t) > 0,$$

or when

$$\beta'\big(t-\tau_{1,i}(t)\big) \leq y_i \leq \alpha'\big(t-\tau_{1,i}(t)\big), \quad t-\tau_{1,i}(t) \leq 0, i = 0,1,\ldots,n,$$

$$
\begin{aligned}
&f\big(t,x_0,\ldots,x_n,y_0,\ldots,\alpha'\big(t-\tau_{1,i}(t)\big),\ldots,y_n,z_0,\ldots,z_n,w\big)\\
&\quad\leq f(t,x_0,\ldots,x_n,y_0,\ldots,y_i,\ldots,y_n,z_0,\ldots,z_n,w)\\
&\quad\leq f\big(t,x_0,\ldots,x_n,y_0,\ldots,\beta'\big(t-\tau_{1,i}(t)\big),\ldots,y_n,z_0,\ldots,z_n,w\big).
\end{aligned}
$$

(H_5) *For any fixed* $t\in[0,+\infty)$, $x_i,y_i,w\in\mathbb{R}$, $i=0,\ldots,n$ *when*

$$
\begin{aligned}
&\alpha''\big(t-\tau_{2,i}(t)\big)\leq z_i\leq\beta''\big(t-\tau_{2,i}(t)\big),\quad i=0,1,\ldots,n,\\
&f\big(t,x_0,\ldots,x_n,y_0,\ldots,y_n,z_0,\ldots,\alpha''\big(t-\tau_{2,i}(t)\big),\ldots,z_n,w\big)\\
&\quad\leq f(t,x_0,\ldots,x_n,y_0,\ldots,y_n,z_0,\ldots,z_i,\ldots,z_n,w)\\
&\quad\leq f\big(t,x_0,\ldots,x_n,y_0,\ldots,y_n,z_0,\ldots,\beta''\big(t-\tau_{2,i}(t)\big),\ldots,z_n,w\big),
\end{aligned}
$$

where $\tau_{0,0}=\tau_{1,0}=\tau_{2,0}=0$.

(H_6)

$$
\int_0^\infty\max\{s,1\}q(s)\,ds<+\infty,\qquad\int_0^\infty\max\{s,1\}q(s)\varphi(s)\,ds<+\infty.
$$

Then BVP (1.1)-(1.2) *has at least one solution* $u\in X\cap\mathcal{C}^4(0,+\infty)$ *satisfying* (3.1)-(3.2) *and* $\|u\|_\infty^3<R$.

Proof Let R be a positive number as in Lemma 3.1 and define the auxiliary functions,

$$
F_0,F_1,F_2,F_3:[0,+\infty)\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\times\mathbb{R}\to\mathbb{R}
$$

as follows:

$$
\begin{aligned}
&F_0(t,x_0,\ldots,x_n,y_0,\ldots,y_n,z_0,\ldots,z_n,w)\\
&\quad=\begin{cases}f(t,\beta,\widetilde{x}_1,\ldots,\widetilde{x}_n,y_0,\ldots,y_n,z_0,\ldots,z_n,w), & x_0>\beta(t),\\ f(t,x_0,\widetilde{x}_1,\ldots,\widetilde{x}_n,y_0,\ldots,y_n,z_0,\ldots,z_n,w), & \alpha(t)\leq x_0\leq\beta(t),\\ f(t,\alpha,\widetilde{x}_1,\ldots,\widetilde{x}_n,y_0,\ldots,y_n,z_0,\ldots,z_n,w), & x_0<\alpha(t),\end{cases}\\
&F_1(t,x_0,\ldots,x_n,y_0,\ldots,y_n,z_0,\ldots,z_n,w)\\
&\quad=\begin{cases}F_0(t,x_0,\ldots,x_n,\beta',\widetilde{y}_1,\ldots,\widetilde{y}_n,z_0,\ldots,z_n,w), & y_0>\beta'(t),\\ F_0(t,x_0,\ldots,x_n,y_0,\widetilde{y}_1,\ldots,\widetilde{y}_n,z_0,\ldots,z_n,w), & \alpha'(t)\leq y_0\leq\beta'(t),\\ F_0(t,x_0,\ldots,x_n,\alpha',\widetilde{y}_1,\ldots,\widetilde{y}_n,z_0,\ldots,z_n,w), & y_0<\alpha'(t),\end{cases}\\
&F_2(t,x_0,\ldots,x_n,y_0,\ldots,y_n,z_0,\ldots,z_n,w)\\
&\quad=\begin{cases}F_1(t,x_0,\ldots,x_n,y_0,\ldots,y_n,\beta'',\widetilde{z}_1,\ldots,\widetilde{z}_n,w)-\frac{z_0-\beta''}{1+|z_0-\beta''|}, & z_0>\beta''(t),\\ F_1(t,x_0,\ldots,x_n,y_0,\ldots,y_n,z_0,\widetilde{z}_1,\ldots,\widetilde{z}_n,w), & \alpha''(t)\leq z_0\leq\beta''(t),\\ F_1(t,x_0,\ldots,x_n,y_0,\ldots,y_n,\alpha'',\widetilde{z}_1,\ldots,\widetilde{z}_n,w)+\frac{z_0-\alpha''}{1+|z_0-\alpha''|}, & z_0<\alpha''(t),\end{cases}\\
&F_3(t,x_0,x_1,\ldots,x_n,y_0,y_1,\ldots,y_n,z_0,\ldots,z_n,w)\\
&\quad=\begin{cases}F_2(t,x_0,\ldots,x_n,y_0,\ldots,y_n,z_0,\ldots,z_n,R), & w>R,\\ F_2(t,x_0,\ldots,x_n,y_0,\ldots,y_n,z_0,\ldots,z_n,w), & -R\leq w\leq R,\\ F_2(t,x_0,\ldots,x_n,y_0,\ldots,y_n,z_0,\ldots,z_n,-R), & w<R,\end{cases}
\end{aligned}
$$

where, for $i = 1, 2, \dots, n$,

$$\widetilde{x}_i = \begin{cases} \beta, & x_i > \beta(t - \tau_{0,i}(t)); \\ x_i, & \alpha(t - \tau_{0,i}(t)) \le x_i \le \beta(t - \tau_{0,i}(t)); \\ \alpha, & x_i < \alpha(t - \tau_{0,i}(t)), \end{cases}$$

if $t - \tau_{1,i}(t) > 0$,

$$\widetilde{y}_i = \begin{cases} \beta', & y_i > \beta'(t - \tau_{1,i}(t)); \\ y_i, & \alpha'(t - \tau_{1,i}(t)) \le y_i \le \beta'(t - \tau_{1,i}(t)); \\ \alpha', & y_i < \alpha'(t - \tau_{1,i}(t)), \end{cases}$$

if $t - \tau_{1,i}(t) \le 0$,

$$\widetilde{y}_i = \begin{cases} \alpha', & y_i > \alpha'(t - \tau_{1,i}(t)); \\ y_i, & \beta'(t - \tau_{1,i}(t)) \le y_i \le \alpha'(t - \tau_{1,i}(t)); \\ \beta', & y_i < \beta'(t - \tau_{1,i}(t)), \end{cases}$$

and

$$\widetilde{z}_i = \begin{cases} \beta'', & z_i > \beta''(t - \tau_{2,i}(t)); \\ z_i, & \alpha''(t - \tau_{2,i}(t)) \le z_i \le \beta''(t - \tau_{2,i}(t)); \\ \alpha'', & z_i < \alpha''(t - \tau_{2,i}(t)). \end{cases}$$

We consider the modified boundary value problem

$$u^{(4)}(t) + q(t)F_3\bigl(t, [u(t)], [u'(t)], [u''(t)], u'''(t)\bigr) = 0, \quad 0 < t < +\infty, \tag{3.3}$$

with the boundary conditions (1.2). We will show that the problem (3.3)-(1.2) has at least one solution u in X. Now for $u \in X$, we define two operators $\widetilde{T}_1$, T_1 by

$$\widetilde{T}_1 u(t) = \int_0^\infty G(t,s)q(s)F_3\bigl(s, [u(s)], [u'(s)], [u''(s)], u'''(s)\bigr)\, ds$$

and

$$T_1 u(t) = \begin{cases} \psi_1(t), & -\tau \le t \le 0; \\ l(t) + \widetilde{T}_1 u(t), & 0 \le t < +\infty, \end{cases} \tag{3.4}$$

where $l(t)$ is as in (2.7) and

$$\begin{aligned} \psi_1(t) = A + Bt + \biggl(\theta(0) + aC + a\int_0^\infty q(s)F_3\bigl(s, [u(s)], [u'(s)], [u''(s)], u'''(s)\bigr)\, ds\biggr) \\ \times \bigl(-at - a^2 + a^2 e^{\frac{t}{a}}\bigr) + \int_t^0 \bigl(s - a - t + ae^{\frac{t-s}{a}}\bigr)\theta(s)\, ds. \end{aligned}$$

We want to show that the operator T_1 is completely continuous. We split the proof in the following parts:

(1) $T_1 : X \to X$ is well defined. Obviously, for any $u \in X$ by direct calculation, we have

$$(T_1u)''(t) - a(T_1u)'''(t) = \theta(t) \quad \text{for } t \in [-\tau, 0] \quad \text{and}$$
$$(T_1u)'(0) = B, \qquad (T_1u)(0) = A$$

and for $t \in (0, +\infty)$,

$$(T_1u)'(t) = l'(t) + \int_0^\infty \frac{\partial G(t,s)}{\partial t} q(s)F_3\bigl(s, [u(s)], [u'(s)], [u''(s)], u'''(s)\bigr)\,ds,$$
$$(T_1u)''(t) = l''(t) + \int_0^\infty \frac{\partial^2 G(t,s)}{\partial t^2} q(s)F_3\bigl(s, [u(s)], [u'(s)], [u''(s)], u'''(s)\bigr)\,ds,$$
$$(T_1u)'''(t) = C + \int_t^\infty q(s)F_3\bigl(s, [u(s)], [u'(s)], [u''(s)], u'''(s)\bigr)\,ds,$$

which show that $T_1u(t) \in C^3[-\tau, +\infty)$. Further, we have

$$\left|\int_0^\infty q(s)F_3\bigl(s, [u(s)], [u'(s)], [u''(s)], u'''(s)\bigr)\,ds\right| \leq \int_0^\infty \max\{s, 1\}q(s)\bigl(H\varphi(s) + 1\bigr)\,ds < +\infty, \tag{3.5}$$

where $H = \max_{0 \leq s \leq \sup_{t\in[0,+\infty)} |u'''(t)|} h(s)$. Now from (3.5) it follows that

$$\int_1^\infty sq(s)\bigl(H\varphi(s) + 1\bigr)\,ds \leq \int_0^\infty \max\{s, 1\}q(s)\bigl(H\varphi(s) + 1\bigr)\,ds < +\infty,$$

which implies

$$\lim_{t\to+\infty} tq(t)\bigl(H\varphi(t) + 1\bigr) = 0. \tag{3.6}$$

Next since

$$\int_t^\infty q(s)\bigl(H\varphi(s) + 1\bigr)\,ds \leq \int_t^\infty sq(s)\bigl(H\varphi(s) + 1\bigr)\,ds < +\infty, \quad t \geq 1,$$

we also have

$$\lim_{t\to+\infty} \int_t^\infty q(s)\bigl(H\varphi(s) + 1\bigr)\,ds = 0. \tag{3.7}$$

By Lebesgue's dominated convergent theorem, L'Hopital's rule, and (3.6), (3.7), we obtain

$$\left|\lim_{t\to+\infty} \frac{(\widetilde{T_1}u)(t)}{1 + t^3}\right| \leq \lim_{t\to+\infty} \int_0^\infty \frac{|G(t,s)|}{1 + t^3} q(s)\bigl|F_3\bigl(s, [u(s)], [u'(s)], [u''(s)], u'''(s)\bigr)\bigr|\,ds$$
$$\leq \lim_{t\to+\infty} \left(\int_0^\infty \frac{|G(t,s)|}{1 + t^3} q(s)\bigl(H\varphi(s) + 1\bigr)\,ds\right)$$

$$
\begin{aligned}
&= \lim_{t\to+\infty}\biggl[\int_0^t \frac{(\frac{a}{2}t^2 + \frac{st^2}{2} - \frac{s^2t}{2} + \frac{s^3}{6})}{1+t^3} q(s)\bigl(H\varphi(s)+1\bigr)\,ds \\
&\quad + \int_t^\infty \frac{(\frac{a}{2}t^2 + \frac{t^3}{6})}{1+t^3} q(s)\bigl(H\varphi(s)+1\bigr)\,ds\biggr] \\
&= \lim_{t\to+\infty}\biggl[\int_0^t \frac{(at + st - \frac{s^2}{2})}{3t^2} q(s)\bigl(H\varphi(s)+1\bigr)\,ds + \frac{(\frac{a}{2}t^2 + \frac{t^3}{6})}{3t^2} q(t)\bigl(H\varphi(t)+1\bigr)\biggr] \\
&\quad + \lim_{t\to+\infty}\biggl[\int_t^\infty \frac{(at + \frac{t^2}{2})}{3t^2} q(s)\bigl(H\varphi(s)+1\bigr)\,ds - \frac{(\frac{a}{2}t^2 + \frac{t^3}{6})}{3t^2} q(t)\bigl(H\varphi(t)+1\bigr)\biggr] \\
&= \lim_{t\to+\infty}\int_0^t \frac{(at + st - \frac{s^2}{2})}{3t^2} q(s)\bigl(H\varphi(s)+1\bigr)\,ds + \lim_{t\to+\infty}\frac{(\frac{a}{2}t + \frac{t^2}{6})}{3t^2} tq(t)\bigl(H\varphi(t)+1\bigr) \\
&\quad + \lim_{t\to+\infty}\int_t^\infty \frac{(at + \frac{t^2}{2})}{3t^2} q(s)\bigl(H\varphi(s)+1\bigr)\,ds - \lim_{t\to+\infty}\frac{(\frac{a}{2}t + \frac{t^2}{6})}{3t^2} tq(t)\bigl(H\varphi(t)+1\bigr) \\
&= \lim_{t\to+\infty}\int_0^t \biggl[\frac{(a+s)}{6t} q(s)\bigl(H\varphi(s)+1\bigr)\,ds + \frac{(at + \frac{t^2}{2})}{6t} q(t)\bigl(H\varphi(t)+1\bigr)\biggr] \\
&\quad + \lim_{t\to+\infty}\biggl[\int_t^\infty \frac{a+t}{6t} q(s)\bigl(H\varphi(s)+1\bigr)\,ds - \frac{(at + \frac{t^2}{2})}{6t} q(t)\bigl(H\varphi(t)+1\bigr)\biggr] \\
&= \lim_{t\to+\infty}\int_0^t \frac{(a+s)}{6t} q(s)\bigl(H\varphi(s)+1\bigr)\,ds + \lim_{t\to+\infty}\frac{(a + \frac{t}{2})}{6t} tq(t)\bigl(H\varphi(t)+1\bigr) \\
&\quad + \lim_{t\to+\infty}\int_t^\infty \frac{a+t}{6t} q(s)\bigl(H\varphi(s)+1\bigr)\,ds - \lim_{t\to+\infty}\frac{(a + \frac{t}{2})}{6t} tq(t)\bigl(H\varphi(t)+1\bigr) \\
&= \lim_{t\to+\infty}\int_t^\infty q(s)\bigl(H\varphi(s)+1\bigr)\,ds - \lim_{t\to+\infty}\frac{(a+t)}{6} q(t)\bigl(H\varphi(t)+1\bigr) \\
&= \lim_{t\to+\infty}\int_t^\infty q(s)\bigl(H\varphi(s)+1\bigr)\,ds - \lim_{t\to+\infty}\frac{(a+t)}{6t} tq(t)\bigl(H\varphi(t)+1\bigr) \\
&= \frac{1}{6}\lim_{t\to+\infty}\int_t^\infty q(s)\bigl(H\varphi(s)+1\bigr)\,ds = 0,
\end{aligned}
$$

that is, $\lim_{t\to+\infty}\frac{(\widetilde{T}_1u)(t)}{1+t^3} = 0$, and

$$
\begin{aligned}
\lim_{t\to+\infty}\frac{(T_1u)(t)}{1+t^3} &= \lim_{t\to+\infty}\frac{l(t)}{1+t^3} + \lim_{t\to+\infty}\frac{(\widetilde{T}_1u)(t)}{1+t^3} \\
&= \lim_{t\to+\infty}\frac{A + Bt + (aC + \theta(0))\frac{t^2}{2} + C\frac{t^3}{3!}}{1+t^3} = \frac{C}{6},
\end{aligned}
$$

which implies that $\sup_{t\in[0,+\infty)}\frac{|(T_1u)(t)|}{1+t^3} < +\infty$. Similarly, we have

$$
\begin{aligned}
&\left|\lim_{t\to+\infty}\frac{(\widetilde{T}_1u)'(t)}{1+t^2}\right| \\
&\quad \le \lim_{t\to+\infty}\frac{1}{1+t^2}\int_0^\infty \frac{\partial G(t,s)}{\partial t} q(s)\bigl|F_3\bigl(s,[u(s)],[u'(s)],[u''(s)],u'''(s)\bigr)\bigr|\,ds \\
&\quad \le \frac{1}{2}\lim_{t\to+\infty}\int_t^\infty q(s)\bigl(H\varphi(s)+1\bigr)\,ds = 0,
\end{aligned}
$$

that is, $\lim_{t\to+\infty}\frac{(\widetilde{T}_1u)'(t)}{1+t^2}=0$ and $\lim_{t\to+\infty}\frac{(\widetilde{T}_1u)'(t)}{1+t^2}=\lim_{t\to+\infty}\frac{l'(t)}{1+t^2}+\lim_{t\to+\infty}\frac{(T_1u)'(t)}{1+t^2}=\frac{C}{2}$, which implies that $\sup_{t\in[0,+\infty)}\frac{|(T_1u)'(t)|}{1+t^2}<+\infty$,

$$\left|\lim_{t\to+\infty}\frac{(\widetilde{T}_1u)''(t)}{1+t}\right|\le\lim_{t\to+\infty}\frac{1}{1+t}\int_0^\infty\frac{\partial^2G(t,s)}{\partial t^2}q(s)\bigl|F_3\bigl(s,\bigl[u(s)\bigr],\bigl[u'(s)\bigr],\bigl[u''(s)\bigr],u'''(s)\bigr)\bigr|\,ds$$
$$\le\lim_{t\to+\infty}\int_t^\infty q(s)\bigl(H\varphi(s)+1\bigr)\,ds=0,$$

that is, $\lim_{t\to+\infty}\frac{(\widetilde{T}_1u)''(t)}{1+t}=0$ and $\lim_{t\to+\infty}\frac{(T_1u)''(t)}{1+t}=\lim_{t\to+\infty}\frac{l''(t)}{1+t}+\lim_{t\to+\infty}\frac{(\widetilde{T}_1u)''(t)}{1+t}=C$, which implies that $\sup_{t\in[0,+\infty)}\frac{|(T_1u)''(t)|}{1+t}<+\infty$, and by (3.5)

$$\left|\lim_{t\to+\infty}\int_t^\infty q(s)F_3\bigl(s,\bigl[u(s)\bigr],\bigl[u'(s)\bigr],\bigl[u''(s)\bigr],u'''(s)\bigr)\,ds\right|$$
$$\le\lim_{t\to+\infty}\int_t^\infty q(s)\bigl(H\varphi(s)+1\bigr)\,ds=0,$$

then

$$\lim_{t\to+\infty}(T_1u)'''(t)=\lim_{t\to+\infty}\left(C+\int_t^\infty q(s)F_3\bigl(s,\bigl[u(s)\bigr],\bigl[u'(s)\bigr],\bigl[u''(s)\bigr],u'''(s)\bigr)\,ds\right)$$
$$=C<+\infty.$$

Therefore $T_1u\in X$.

(2) $T_1:X\to X$ is continuous. For any convergent sequence $u_m\to u$ in X, we have

$$u_m(t)\to u(t),\qquad u'_m(t)\to u'(t),\qquad u''_m(t)\to u''(t),$$
$$u'''_m(t)\to u'''(t),\quad m\to+\infty,t\in[-\tau,+\infty).$$

Thus the continuity of F_3 implies that

$$\bigl|F_3\bigl(s,\bigl[u_m(s)\bigr],\bigl[u'_m(s)\bigr],\bigl[u''_m(s)\bigr],u'''_m(s)\bigr)-F_3\bigl(s,\bigl[u(s)\bigr],\bigl[u'(s)\bigr],\bigl[u''(s)\bigr],u'''(s)\bigr)\bigr|\to0,$$
$$m\to+\infty.$$

Since $u'''_m(t)\to u'''(t)$, $\sigma=\sup\{s_1:s_1=\sup_{t\in[0,+\infty)}|u'''_m(t)|,m\in N\}<+\infty$.

Let $H_1=\max_{0\le s\le\max\{\sup_{t\in[0,+\infty)}|u'''(t)|,\sigma\}}h(s)$. Then we have

$$\int_0^\infty q(s)\bigl|\bigl(F_3\bigl(s,\bigl[u_m(s)\bigr],\bigl[u'_m(s)\bigr],\bigl[u''_m(s)\bigr],u'''_m(s)\bigr)$$
$$-F_3\bigl(s,\bigl[u(s)\bigr],\bigl[u'(s)\bigr],\bigl[u''(s)\bigr],u'''(s)\bigr)\bigr)\bigr|\,ds$$
$$\le2\int_0^\infty q(s)\bigl(H_1\phi(s)+1\bigr)\,ds<+\infty.$$

Thus, we find

$$\|T_1u_m-T_1u\|_0$$
$$=\max_{t\in[-\tau,0]}\bigl|(T_1u_m)(t)-(T_1u)(t)\bigr|$$

$$= \max_{t\in[-\tau,0]}\left|\int_0^\infty \left(-a^3 - a^2 t + a^3 e^{\frac{t}{a}}\right) q(s)\left(F_3\left(s,\left[u_m(s)\right],\left[u'_m(s)\right],\left[u''_m(s)\right],u'''_m(s)\right)\right.\right.$$
$$\left.\left. - F_3\left(s,\left[u(s)\right],\left[u'(s)\right],\left[u''(s)\right],u'''(s)\right)\right) ds\right|$$
$$\le \left|-a^3 + a^2\tau + a^3 e^{\frac{-\tau}{a}}\right| \int_0^\infty q(s)\left|F_3\left(s,\left[u_m(s)\right],\left[u'_m(s)\right],\left[u''_m(s)\right],u'''_m(s)\right)\right.$$
$$\left. - F_3\left(s,\left[u(s)\right],\left[u'(s)\right],\left[u''(s)\right],u'''(s)\right)\right| ds,$$

that is,

$$\|T_1 u_m - T_1 u\|_0 \to 0, \tag{3.8}$$

as $m \to +\infty$.

$$\|T_1 u_m - T_1 u\|_\infty^0 = \sup_{t\in[0,+\infty)}\left|\frac{(\widetilde{T}_1 u_m)(t)}{1+t^3} - \frac{(\widetilde{T}_1 u)(t)}{1+t^3}\right|$$
$$= \sup_{t\in[0,+\infty)}\left|\int_0^\infty \frac{G(t,s)}{1+t^3} q(s)\left(F_3\left(s,\left[u_m(s)\right],\left[u'_m(s)\right],\left[u''_m(s)\right],u'''_m(s)\right)\right.\right.$$
$$\left.\left. - F_3\left(s,\left[u(s)\right],\left[u'(s)\right],\left[u''(s)\right],u'''(s)\right)\right) ds\right|$$
$$\le \int_0^\infty \left(\frac{a\sqrt[3]{4}+1}{6}\right) q(s)\left|F_3\left(s,\left[u_m(s)\right],\left[u'_m(s)\right],\left[u''_m(s)\right],u'''_m(s)\right)\right.$$
$$\left. - F_3\left(s,\left[u(s)\right],\left[u'(s)\right],\left[u''(s)\right],u'''(s)\right)\right| ds,$$

that is,

$$\|T_1 u_m - T_1 u\|_\infty^0 \to 0, \tag{3.9}$$

as $m \to +\infty$.

$$\|T_1 u_m - T_1 u\|_1 = \max_{t\in[-\tau,0]}\left|(T_1 u_m)'(t) - (T_1 u)'(t)\right|$$
$$= \max_{t\in[-\tau,0]}\left|\int_0^\infty \left(-a^2 + a^2 e^{\frac{t}{a}}\right) q(s)\left(F_3\left(s,\left[u_m(s)\right],\left[u'_m(s)\right],\left[u''_m(s)\right],u'''_m(s)\right)\right.\right.$$
$$\left.\left. - F_3\left(s,\left[u(s)\right],\left[u'(s)\right],\left[u''(s)\right],u'''(s)\right)\right) ds\right|$$
$$\le \left|-a^2 + a^2 e^{\frac{-\tau}{a}}\right| \int_0^\infty q(s)\left|F_3\left(s,\left[u_m(s)\right],\left[u'_m(s)\right],\left[u''_m(s)\right],u'''_m(s)\right)\right.$$
$$\left. - F_3\left(s,\left[u(s)\right],\left[u'(s)\right],\left[u''(s)\right],u'''(s)\right)\right| ds,$$

that is,

$$\|T_1 u_m - T_1 u\|_1 \to 0, \tag{3.10}$$

as $m \to +\infty$.

$$\begin{aligned}
&\|T_1u_m - T_1u\|_\infty^1 \\
&\quad= \sup_{t\in[0,+\infty)}\left|\frac{(\widetilde{T}_1u_m)'(t)}{1+t^2} - \frac{(\widetilde{T}_1u)'(t)}{1+t^2}\right| \\
&\quad= \sup_{t\in[0,+\infty)}\left|\frac{1}{1+t^2}\int_0^\infty \frac{\partial G(t,s)}{\partial t}q(s)\big(F_3\big(s,\big[u_m(s)\big],\big[u_m'(s)\big],\big[u_m''(s)\big],u_m'''(s)\big)\right. \\
&\qquad\left. - F_3\big(s,\big[u(s)\big],\big[u'(s)\big],\big[u''(s)\big],u'''(s)\big)\big)\,ds\right| \\
&\quad\le \int_0^\infty \left(\frac{a+1}{2}\right)q(s)\big|F_3\big(s,\big[u_m(s)\big],\big[u_m'(s)\big],\big[u_m''(s)\big],u_m'''(s)\big) \\
&\qquad - F_3\big(s,\big[u(s)\big],\big[u'(s)\big],\big[u''(s)\big],u'''(s)\big)\big|\,ds,
\end{aligned}$$

that is,

$$\|T_1u_m - T_1u\|_\infty^1 \to 0, \tag{3.11}$$

as $m \to +\infty$.

$$\begin{aligned}
\|T_1u_m - T_1u\|_2 &= \max_{t\in[-\tau,0]}\big|(T_1u_m)''(t) - (T_1u)''(t)\big| \\
&= \max_{t\in[-\tau,0]}\left|\int_0^\infty ae^{\frac{t}{a}}q(s)\big(F_3\big(s,\big[u_m(s)\big],\big[u_m'(s)\big],\big[u_m''(s)\big],u_m'''(s)\big)\right. \\
&\quad\left. - F_3\big(s,\big[u(s)\big],\big[u'(s)\big],\big[u''(s)\big],u'''(s)\big)\big)\,ds\right| \\
&\le a\int_0^\infty q(s)\big|F_3\big(s,\big[u_m(s)\big],\big[u_m'(s)\big],\big[u_m''(s)\big],u_m'''(s)\big) \\
&\quad - F_3\big(s,\big[u(s)\big],\big[u'(s)\big],\big[u''(s)\big],u'''(s)\big)\big|\,ds,
\end{aligned}$$

that is,

$$\|T_1u_m - T_1u\|_2 \to 0, \tag{3.12}$$

as $m \to +\infty$.

$$\begin{aligned}
&\|T_1u_m - T_1u\|_\infty^2 \\
&\quad= \sup_{t\in[0,+\infty)}\left|\frac{(\widetilde{T}_1u_m)''(t)}{1+t} - \frac{(\widetilde{T}_1u)''(t)}{1+t}\right| \\
&\quad= \sup_{t\in[0,+\infty)}\left|\frac{1}{1+t}\int_0^\infty \frac{\partial^2 G(t,s)}{\partial t^2}q(s)\big(F_3\big(s,\big[u_m(s)\big],\big[u_m'(s)\big],\big[u_m''(s)\big],u_m'''(s)\big)\right. \\
&\qquad\left. - F_3\big(s,\big[u(s)\big],\big[u'(s)\big],\big[u''(s)\big],u'''(s)\big)\big)\,ds\right| \\
&\quad\le \int_0^\infty (a+1)q(s)\big|F_3\big(s,\big[u_m(s)\big],\big[u_m'(s)\big],\big[u_m''(s)\big],u_m'''(s)\big) \\
&\qquad - F_3\big(s,\big[u(s)\big],\big[u'(s)\big],\big[u''(s)\big],u'''(s)\big)\big|\,ds,
\end{aligned}$$

that is,

$$\|T_1u_m - T_1u\|_\infty^2 \to 0, \tag{3.13}$$

as $m \to +\infty$.

To show $\|T_1u_m - T_1u\|_\infty^3 \to 0$, as $m \to +\infty$, we need the following:

$$\begin{aligned}
&\sup_{t\in[0,+\infty)} \bigl|(T_1u_m)'''(t) - (T_1u)'''(t)\bigr| \\
&\quad= \sup_{t\in[0,+\infty)} \bigl|(\widetilde{T}_1u_m)'''(t) - (\widetilde{T}_1u)'''(t)\bigr| \\
&\quad= \sup_{t\in[0,+\infty)} \biggl|\int_t^\infty q(s)\bigl(F_3\bigl(s,\bigl[u_m(s)\bigr],\bigl[u_m'(s)\bigr],\bigl[u_m''(s)\bigr],u_m'''(s)\bigr) \\
&\qquad - F_3\bigl(s,\bigl[u(s)\bigr],\bigl[u'(s)\bigr],\bigl[u''(s)\bigr],u'''(s)\bigr)\bigr)\,ds\biggr| \\
&\quad\le \int_0^\infty q(s)\bigl|F_3\bigl(s,\bigl[u_m(s)\bigr],\bigl[u_m'(s)\bigr],\bigl[u_m''(s)\bigr],u_m'''(s)\bigr) \\
&\qquad - F_3\bigl(s,\bigl[u(s)\bigr],\bigl[u'(s)\bigr],\bigl[u''(s)\bigr],u'''(s)\bigr)\bigr|\,ds
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{t\in[-\tau,0)} \bigl|(T_1u_m)'''(t) - (T_1u)'''(t)\bigr| \\
&\quad= \sup_{t\in[-\tau,0)} \frac{1}{a}\bigl|(T_1u_m)''(t) - (T_1u)''(t)\bigr| \\
&\quad\le \int_0^\infty q(s)\bigl|F_3\bigl(s,\bigl[u_m(s)\bigr],\bigl[u_m'(s)\bigr],\bigl[u_m''(s)\bigr],u_m'''(s)\bigr) \\
&\qquad - F_3\bigl(s,\bigl[u(s)\bigr],\bigl[u'(s)\bigr],\bigl[u''(s)\bigr],u'''(s)\bigr)\bigr|\,ds.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&\|T_1u_m - T_1u\|_\infty^3 \\
&\quad= \sup_{t\in[-\tau,+\infty)} \bigl|(T_1u_m)'''(t) - (T_1u)'''(t)\bigr| \\
&\quad\le \sup_{t\in[0,+\infty)} \bigl|(T_1u_m)'''(t) - T_1u)'''(t)\bigr| + \sup_{t\in[-\tau,0)} \bigl|(T_1u_m)'''(t) - (T_1u)'''(t)\bigr| \\
&\quad\le 2\int_0^\infty q(s)\bigl|F_3\bigl(s,\bigl[u_m(s)\bigr],\bigl[u_m'(s)\bigr],\bigl[u_m''(s)\bigr],u_m'''(s)\bigr) \\
&\qquad - F_3\bigl(s,\bigl[u(s)\bigr],\bigl[u'(s)\bigr],\bigl[u''(s)\bigr],u'''(s)\bigr)\bigr|\,ds,
\end{aligned}$$

that is,

$$\|T_1u_m - T_1u\|_\infty^3 \to 0, \tag{3.14}$$

as $m \to +\infty$.

Combining (3.8)-(3.14), we find $\|(T_1u_m) - (T_1u)\| \to 0$, as $m \to +\infty$; so $T_1 : X \to X$ is continuous.

(3) $T_1 : X \to X$ is compact. The operator T_1 is compact if T_1 maps bounded subsets of X into relatively compact sets. Let K be any bounded subset of X, then $r_3 = \sup_{0\leq s\leq\{\sup_{t\in[0,+\infty)}|u'''(t)|,u\in K\}} h(s) < +\infty$. For any $u \in K$, we have the following:

$$\begin{aligned}
\|T_1u\|_0 &\leq |A| + \tau|B| + \bigl(\bigl|a^2C + a\theta(0)\bigr| + \tau\|\theta\|_0\bigr)\bigl(-a + \tau + ae^{\frac{-\tau}{a}}\bigr) \\
&\quad + \bigl(-a^3 + a^2\tau + a^3e^{\frac{-\tau}{a}}\bigr)\int_0^\infty q(s)\bigl(r_3\varphi(s) + 1\bigr)\,ds, \\
\|T_1u\|_\infty^0 &\leq |A| + |B| + \bigl|aC + \theta(0)\bigr| + C + \left(\frac{a\sqrt[3]{4} + 1}{6}\right)\int_0^\infty q(s)\bigl(r_3\varphi(s) + 1\bigr)\,ds, \\
\bigl\|(T_1u)\bigr\|_1 &\leq |B| + \bigl(\bigl|a^2C + a\theta(0)\bigr| + \tau\|\theta\|_0\bigr)\bigl(1 - e^{\frac{-\tau}{a}}\bigr) \\
&\quad + \bigl(a^2 - a^2e^{\frac{-\tau}{a}}\bigr)\int_0^\infty q(s)\bigl(r_3\varphi(s) + 1\bigr)\,ds, \\
\bigl\|(T_1u)\bigr\|_\infty^1 &\leq |B| + \bigl|aC + \theta(0)\bigr| + C + \left(\frac{a + 2}{2}\right)\int_0^\infty q(s)\bigl(r_3\varphi(s) + 1\bigr)\,ds, \\
\bigl\|(T_1u)\bigr\|_2 &\leq \bigl|aC + \theta(0)\bigr| + \|\theta\|_0 + a\int_0^\infty q(s)\bigl(r_3\varphi(s) + 1\bigr)\,ds, \\
\bigl\|(T_1u)\bigr\|_\infty^2 &\leq \bigl|aC + \theta(0)\bigr| + C + (a + 1)\int_0^\infty q(s)\bigl(r_3\varphi(s) + 1\bigr)\,ds, \\
\bigl\|(T_1u)\bigr\|_\infty^3 &\leq \frac{1}{a}\bigl|aC + \theta(0)\bigr| + \frac{2}{a}\|\theta\|_0 + C + 2\int_0^\infty q(s)\bigl(r_3\varphi(s) + 1\bigr)\,ds,
\end{aligned}$$

which implies that

$$\|T_1u\| \leq |A| + \xi|B| + \upsilon\bigl|aC + \theta(0)\bigr| + C + \gamma\|\theta\|_0 + \chi\int_0^\infty q(s)\bigl(r_3\varphi(s) + 1\bigr)\,ds,$$

where

$$\begin{aligned}
\chi &= \max\left\{\bigl(-a^3 + a^2\tau + a^3e^{\frac{-\tau}{a}}\bigr), \left(\frac{a\sqrt[3]{4} + 1}{6}\right), \bigl(a^2 - a^2e^{\frac{-\tau}{a}}\bigr), (a + 1), 2\right\}, \\
\upsilon &= \max\left\{\bigl(-a^2 + a\tau + a^2e^{\frac{-\tau}{a}}\bigr), \bigl(a - ae^{\frac{-\tau}{a}}\bigr), \frac{1}{a}, 1\right\}, \qquad \xi = \max\{\tau, 1\}, \\
\gamma &= \max\left\{\bigl(-a\tau + \tau^2 + a\tau e^{\frac{-\tau}{a}}\bigr), \bigl(\tau - \tau e^{\frac{-\tau}{a}}\bigr), 1, \frac{2}{a}\right\}.
\end{aligned}$$

Therefore, T_1K is uniformly bounded. We also know that $\psi_1(t)$ and $\psi_1'(t)$ are continuous on $[-\tau, 0]$. Thus in view of $[-\tau, 0]$ compact, $\psi_1(t)$ and $\psi_1'(t)$ are also uniformly continuous. Thus it follows that for $t_1, t_2 \in [-\tau, 0]$,

$$\begin{aligned}
&\bigl|T_1u(t_1) - T_1u(t_2)\bigr| = \bigl|\psi_1(t_1) - \psi_1(t_2)\bigr| \to 0 \quad \text{as } t_1 \to t_2, \\
&\bigl|(T_1u)'(t_1) - (T_1u)'(t_2)\bigr| = \bigl|\psi_1'(t_1) - \psi_1'(t_2)\bigr| \to 0 \quad \text{as } t_1 \to t_2,
\end{aligned}$$

further since

$$
\begin{aligned}
(T_1u)''(t) &= \psi_1''(t) \\
&= \left(\theta(0) + aC + a\int_0^\infty q(s)F_3\bigl(s,[u(s)],[u'(s)],[u''(s)],u'''(s)\bigr)\,ds\right)e^{\frac{t}{a}} \\
&\quad + \frac{1}{a}\int_t^0 e^{\frac{t-s}{a}}\theta(s)\,ds
\end{aligned}
$$

is continuous on $[-\tau,0]$, we find

$$
\bigl|(T_1u)''(t_1) - (T_1u)''(t_2)\bigr| = \bigl|\psi_1''(t_1) - \psi_1''(t_2)\bigr| \to 0 \quad \text{as } t_1 \to t_2.
$$

Next, for $t_1, t_2 \in [0,\varepsilon]$ with $\varepsilon > 0$ a constant, we have

$$
\begin{aligned}
&\left|\frac{(T_1u)(t_1)}{1+t_1^3} - \frac{(T_1u)(t_2)}{1+t_2^3}\right| \\
&\quad = \left|\frac{l(t_1)}{1+t_1^3} - \frac{l(t_2)}{1+t_2^3} + \int_0^\infty \left(\frac{G(t_1,s)}{1+t_1^3} - \frac{G(t_2,s)}{1+t_2^3}\right)q(s)\right. \\
&\qquad \left.\times F_3\bigl(s,[u(s)],[u'(s)],[u''(s)],u'''(s)\bigr)\,ds\right| \\
&\quad \le \left|\frac{l(t_1)}{1+t_1^3} - \frac{l(t_2)}{1+t_2^3}\right| + \int_0^\infty \left|\frac{G(t_1,s)}{1+t_1^3} - \frac{G(t_2,s)}{1+t_2^3}\right| q(s)\bigl(r_3\varphi(s)+1\bigr)\,ds \\
&\quad \to 0 \quad \text{as } t_1 \to t_2, \\
&\left|\frac{(T_1u)'(t_1)}{1+t_1^2} - \frac{(T_1u)'(t_2)}{1+t_2^2}\right| \\
&\quad = \left|\frac{l'(t_1)}{1+t_1^2} - \frac{l'(t_2)}{1+t_2^2} + \int_0^\infty \left(\frac{\frac{\partial G(t_1,s)}{\partial t}}{1+t_1^2} - \frac{\frac{\partial G(t_2,s)}{\partial t}}{1+t_2^2}\right)q(s)\right. \\
&\qquad \left.\times F_3\bigl(s,[u(s)],[u'(s)],[u''(s)],u'''(s)\bigr)\,ds\right| \\
&\quad \le \left|\frac{l'(t_1)}{1+t_1^2} - \frac{l'(t_2)}{1+t_2^2}\right| + \int_0^\infty \left|\frac{\frac{\partial G(t_1,s)}{\partial t}}{1+t_1^2} - \frac{\frac{\partial G(t_2,s)}{\partial t}}{1+t_2^2}\right| q(s)\bigl(r_3\varphi(s)+1\bigr)\,ds \\
&\quad \to 0 \quad \text{as } t_1 \to t_2, \\
&\left|\frac{(T_1u)''(t_1)}{1+t_1} - \frac{(T_1u)''(t_2)}{1+t_2}\right| \\
&\quad = \left|\frac{l''(t_1)}{1+t_1} - \frac{l''(t_2)}{1+t_2} + \int_0^\infty \left(\frac{\frac{\partial^2 G(t_1,s)}{\partial t^2}}{1+t_1^2} - \frac{\frac{\partial^2 G(t_2,s)}{\partial}t^2}{1+t_2^2}\right)q(s)\right. \\
&\qquad \left.\times F_3\bigl(s,[u(s)],[u'(s)],[u''(s)],u'''(s)\bigr)\,ds\right| \\
&\quad \le \left|\frac{l''(t_1)}{1+t_1} - \frac{l''(t_2)}{1+t_2}\right| + \int_0^\infty \left|\frac{\frac{\partial^2 G(t_1,s)}{\partial t^2}}{1+t_1} - \frac{\frac{\partial^2 G(t_2,s)}{\partial t^2}}{1+t_2}\right| q(s)\bigl(r_3\varphi(s)+1\bigr)\,ds \\
&\quad \to 0 \quad \text{as } t_1 \to t_2,
\end{aligned}
$$

and

$$\begin{aligned}
&\bigl|(T_1u)'''(t_1) - (T_1u)'''(t_2)\bigr| \\
&\quad = \left| \int_{t_1}^{\infty} q(s)F_3\bigl(s, \bigl[u(s)\bigr], \bigl[u'(s)\bigr], \bigl[u''(s)\bigr], u'''(s)\bigr)\, ds \right. \\
&\qquad \left. - \int_{t_2}^{\infty} q(s)F_3\bigl(s, \bigl[u(s)\bigr], \bigl[u'(s)\bigr], \bigl[u''(s)\bigr], u'''(s)\bigr)\, ds \right| \\
&\quad \le \int_{t_1}^{t_2} q(s)\bigl(r_3\varphi(s) + 1\bigr)\, ds \\
&\quad \to 0 \quad \text{as } t_1 \to t_2.
\end{aligned}$$

Thus, T_1K is equi-continuous. Finally, we will show that T_1K is equi-convergent at infinity. In fact, when $t > 0$ we have

$$\begin{aligned}
&\left| \frac{(T_1u)(t)}{1+t^3} - \lim_{t\to+\infty} \frac{(T_1u)(t)}{1+t^3} \right| = \left| \frac{(T_1u)(t)}{1+t^3} - \frac{C}{6} \right| \to 0, \quad \text{as } t \to +\infty, \\
&\left| \frac{(T_1u)'(t)}{1+t^2} - \lim_{t\to+\infty} \frac{(T_1u)'(t)}{1+t^2} \right| = \left| \frac{(T_1u)'(t)}{1+t^2} - \frac{C}{2} \right| \to 0, \quad \text{as } t \to +\infty, \\
&\left| \frac{(T_1u)''(t)}{1+t} - \lim_{t\to+\infty} \frac{(T_1u)''(t)}{1+t} \right| = \left| \frac{(T_1u)''(t)}{1+t} - C \right| \to 0, \quad \text{as } t \to +\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\Bigl|(T_1u)'''(t) - \lim_{t\to+\infty}(T_1u)'''(t)\Bigr| = \bigl|(T_1u)'''(t) - C\bigr| \le \left| \int_t^{\infty} q(s)\bigl(r_3\varphi(s) + 1\bigr)\, ds \right| \to 0 \\
&\quad \text{as } t \to +\infty.
\end{aligned}$$

Hence all conditions of Lemma 2.5 are fulfilled, so T_1K is relatively compact. Therefore, $T_1 : X \to X$ is completely continuous.

(4) $T_1 : X \to X$ has at least one fixed point. Let $\Omega = \{u \in X, \|u\| \le N\}$ where

$$N = |A| + \xi|B| + \upsilon\bigl|aC + \theta(0)\bigr| + C + \gamma\|\theta\|_0 + \chi \int_0^{\infty} q(s)\bigl(H\varphi(s) + 1\bigr)\, ds. \tag{3.15}$$

For any $u \in \Omega$, it is easy to see that $\|T_1u\| \le \Omega$, and thus $T_1\Omega \subset \Omega$. The Schäuder fixed point theorem now guarantees that the operator T_1 has at least one fixed point in Ω, which is a solution of BVP (3.3)-(1.2). Now we shall show that this solution u satisfies the inequalities (3.1) and (3.2) which in view of the definitions of F_3, F_2, F_1, and F_0 will imply that u is in fact a solution of (1.1)-(1.2). For this, we only prove that $u''(t) \le \beta''(t)$, $t \in [-\tau, +\infty)$. A similar argument can be used to prove $\alpha''(t) \le u''(t)$, $t \in [-\tau, +\infty)$. If not true, we set $\omega(t) = u''(t) - \beta''(t)$, then there exists $t^* \in [-\tau, +\infty)$ such that $\omega(t^*) = \sup_{-\tau\le t<+\infty} \omega(t) > 0$. Obviously, if $t^* = -\tau$ then $\omega'(t^*) \le 0$, and if $t^* \in (-\tau, 0]$ then $\omega'(t^*) = 0$. However, from the boundary condition, we have $\omega'(t^*) = \frac{1}{a}\omega(t^*) > 0$, which gives a contraction. If $t^* \in (0, +\infty)$, then we have

$$\omega\bigl(t^*\bigr) > 0, \qquad \omega'\bigl(t^*\bigr) = 0, \qquad \omega''\bigl(t^*\bigr) \le 0. \tag{3.16}$$

By the definition of auxiliary functions and $R > \sup_{t\in[0,+\infty)} |\beta'''(t)|$, we have

$$\begin{aligned} u^{(4)}(t^*) &= -q(t^*)F_3(t^*, [u(t^*)], [u'(t^*)], [u''(t^*)], u'''(t^*)) \\ &= -q(t^*)F_2(t^*, [u(t^*)], [u'(t^*)], [u''(t^*)], \beta'''(t^*)) \\ &= -q(t^*)\biggl[F_1(t^*, [u(t^*)], [u'(t^*)], \beta''(t^*), u''(t^* - \tau_{2,1}(t^*)), \dots, \beta'''(t^*)) \\ &\quad - \frac{u''(t^*) - \beta''(t^*)}{1 + |u''(t^*) - \beta''(t^*)|}\biggr]. \end{aligned}$$

Now if $u''(t^* - \tau_{2,1}(t^*)) > \beta''(t^* - \tau_{2,1}(t^*))$ from the definition of $\widetilde{z}_1$ it follows that

$$\begin{aligned} u^{(4)}(t^*) &= -q(t^*)F_1(t^*, [u(t^*)], [u'(t^*)], \beta''(t^*), \beta''(t^* - \tau_{2,1}(t^*)), u''(t^* - \tau_{2,2}(t^*)), \dots, \\ &\quad \beta'''(t^*)) + q(t^*)\frac{u''(t^*) - \beta''(t^*)}{1 + |u''(t^*) - \beta''(t^*)|} \end{aligned}$$

and if $u''(t^* - \tau_{2,1}(t^*)) \leq \beta''(t^* - \tau_{2,1}(t^*))$ from the condition (H_5), we have

$$\begin{aligned} u^{(4)}(t^*) &\geq -q(t^*)F_1(t^*, [u(t^*)], [u'(t^*)], \beta''(t^*), \beta''(t^* - \tau_{2,1}(t^*)), u''(t^* - \tau_{2,2}(t^*)), \dots, \\ &\quad \beta'''(t^*)) + q(t^*)\frac{u''(t^*) - \beta''(t^*)}{1 + |u''(t^*) - \beta''(t^*)|}. \end{aligned}$$

Similarly, we consider the cases $u''(t^* - \tau_{2,i}(t^*)) > \beta''(t^* - \tau_{2,i}(t^*))$ or $u''(t^* - \tau_{2,i}(t^*)) \leq \beta''(t^* - \tau_{2,i}(t^*))$, $i = 2, 3, \dots, n$, and obtain the inequality

$$u^{(4)}(t^*) \geq -q(t^*)F_1(t^*, [u(t^*)], [u'(t^*)], [\beta''(t^*)], \beta'''(t^*)) + q(t^*)\frac{u''(t^*) - \beta''(t^*)}{1 + |u''(t^*) - \beta''(t^*)|}.$$

Next, if $u'(t^*) > \beta'(t^*)$ from the definition of F_1 it follows that

$$\begin{aligned} u^{(4)}(t^*) &\geq -q(t^*)F_0(t^*, [u(t^*)], \beta'(t^*), u'(t^* - \tau_{1,1}(t^*)), \dots, [\beta''(t^*)], \beta'''(t^*)) \\ &\quad + q(t^*)\frac{u''(t^*) - \beta''(t^*)}{1 + |u''(t^*) - \beta''(t^*)|} \end{aligned}$$

and if $u'(t^*) \leq \beta'(t^*)$ from the definition of F_1 and the condition (H_4) we have

$$\begin{aligned} u^{(4)}(t^*) &\geq -q(t^*)F_0(t^*, [u(t^*)], \beta'(t^*), u'(t^* - \tau_{1,1}(t^*)), \dots, [\beta''(t^*)], \beta'''(t^*)) \\ &\quad + q(t^*)\frac{u''(t^*) - \beta''(t^*)}{1 + |u''(t^*) - \beta''(t^*)|}. \end{aligned}$$

If $t^* - \tau_{1,1}(t^*) > 0$, while discussing the cases $u'(t^* - \tau_{1,1}(t^*)) > \beta'(t^* - \tau_{1,1}(t^*))$ we use the definition of $\widetilde{y}_1$, and when discussing the cases $u'(t^* - \tau_{1,1}(t^*)) \leq \beta'(t^* - \tau_{1,1}(t^*))$ we use (H_4), and obtain

$$\begin{aligned} u^{(4)}(t^*) &\geq -q(t^*)F_0(t^*, [u(t^*)], \beta'(t^*), \beta'(t^* - \tau_{1,1}(t^*)), \dots, [\beta''(t^*)], \beta'''(t^*)) \\ &\quad + q(t^*)\frac{u''(t^*) - \beta''(t^*)}{1 + |u''(t^*) - \beta''(t^*)|}. \end{aligned}$$

Similarly, if $t^* - \tau_{1,1}(t^*) \leq 0$, while discussing the cases $u'(t^* - \tau_{1,1}(t^*)) \geq \beta'(t^* - \tau_{1,1}(t^*))$ we use (H_4), while when discussing the cases $u'(t^* - \tau_{1,1}(t^*)) < \beta'(t^* - \tau_{1,1}(t^*))$ we use the definition of $\widetilde{y}_1$, to again find

$$u^{(4)}(t^*) \geq -q(t^*)F_0(t^*, [u(t^*)], \beta'(t^*), \beta'(t^* - \tau_{1,1}(t^*)), \dots, [\beta''(t^*)], \beta'''(t^*)) + q(t^*)\frac{u''(t^*) - \beta''(t^*)}{1 + |u''(t^*) - \beta''(t^*)|}.$$

Following exactly as above, using the definition of $\widetilde{y}_i$ and (H_4), we consider the cases $i = 2, \dots, n$ to finally obtain

$$u^{(4)}(t^*) \geq -q(t^*)F_0(t^*, [u(t^*)], [\beta'(t^*)], [\beta''(t^*)], \beta'''(t^*)) + q(t^*)\frac{u''(t^*) - \beta''(t^*)}{1 + |u''(t^*) - \beta''(t^*)|}.$$

Next, if $u(t^*) > \beta(t^*)$ from the definition of F_0 it follows that

$$u^{(4)}(t^*) \geq -q(t^*)f(t^*, \beta(t^*), u(t^* - \tau_{0,1}(t^*)), \dots, [u(t^*)], [\beta''(t^*)], \beta'''(t^*)) + q(t^*)\frac{u''(t^*) - \beta''(t^*)}{1 + |u''(t^*) - \beta''(t^*)|}$$

and if $u(t^*) \leq \beta(t^*)$ from the definition of F_0 and the condition (H_3) we have

$$u^{(4)}(t^*) \geq -q(t^*)f(t^*, \beta(t^*), u(t^* - \tau_{0,1}(t^*)), \dots, [u(t^*)], [\beta''(t^*)], \beta'''(t^*)) + q(t^*)\frac{u''(t^*) - \beta''(t^*)}{1 + |u''(t^*) - \beta''(t^*)|}.$$

Similarly, we use the definition of $\widetilde{x}_i$ and (H_3) while discussing the cases $u(t^* - \tau_{0,i}(t^*)) > \beta(t^* - \tau_{0,i}(t^*))$ or $u(t^* - \tau_{0,i}(t^*)) \leq \beta(t^* - \tau_{0,i}(t^*))$, $i = 1, \dots, n$, to get

$$u^{(4)}(t^*) \geq -q(t^*)f(t^*, [\beta(t^*)], [\beta'(t^*)], [\beta''(t^*)], \beta'''(t^*)) + q(t^*)\frac{u''(t^*) - \beta''(t^*)}{1 + |u''(t^*) - \beta''(t^*)|},$$

which implies that

$$\omega''(t^*) \geq q(t^*)\frac{u''(t^*) - \beta''(t^*)}{1 + |u''(t^*) - \beta''(t^*)|} > 0,$$

which is a contractions.

If $t^* = +\infty$ then $\omega(+\infty) = \sup_{t\in[-\tau,+\infty)} \omega(t) > 0$. From the boundary conditions, we also have $\omega'(+\infty) = u'''(+\infty) - \beta'''(+\infty) \leq 0$. But this implies that $\omega''(+\infty) \leq 0$ and $\omega'(+\infty) = 0$. However, now following as above, we find $\omega''(+\infty) > 0$, which is a contradiction. Thus, $u''(t) \leq \beta''(t)$, $t \in [-\tau, +\infty)$.

Consequently, we have

$$\alpha''(t) \leq u''(t) \leq \beta''(t), \quad t \in [-\tau, +\infty),$$

which on integration and using boundary conditions gives

$$\beta'(t) \leq u'(t) \leq \alpha'(t), \quad t \in [-\tau, 0), \quad \text{and} \quad \alpha'(t) \leq u'(t) \leq \beta'(t), \quad t \in [0, +\infty)$$

and now a further integration leads to

$$\alpha(t) \leq u(t) \leq \beta(t), \quad t \in [-\tau, +\infty).$$

Further, since all conditions of the Lemma 3.1 are satisfied, $\|u\|_\infty^3 < R$. Consequently, we have

$$\begin{aligned} u^{(4)}(t) &= -q(t)F_3\bigl(t, [u(t)], [u'(t)], [u''(t)], u'''(t)\bigr) \\ &= -q(t)f\bigl(t, [u(t)], [u'(t)], [u''(t)], u'''(t)\bigr) \end{aligned}$$

and hence, u is a solution of (1.1)-(1.2). □

Theorem 3.3 *Assume that there exist two pairs of upper and lower solutions* $\beta_k, \alpha_k, k = 1, 2$ *of BVP* (1.1)-(1.2), *where* α_2, β_1 *are strict and*

$$\begin{aligned} &\alpha_2^{(i)}(t) \nleq \beta_1^{(i)}(t), \quad i = 0, 1, 2, \\ &\alpha_1^{(i)}(t) \leq \alpha_2^{(i)}(t) \leq \beta_2^{(i)}(t), \qquad \alpha_1^{(i)}(t) \leq \beta_1^{(i)}(t) \leq \beta_2^{(i)}(t), \quad t \in [-\tau, +\infty), i = 0, 2, \\ &\beta_2'(t) \leq \alpha_2'(t) \leq \alpha_1'(t), \qquad \beta_2'(t) \leq \beta_1'(t) \leq \alpha_1'(t), \quad t \in [-\tau, 0), \\ &\alpha_1'(t) \leq \alpha_2'(t) \leq \beta_2'(t), \qquad \alpha_1'(t) \leq \beta_1'(t) \leq \beta_2'(t), \quad t \in [0, +\infty), \end{aligned} \tag{3.17}$$

and f *satisfies Nagumo's condition with respect to* α_1, β_2. *Suppose further that conditions* (H_2)-(H_6) *hold with* α *and* β *replaced by* α_1 *and* β_2, *respectively. Then the problem* (1.1)-(1.2) *has at least three solutions* u_1, u_2, *and* u_3 *such that*

$$\begin{aligned} &\alpha_k''(t) \leq u_k''(t) \leq \beta_k''(t), \qquad \alpha_k(t) \leq u_k(t) \leq \beta_k(t), \quad t \in [-\tau, +\infty), k = 1, 2, \\ &\beta_k'(t) \leq u_k'(t) \leq \alpha_k'(t), \quad t \in [-\tau, 0), \qquad \alpha_k'(t) \leq u_k'(t) \leq \beta_k'(t), \quad t \in [0, +\infty), k = 1, 2, \\ &u_3^{(i)}(t) \nleq \beta_1^{(i)}(t), \qquad u_3^{(i)}(t) \ngeq \alpha_2^{(i)}(t), \quad t \in [-\tau, +\infty), i = 0, 1, 2. \end{aligned}$$

Proof First we define the truncated functions $\widetilde{F}_0, \widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3$ the same as F_0, F_1, F_2, F_3 in Theorem 3.2 with α, β replaced by α_1 and β_2, respectively. Consider the modified differential equation

$$u^{(4)}(t) + q(t)\widetilde{F}_3\bigl(t, [u(t)], [u'(t)], [u''(t)], u'''(t)\bigr) = 0, \quad 0 \leq t < +\infty. \tag{3.18}$$

To show that (3.18)-(1.2) has at least three solutions, we define operators $\widetilde{T}_2$, T_2 as

$$\widetilde{T}_2u(t) = \int_0^\infty G(t,s)q(s)\widetilde{F}_3\bigl(s, [u(s)], [u'(s)], [u''(s)], u'''(s)\bigr)\, ds$$

and

$$T_2u(t) = \begin{cases} \psi_2(t), & -\tau \leq t \leq 0; \\ l(t) + \widetilde{T}_2u(t), & 0 \leq t < +\infty, \end{cases}$$

where $l(t)$ is as in (2.7) and

$$\begin{aligned}\psi_2(t) &= A + Bt\\ &\quad + \left(\theta(0) + aC + a\int_0^{\infty} q(s)\widetilde{F}_3\big(s,[u(s)],[u'(s)],[u''(s)],u'''(s)\big)\,ds\right)\\ &\quad \times \left(-at - a^2 + a^2 e^{\frac{t}{a}}\right) + \int_t^0 \left(s - a - t + ae^{\frac{t-s}{a}}\right)\theta(s)\,ds.\end{aligned}$$

As in Theorem 3.2, T_2 is completely continuous. By using the degree theory, we will show that T_2 has at least three fixed points which are solutions of (3.18)-(1.2). We note that R in Lemma 3.1 instead of α, β now depends on α_1, β_2. Set $\Omega_2 = \{u \in X, \|u\| < N\}$ where N is as in (3.15) then for any $u \in \overline{\Omega}_2$, it follows that $\|T_2 u\| < N$, thus $T_2\overline{\Omega}_2 \subset \Omega_2$, and so we have $\deg(I - T_2, \Omega_2, 0) = 1$. Set

$$\begin{aligned}\Omega_{\alpha_2} &= \big\{u \in \Omega_2 : u''(t) > \alpha_2''(t), t \in [-\tau, +\infty)\big\},\\ \Omega^{\beta_1} &= \big\{u \in \Omega_2 : u''(t) < \beta_1''(t), t \in [-\tau, +\infty)\big\}.\end{aligned}$$

Since $\alpha_2'' \nleq \beta_1''$, $\alpha_1''(t) \le \alpha_2''(t) \le \beta_2''(t)$, $\alpha_1''(t) \le \beta_1''(t) \le \beta_2''(t)$, we find $\Omega_{\alpha_2} \neq \emptyset \neq \Omega^{\beta_1}$, $\overline{\Omega}_{\alpha_2} \cap \overline{\Omega^{\beta_1}} = \emptyset$, $\Omega_2 \backslash \overline{\Omega_{\alpha_2} \cup \Omega^{\beta_1}} \neq \emptyset$. Now since α_2, β_1 are strict lower and upper solutions there is no solution in $\partial\Omega_{\alpha_2} \cup \partial\Omega^{\beta_1}$. The additivity of degree implies that

$$\begin{aligned}\deg(I - T_2, \Omega_2, 0) = {} & \deg\big(I - T_2, \Omega_2 \backslash \overline{\Omega_{\alpha_2} \cup \Omega^{\beta_1}}, 0\big)\\ & + \deg(I - T_2, \Omega_{\alpha_2}, 0) + \deg\big(I - T_2, \Omega^{\beta_1}, 0\big).\end{aligned}$$

We will show that $\deg(I - T_2, \Omega_{\alpha_2}, 0) = \deg(I - T_2, \Omega^{\beta_1}, 0) = 1$. For this, we define new operators $\widetilde{T}_3 : \overline{\Omega}_2 \to \overline{\Omega}_2$ and $T_3 : \overline{\Omega}_2 \to \overline{\Omega}_2$ as

$$\widetilde{T}_3 u(t) = \int_0^{\infty} q(s)\widehat{F}_3\big(s,[u(s)],[u'(s)],[u''(s)],u'''(s)\big)\,ds$$

and

$$T_3 u(t) = \begin{cases}\psi_3(t), & -\tau \le t \le 0;\\ l(t) + \widetilde{T}_3 u(t), & 0 \le t < +\infty,\end{cases}$$

where $l(t)$ is as in (2.7) and

$$\begin{aligned}\psi_3(t) &= A + Bt\\ &\quad + \left(\theta(0) + aC + a\int_0^{\infty} q(s)\widehat{F}_3\big(s,[u(s)],[u'(s)],[u''(s)],u'''(s)\big)\,ds\right)\\ &\quad \times \left(-at - a^2 + a^2 e^{\frac{t}{a}}\right) + \int_t^0 \left(s - a - t + ae^{\frac{t-s}{a}}\right)\theta(s)\,ds.\end{aligned}$$

Here the functions $\widehat{F}_0$, $\widehat{F}_1$, $\widehat{F}_2$, $\widehat{F}_3$ are same as $\widetilde{F}_0$, $\widetilde{F}_1$, $\widetilde{F}_2$, $\widetilde{F}_3$ except that α_1 is replaced by α_2. Now similar to the proof of Theorem 3.2 we find that u is a fixed point of T_3 only when $\alpha_2''(t) \leq u''(t) \leq \beta_2''(t)$, $t \in [-\tau, +\infty)$. Since the lower solution α_2 is strict, $\alpha_2''(t) \neq u''(t)$, $t \in (-\tau, +\infty)$. Therefore, $u \in \Omega_{\alpha_2}$. Hence, it follows that

$$\deg(I - T_3, \Omega_2 \backslash \overline{\Omega}_{\alpha_2}, 0) = 0.$$

Also, $T_3\overline{\Omega}_2 \subset \Omega_2$, so that we have

$$\deg(I - T_3, \Omega_2, 0) = 1.$$

Therefore,

$$\begin{aligned}\deg(I - T_2, \Omega_{\alpha_2}, 0) &= \deg(I - T_3, \Omega_{\alpha_2}, 0) \\ &= \deg(I - T_3, \Omega_{\alpha_2}, 0) + \deg(I - T_3, \Omega_2 \backslash \overline{\Omega}_{\alpha_2}, 0) \\ &= \deg(I - T_3, \Omega_2, 0) = 1.\end{aligned}$$

Similarly, we have

$$\deg\bigl(I - T_2, \Omega^{\beta_1}, 0\bigr) = 1,$$

and this leads to

$$\deg\bigl(I - T_2, \Omega_2 \backslash \overline{\Omega_{\alpha_2} \cup \Omega^{\beta_1}}, 0\bigr) = -1.$$

Finally, using the properties of the degree, we conclude that T_2 has at least three fixed points

$$u_1 \in \Omega_{\alpha_2}, \qquad u_2 \in \Omega^{\beta_1}, \qquad u_3 \in \Omega_2 \backslash \overline{\Omega_{\alpha_2} \cup \Omega^{\beta_1}}$$

which are the claimed solutions of the BVP (1.1)-(1.2). □

4 Examples

Example 4.1 Consider the fourth-order nonlinear differential equation on the half-line with deviating arguments

$$u^{(4)}(t) + \frac{1}{(1+t)^3} f\bigl(t, \bigl[u(t)\bigr], \bigl[u'(t)\bigr], \bigl[u''(t)\bigr], u'''(t)\bigr) = 0, \quad t \in (0, +\infty), \tag{4.1}$$

where

$$\begin{aligned} &f\bigl(t, \bigl[u(t)\bigr], \bigl[u'(t)\bigr], \bigl[u''(t)\bigr], u'''(t)\bigr) \\ &\quad = \frac{(u'''(t) - 1)^2}{(1+t)^4}\bigl[\bigl(2t + u''(t)\bigr) + \bigl(t + u''(t-1)\bigr) + \bigl(u'(t/3 - 1/3) - 1\bigr)^2 \\ &\qquad + \bigl(t^3 + u(t/2 - 1/2)\bigr)\bigr]. \end{aligned}$$

Clearly, (4.1) is a particular case of (1.1) with $q(t) = \frac{1}{(1+t)^3}$,

$$\bigl[u(t)\bigr] = \bigl(u(t), u(t - t/2 - 1/2)\bigr), \qquad \bigl[u'(t)\bigr] = \bigl(u'(t), u'(t - 2t/3 - 1/3)\bigr),$$

$$[u''(t)] = (u''(t), u''(t-1)),$$
$$\tau_{2,1}(t) = 1, \qquad \tau_{1,1}(t) = \frac{2t}{3} + \frac{1}{3}, \qquad \tau_{0,1}(t) = \frac{t}{2} + \frac{1}{2}.$$

It follows that

$$\tau = -\min_{0\le j\le 2}\min_{t\ge 0}(t - \tau_{j,1}(t)) = 1.$$

We consider (4.1) together with the following boundary conditions:

$$\begin{cases} u(0) = 2, \qquad u'(0) = 0, \qquad u''(t) - \frac{1}{3}u'''(t) = \frac{4}{3} \quad \text{for } t \in [-1, 0]; \\ u'''(+\infty) = 0. \end{cases} \tag{4.2}$$

Comparing this with (1.2), we find $\theta(t) = \frac{4}{3}$, $a = \frac{1}{3}$, $A = 2$, $B = 0$, $C = 0$.

For (4.1)-(4.2) a direct substitution shows that

$$\beta(t) = \frac{t^3}{6} + \frac{4t^2}{3} + \frac{11}{3}, \qquad \alpha(t) = -\frac{t^3}{6}$$

are upper and lower solutions such that $\beta, \alpha \in X \cap \mathcal{C}^4(0, +\infty)$. Further, for these functions we have

$$\alpha''(t) = -t \le \beta''(t) = t + \frac{8}{3}, \quad t \in [-1, +\infty).$$

We also note that when $t \in [0, +\infty)$ and

$$\alpha(t/2 - 1/2) = \frac{1}{48} - \frac{t}{16} + \frac{t^2}{16} - \frac{t^3}{48} \le x_1 \le \beta(t/2 - 1/2) = \frac{191}{48} - \frac{29t}{48} + \frac{13t^2}{48} + \frac{t^3}{48},$$
$$\beta'(t/3 - 1/3) = -\frac{5}{6} + \frac{7t}{9} + \frac{t^2}{18} \le y_1 \le \alpha'(t/3 - 1/3) = -\frac{1}{18} + \frac{t}{9} - \frac{t^2}{18}, \quad t \in [0, 1),$$
$$\alpha'(t/3 - 1/3) = -\frac{1}{18} + \frac{t}{9} - \frac{t^2}{18} \le y_1 \le \beta'(t/3 - 1/3) = -\frac{5}{6} + \frac{7t}{9} + \frac{t^2}{18}, \quad t \in [1, +\infty),$$
$$\alpha''(t) = -t \le z_0 \le \beta''(t) = t + \frac{8}{3}, \qquad \alpha''(t-1) = -t + 1 \le z_1 \le \beta''(t-1) = t + \frac{5}{3},$$

the function f is continuous and satisfies Nagumo's condition with respect to α and β, that is,

$$\begin{aligned} &|f(t, x_0, x_1, y_0, y_1, z_0, z_1, w)| \\ &\quad = \left|(w-1)^2 \frac{(2t + z_0) + (t + z_1) + (y_1 - 1)^2 + (t^3 + x_1)}{(1+t)^4}\right| \\ &\quad \le \left(\sup_{t\in[0,+\infty)} \frac{\frac{t^4}{324} + \frac{1{,}435t^3}{1{,}296} + \frac{871t^2}{1{,}296} + \frac{5{,}393t}{1{,}296} + \frac{1{,}681}{144}}{(1+t)^4}\right)(|w| + 1)^2 \\ &\quad \le 12(|w| + 1)^2. \end{aligned}$$

Hence we can take $\varphi(t) = 12$ and $h(w) = (w + 1)^2$. Now if $1 < \gamma \le 3$, then

$$\sup_{t\in[0,+\infty)} (1+t)^\gamma \frac{12}{(1+t)^3} = \sup_{t\in[0,+\infty)} \frac{12}{(1+t)^{3-\gamma}} \le 12 < +\infty$$

and

$$\int_0^\infty \frac{1}{(1+s)^3}\,ds < +\infty, \qquad \int_0^\infty \frac{s}{(1+s)^3}\,ds < +\infty,$$
$$\int_0^\infty \frac{s}{h(s)}\,ds = \int_0^\infty \frac{s}{(s+1)^2}\,ds = +\infty,$$

and these imply that conditions (H_1), (H_2), and (H_6) are fulfilled. Now we will show that f satisfies conditions (H_3)-(H_5) of Theorem 3.2. For $t \in [0, +\infty)$, $y_i, z_i, w \in \mathbb{R}$, $i = 0, 1$, when

$$\alpha\bigl(t - \tau_{0,1}(t)\bigr) = \alpha(t/2 - 1/2) \le x_1 \le \beta(t/2 - 1/2) = \beta\bigl(t - \tau_{0,1}(t)\bigr)$$

since f is increasing with respect to x_1,

$$\begin{aligned} f\bigl(t, x_0, \alpha(t/2 - 1/2), y_0, y_1, z_0, z_1, w\bigr) &\le f(t, x_0, x_1, y_0, y_1, z_0, z_1, w) \\ &\le f\bigl(t, x_0, \beta(t/2 - 1/2), y_0, y_1, z_0, z_1, w\bigr), \end{aligned}$$

for $x_i, z_i, w \in \mathbb{R}$, $i = 0, 1$, when

$$\begin{aligned} &\beta'\bigl(t - \tau_{1,1}(t)\bigr) = \beta'(t/3 - 1/3) \le y_1 \le \alpha'\bigl(t - \tau_{1,1}(t)\bigr) = \alpha'(t/3 - 1/3), \\ &\quad \text{if } t - \tau_{1,1} \le 0, t \in [0, 1), \end{aligned}$$

or

$$\begin{aligned} &\alpha'\bigl(t - \tau_{1,1}(t)\bigr) = \alpha'(t/3 - 1/3) \le y_1 \le \beta'\bigl(t - \tau_{0,1}(t)\bigr) = \beta'(t/3 - 1/3), \\ &\quad \text{if } t - \tau_{1,1} > 0, t \in [1, +\infty), \end{aligned}$$

since f is decreasing on $[\beta'(t - \tau_{1,1}), \alpha'(t - \tau_{1,1}(t))]$ for $t \in [0, 1)$ and increasing on $[\alpha'(t - \tau_{1,1}(t)), \beta'(t - \tau_{1,1})]$ for $t \in [1, +\infty)$ with respect to y_1,

$$\begin{aligned} f\bigl(t, x_0, x_1, y_0, \alpha'(t/3 - 1/3), z_0, z_1, w\bigr) &\le f(t, x_0, x_1, y_0, y_1, z_0, z_1, w) \\ &\le f\bigl(t, x_0, x_1, y_0, \beta'(t/3 - 1/3), z_0, z_1, w\bigr), \end{aligned}$$

and for $t \in [0, +\infty)$, $x_i, y_i, w \in \mathbb{R}$, $i = 0, 1$, when

$$\alpha''(t) \le z_0 \le \beta''(t),$$

since f is increasing with respect to z_0,

$$f\bigl(t, x_0, x_1, y_0, y_1, \alpha''(t), z_1, w\bigr) \le f(t, x_0, x_1, y_0, y_1, z_0, z_1, w) \le f\bigl(t, x_0, x_1, y_0, y_1, \beta''(t), z_1, w\bigr),$$

also when

$$\alpha''\bigl(t - \tau_{2,1}(t)\bigr) = \alpha''(t - 1) \le z_1 \le \beta'\bigl(t - \tau_{2,1}(t)\bigr) = \beta''(t - 1),$$

since f is increasing respect to z_1,

$$\begin{aligned} f\big(t,x_0,x_1,y_0,y_1,z_0,\alpha''(t-1),w\big) &\leq f(t,x_0,x_1,y_0,y_1,z_0,z_1,w) \\ &\leq f\big(t,x_0,x_1,y_0,y_1,z_0,\beta''(t-1),w\big). \end{aligned}$$

Theorem 3.2 now ensures that the BVP (4.1)-(4.2) has at least one solution $u(t)$ such that

$$-\frac{t^3}{6} \leq u(t) \leq \frac{t^3}{6} + \frac{4t^2}{3} + \frac{11}{3}, \quad -t \leq u''(t) \leq t, \text{for all } t \in [-1,+\infty),$$

and

$$\begin{aligned} &\frac{t^2}{2} + \frac{8t}{3} \leq u'(t) \leq -\frac{t^2}{2}, \quad \text{for all } t \in [-1,0), \\ &-\frac{t^2}{2} \leq u'(t) \leq \frac{t^2}{2} + \frac{8t}{3}, \quad \text{for all } t \in [0,+\infty), \end{aligned}$$

also $\|u\|_\infty^3 < R$ where $R > \sqrt{\exp(192)(1+\eta^2)}$, $\eta \geq 4$, and $\gamma = 2$.

Example 4.2 Consider the fourth-order nonlinear differential equation on the half-line with deviating arguments

$$u^{(4)}(t) + \frac{1}{(1+t)^3} f\big(t,\big[u(t)\big],\big[u'(t)\big],\big[u''(t)\big],u'''(t)\big) = 0, \quad t \in (0,+\infty), \tag{4.3}$$

where

$$\begin{aligned} &f\big(t,\big[u(t)\big],\big[u'(t)\big],\big[u''(t)\big],u'''(t)\big) \\ &\quad = \left(\frac{23}{28} - u'''(t)\right) \\ &\qquad + \frac{(1-(u'''(t))^2)^2(\frac{9}{16}-(u'''(t))^2)^2(\frac{6}{7}-u'''(t))^2[(1+u''(t-1))+(u'(t-\frac{1}{2})-\frac{1}{2})^2+(u(t-\frac{1}{3})+1)]}{(u'''(t)^2+1)^4(1+t)^4}. \end{aligned}$$

Clearly, (4.3) is a particular case of (1.1) with $q(t) = \frac{1}{(1+t)^3}$,

$$\begin{aligned} &\big[u(t)\big] = \big(u(t),u(t-1/3)\big), \qquad \big[u'(t)\big] = \big(u'(t),u'(t-1/2)\big), \\ &\big[u''(t)\big] = \big(u''(t),u''(t-1)\big), \\ &\tau_{2,1}(t) = 1, \qquad \tau_{1,1}(t) = 1/2, \qquad \tau_{0,1}(t) = 1/3. \end{aligned}$$

It follows that

$$\tau = -\min_{0\leq j\leq 2}\min_{t\geq 0}(t-\tau_{j,1}) = 1.$$

We consider (4.3) together with the following boundary conditions:

$$\begin{cases} u(0) = \frac{13}{2}, \qquad u'(0) = 4, \qquad u''(t) - 2u'''(t) = t-1 \quad t \in [-1,0]; \\ u'''(+\infty) = \frac{23}{28}. \end{cases} \tag{4.4}$$

Comparing this with (1.2), we find $\theta(t) = t-1$, $a = \frac{1}{3}$, $A = \frac{13}{2}$, $B = 4$, $C = \frac{23}{28}$.

For (4.3)-(4.4) we take α_1, α_2 and β_1, β_2 as follows:

$$\alpha_1(t) = -\frac{t^3}{6} - 2t^2 + 4t, \qquad \alpha_2(t) = \frac{t^3}{8} + t^2 + 4t + 6$$

and

$$\beta_1(t) = \frac{t^3}{7} + 4t + 7, \qquad \beta_2(t) = \frac{t^3}{6} + 2t^2 + 4t + 7,$$

and by direct substitution verify that $\alpha_2, \beta_1 \in X \cap \mathcal{C}^4(0,+\infty)$ are its strict lower and upper solutions, and $\alpha_1, \beta_2 \in X \cap \mathcal{C}^4(0,+\infty)$ are lower and upper solutions, and satisfy the assumption (3.17).

We also verify that for every $t \in [0,+\infty)$, $w \in \mathbb{R}$,

$$\begin{aligned}
&-t-3 \le z_1 \le t+3, \qquad -\frac{t^3}{6} - \frac{11t^2}{6} + \frac{95t}{18} - \frac{251}{162} \le x_1 \le \frac{t^3}{6} + \frac{11t^2}{6} + \frac{49t}{18} + \frac{953}{162}, \\
&\frac{t^2}{2} + \frac{7t}{2} + \frac{17}{8} \le y_1 \le -\frac{t^2}{2} - \frac{7t}{2} + \frac{47}{8}, \quad t \in [0,1/2) \quad \text{and} \\
&-\frac{t^2}{2} - \frac{7t}{2} + \frac{47}{8} \le y_1 \le \frac{t^2}{2} + \frac{7t}{2} + \frac{17}{8}, \quad t \in [1/2,+\infty),
\end{aligned}$$

we have

$$\begin{aligned}
&\bigl|f(t,x_0,x_1,y_0,y_1,z_0,z_1,w)\bigr| \\
&\quad = \left|(23/28 - w) + \frac{(1-w^2)^2(\frac{9}{16} - w^2)^2(\frac{6}{7} - w)^2[(1+z_1) + (y_1 - \frac{1}{2})^2 + (x_1+1)]}{(w^2+1)^4(1+t)^4}\right| \\
&\quad \le \bigl(1+|w|\bigr) + \bigl(1+|w|\bigr)^2 \frac{\frac{206{,}185}{5{,}184} + \frac{1{,}087t}{72} + \frac{377t^2}{24} + \frac{11t^3}{3} + \frac{t^4}{4}}{(1+t)^4} \\
&\quad \le \bigl(1+|w|\bigr)^2 \left[1 + \sup_{t\in[0,+\infty)} \frac{\frac{206{,}185}{5{,}184} + \frac{1{,}087t}{72} + \frac{377t^2}{24} + \frac{11t^3}{3} + \frac{t^4}{4}}{(1+t)^4}\right] \\
&\quad \le 41\bigl(1+|w|\bigr)^2 = \varphi(t)h\bigl(|w|\bigr).
\end{aligned}$$

Hence the function f satisfies Nagumo's condition with $h(w) = (w+1)^2$ and $\varphi(t) = 41$. Now if $1 < \gamma \le 3$, then

$$\sup_{t\in[0,+\infty)} (1+t)^\gamma \frac{41}{(1+t)^3} = \sup_{t\in[0,+\infty)} \frac{41}{(1+t)^{3-\gamma}} \le 41 < +\infty,$$

and

$$\begin{aligned}
&\int_0^\infty \frac{1}{(1+s)^3}\,ds < +\infty, \qquad \int_0^\infty \frac{s}{(1+s)^3}\,ds < +\infty, \\
&\int_0^\infty \frac{s}{h(s)}\,ds = \int_0^\infty \frac{s}{(s+1)^2}\,ds = +\infty,
\end{aligned}$$

and these imply that conditions (H_1), (H_2), and (H_6) are fulfilled. Now we shall show that f satisfies conditions (H_3)-(H_5) of Theorem 3.2. For $t \in [0,+\infty)$, $y_i, z_i, w \in \mathbb{R}$, $i = 0,1$, when

$$\alpha\big(t-\tau_{0,1}(t)\big) = \alpha(t-1/3) \le x_1 \le \beta(t-1/3) = \beta\big(t-\tau_{0,1}(t)\big),$$

since f is increasing with respect to x_1,

$$\begin{aligned} f\big(t,x_0,\alpha(t-1/3),y_0,y_1,z_0,z_1,w\big) &\le f(t,x_0,x_1,y_0,y_1,z_0,z_1,w) \\ &\le f\big(t,x_0,\beta(t-1/3),y_0,y_1,z_0,z_1,w\big), \end{aligned}$$

for $x_i, z_i, w \in \mathbb{R}$, $i = 0,1$, when

$$\begin{aligned} &\beta'\big(t-\tau_{1,1}(t)\big) = \beta'(t-1/2) \le y_1 \le \alpha'\big(t-\tau_{1,1}(t)\big) = \alpha'(t-1/2), \\ &\quad \text{if } t-\tau_{1,1} \le 0, t \in [0,1/2), \end{aligned}$$

or

$$\begin{aligned} &\alpha'\big(t-\tau_{1,1}(t)\big) = \alpha'(t-1/2) \le y_1 \le \beta'\big(t-\tau_{0,1}(t)\big) = \beta'(t-1/2), \\ &\quad \text{if } t-\tau_{1,1} > 0, t \in [1/2,+\infty). \end{aligned}$$

Since f is decreasing on $[\beta'(t-\tau_{1,1}), \alpha'(t-\tau_{1,1}(t))]$ for $t \in [0,1/2)$ and increasing on $[\alpha'(t-\tau_{1,1}(t)), \beta'(t-\tau_{1,1})]$ for $t \in [1/2,+\infty)$ with respect to y_1,

$$\begin{aligned} f\big(t,x_0,x_1,y_0,\alpha'(t-1/2),z_0,z_1,w\big) &\le f(t,x_0,x_1,y_0,y_1,z_0,z_1,w) \\ &\le f\big(t,x_0,x_1,y_0,\beta'(t-1/2),z_0,z_1,w\big), \end{aligned}$$

and for $t \in [0,+\infty)$, $x_i, y_i, w \in \mathbb{R}$, $i = 0,1$, when

$$\alpha''\big(t-\tau_{2,1}(t)\big) = \alpha''(t-1) \le z_1 \le \beta'\big(t-\tau_{2,1}(t)\big) = \beta''(t-1),$$

since f is increasing with respect to z_1,

$$\begin{aligned} f\big(t,x_0,x_1,y_0,y_1,z_0,\alpha''(t-1),w\big) &\le f(t,x_0,x_1,y_0,y_1,z_0,z_1,w) \\ &\le f\big(t,x_0,x_1,y_0,y_1,z_0,\beta''(t-1),w\big). \end{aligned}$$

This ensures that in Theorem 3.3 all assumptions (H_1)-(H_6) are fulfilled. Therefore, we conclude that the problem (4.3)-(4.4) has at least three solutions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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