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Existence results for hybrid fractional integro-differential equations

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Abstract

In this paper we study existence results for initial value problems for hybrid fractional integro-differential equations. A couple of hybrid fixed point theorems for the sum of three operators are used for proving the main results. Examples illustrating the results are also presented.

MSC: 34A08; 34A12

Keywords: fractional differential equations; hybrid differential equations; fixed point theorems

1 Introduction

Fractional differential equations arise in the mathematical modeling of systems and processes occurring in many engineering and scientific disciplines such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, *etc.* [1–6]. For some recent developments on the topic, see [7–17] and the references therein.

Hybrid fractional differential equations have also been studied by several researchers. This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers [18–26].

In this paper we study existence results for initial value problems for hybrid fractional integro-differential equations. In Section 3 we consider the following initial value problem for hybrid fractional integro-differential equations:

$$\begin{cases} D^{\alpha}\left[\frac{x(t)-\sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right] = g(t,x(t)), & t \in J := [0,T], \\ x(0) = 0, \end{cases} \tag{1.1}$$

where D^{α} denotes the Riemann-Liouville fractional derivative of order α, $0 < \alpha \le 1$, I^{ϕ} is the Riemann-Liouville fractional integral of order $\phi > 0$, $\phi \in \{\beta_1, \beta_2, \dots, \beta_m\}$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in C(J \times \mathbb{R}, \mathbb{R})$ and $h_i \in C(J \times \mathbb{R}, \mathbb{R})$ with $h_i(0,0) = 0$, $i = 1, 2, \dots, m$. An existence result is obtained for the initial value problem (1.1) by using a hybrid fixed point theorem for three operators in a Banach algebra due to Dhage [27].

As a second problem we discuss in Section 4 an initial value problem for hybrid fractional sequential integro-differential equations,

$$\begin{cases} D^{\alpha}\left[\frac{D^{\omega}x(t)-\sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t))}{f(t,x(t))}\right] = g(t,x(t),I^{\gamma}x(t)), & t \in J, \\ x(0) = 0, \qquad D^{\omega}x(0) = 0, \end{cases} \tag{1.2}$$

where $0 < \alpha, \omega \leq 1$, $1 < \alpha + \omega \leq 2$, functions f, h, and constants $\beta_1, \beta_2, \ldots, \beta_m$ are defined as in problem (1.1), $g \in C(J \times \mathbb{R}^2, \mathbb{R})$, and I^{γ} is the Riemann-Liouville fractional integral of order γ. By using a useful generalization of Krasnoselskii's fixed point theorem due to Dhage [28], we prove an existence result for the initial value problem (1.2). Note that if $m = 1$, $\beta_1 = \gamma = 0$, and $h_1(t,x(t)) = -\lambda x(t)$, λ is a constant, then the first equation of (1.2) is reduced to the hybrid fractional Langevin equation (first formulated by Langevin in 1908) of the form

$$D^{\alpha}\left[\frac{(D^{\omega}+\lambda)x(t)}{f(t,x(t))}\right] = g\bigl(t,x(t)\bigr), \quad t \in J, \tag{1.3}$$

which is a generalization of the well-known classical results in [29].

The rest of the paper is organized as follows: In Section 2 we recall some useful preliminaries. In Section 3 we study the existence of the initial value problem (1.1), while in Section 4 we deal with the initial value problem (1.2). Examples illustrating the obtained results are presented in Section 5.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [1, 5] and present preliminary results needed in our proofs later.

Definition 2.1 The Riemann-Liouville fractional derivative of order $q > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$D^{q}f(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^{n}\int_0^t (t-s)^{n-q-1}f(s)\,ds, \quad n-1 < q < n,$$

where $n = [q] + 1$, $[q]$ denotes the integer part of a real number q, provided the right-hand side is point-wise defined on $(0,\infty)$, where Γ is the gamma function defined by $\Gamma(q) = \int_0^{\infty} e^{-s}s^{q-1}\,ds$.

Definition 2.2 The Riemann-Liouville fractional integral of order $p > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$I^{p}f(t) = \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}f(s)\,ds,$$

provided the right-hand side is point-wise defined on $(0,\infty)$.

Lemma 2.1 [1] *Let $q > 0$ and $x \in C(0,T) \cap L(0,T)$. Then the fractional differential equation*

$$D^{q}x(t) = 0$$

has a unique solution

$$x(t) = k_1 t^{q-1} + k_2 t^{q-2} + \cdots + k_n t^{q-n},$$

where $k_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, *and* $n - 1 < q < n$.

Lemma 2.2 [1] *Let* $q > 0$. *Then for* $x \in C(0,T) \cap L(0,T)$ *we have*

$$I^q D^q x(t) = x(t) - \sum_{j=1}^{n} \frac{(I^{n-q}x)^{(n-j)}(0)}{\Gamma(q-j+1)} t^{q-j},$$

where $n - 1 < q < n$.

Let $E = C(J, \mathbb{R})$ be the space of continuous real-valued functions defined on $J = [0, T]$. Define a norm $\|\cdot\|$ and a multiplication in E by

$$\|x\| = \sup_{t \in J} |x(t)| \quad \text{and} \quad (xy)(t) = x(t)y(t), \quad \forall t \in J.$$

Clearly E is a Banach algebra with respect to above supremum norm and the multiplication in it.

3 Hybrid fractional integro-differential equations

In this section we consider the initial value problem (1.1). The following hybrid fixed point theorem for three operators in a Banach algebra E, due to Dhage [27], will be used to prove the existence result for the initial value problem (1.1).

Lemma 3.1 *Let* S *be a nonempty, closed convex and bounded subset of a Banach algebra* E *and let* $A, C : E \to E$ *and* $B : S \to E$ *be three operators satisfying*:

(a_1) A *and* C *are Lipschitzian with Lipschitz constants* δ *and* ρ, *respectively,*
(b_1) B *is compact and continuous,*
(c_1) $x = AxBy + Cx \Rightarrow x \in S$ *for all* $y \in S$,
(d_1) $\delta M + \rho < 1$, *where* $M = \|B(S)\|$.

Then the operator equation $x = AxBx + Cx$ *has a solution.*

Lemma 3.2 *Suppose that* $0 < \alpha \leq 1$ *and functions* f, g, h_i, $i = 1, 2, \ldots, m$ *satisfy problem* (1.1). *Then the unique solution of the hybrid fractional integro-differential problem* (1.1) *is given by*

$$x(t) = \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\bigl(s, x(s)\bigr)\, ds + \sum_{i=1}^{m} I^{\beta_i} h_i\bigl(t, x(t)\bigr), \quad t \in J. \tag{3.1}$$

Proof Applying the Riemann-Liouville fractional integral of order α to both sides of (1.1) and using Lemma 2.2, we have

$$\begin{aligned} &\left[\frac{x(t) - \sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t))}{f(t,x(t))}\right] - \frac{t^{\alpha-1}}{\Gamma(\alpha)} I^{1-\alpha} \left[\frac{x(t) - \sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t))}{f(t,x(t))}\right]_{t=0} \\ &\quad = I^{\alpha} g\bigl(t, x(t)\bigr). \end{aligned}$$

Since $x(0) = 0$, $h(0,0) = 0$, and $f(0,0) \neq 0$, it follows that

$$x(t) = f\big(t, x(t)\big) I^{\alpha} g\big(t, x(t)\big) + \sum_{i=1}^{m} I^{\beta_i} h_i\big(t, x(t)\big).$$

Thus (3.1) holds. The proof is completed. □

Theorem 3.1 *Assume that*:

(H_1) *The functions* $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ *and* $h_i : J \times \mathbb{R} \to \mathbb{R}$, $h_i(0,0) = 0$, $i = 1, 2, \dots, m$, *are continuous and there exist two positive functions* ϕ, ψ_i, $i = 1, 2, \dots, m$ *with bound* $\|\phi\|$ *and* $\|\psi_i\|$, $i = 1, 2, \dots, m$, *respectively, such that*

$$\big|f\big(t, x(t)\big) - f\big(t, y(t)\big)\big| \leq \phi(t)\big|x(t) - y(t)\big| \tag{3.2}$$

and

$$\big|h_i\big(t, x(t)\big) - h_i\big(t, y(t)\big)\big| \leq \psi_i(t)\big|x(t) - y(t)\big|, \quad i = 1, 2, \dots, m, \tag{3.3}$$

for $t \in J$ *and* $x, y \in \mathbb{R}$.

(H_2) *There exist a function* $p \in C(J, \mathbb{R}^+)$ *and a continuous nondecreasing function* $\Psi : [0, \infty) \to (0, \infty)$ *such that*

$$\big|g\big(t, x(t)\big)\big| \leq p(t)\Psi\big(|x|\big), \quad (t, x) \in J \times \mathbb{R}. \tag{3.4}$$

(H_3) *There exists a number* $r > 0$ *such that*

$$r \geq \frac{F_0 \|p\| \Psi(r) \frac{T^{\alpha}}{\Gamma(\alpha+1)} + K_0 \sum_{i=1}^{m} \frac{T^{\beta_i}}{\Gamma(\beta_i+1)}}{1 - \|\phi\| \|p\| \Psi(r) \frac{T^{\alpha}}{\Gamma(\alpha+1)} - \sum_{i=1}^{m} \frac{\|\psi_i\| T^{\beta_i}}{\Gamma(\beta_i+1)}}, \tag{3.5}$$

where $F_0 = \sup_{t \in J} |f(t, 0)|$ *and* $K_0 = \sup_{t \in J} |h_i(t, 0)|$, $i = 1, 2, \dots, m$, *and*

$$\|\phi\| \|p\| \Psi(r) \frac{T^{\alpha}}{\Gamma(\alpha + 1)} + \sum_{i=1}^{m} \frac{\|\psi_i\| T^{\beta_i}}{\Gamma(\beta_i + 1)} < 1. \tag{3.6}$$

Then problem (1.1) *has at least one solution on* J.

Proof Set $E = C(J, \mathbb{R})$ and define a subset S of E as

$$S = \big\{x \in E : \|x\| \leq r\big\},$$

where r satisfies inequality (3.5).

Clearly S is closed, convex, and bounded subset of the Banach space E. By Lemma 3.2, problem (1.1) is equivalent to the integral equation (3.1). Now we define three operators; $\mathcal{A} : E \to E$ by

$$\mathcal{A}x(t) = f\big(t, x(t)\big), \quad t \in J, \tag{3.7}$$

$\mathcal{B} : S \to E$ by

$$\mathcal{B}x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x(s)\bigr)\, ds, \quad t \in J, \tag{3.8}$$

and $\mathcal{C} : E \to E$ by

$$\mathcal{C}x(t) = \sum_{i=1}^{m} I^{\beta_i} h_i\bigl(t, x(t)\bigr) = \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i\bigl(s, x(s)\bigr)\, ds, \quad t \in J. \tag{3.9}$$

We shall show that the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfy all the conditions of Lemma 3.1. This will be achieved in the following series of steps.

Step 1. *We first show that $\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian on E.*

Let $x, y \in E$. Then by (H_1), for $t \in J$ we have

$$\begin{aligned} \bigl|\mathcal{A}x(t) - \mathcal{A}y(t)\bigr| &= \bigl|f\bigl(t, x(t)\bigr) - f\bigl(t, y(t)\bigr)\bigr| \\ &\le \phi(t)\bigl|x(t) - y(t)\bigr| \le \|\phi\| \|x - y\|, \end{aligned}$$

which implies $\|\mathcal{A}x - \mathcal{A}y\| \le \|\phi\| \|x - y\|$ for all $x, y \in E$. Therefore, $\mathcal{A}$ is a Lipschitzian on E with Lipschitz constant $\|\phi\|$.

Analogously, for any $x, y \in E$, we have

$$\begin{aligned} \bigl|\mathcal{C}x(t) - \mathcal{C}y(t)\bigr| &= \left|\sum_{i=1}^{m} I^{\beta_i} h_i\bigl(t, x(t)\bigr) - \sum_{i=1}^{m} I^{\beta_i} h_i\bigl(t, y(t)\bigr)\right| \\ &\le \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} \psi_i(s)\bigl|x(s) - y(s)\bigr|\, ds \\ &\le \|x - y\| \sum_{i=1}^{m} \frac{\|\psi_i\| T^{\beta_i}}{\Gamma(\beta_i + 1)}. \end{aligned}$$

This means that

$$\|\mathcal{C}x - \mathcal{C}y\| \le \sum_{i=1}^{m} \frac{\|\psi_i\| T^{\beta_i}}{\Gamma(\beta_i + 1)} \|x - y\|.$$

Thus, $\mathcal{C}$ is a Lipschitzian on E with Lipschitz constant $\sum_{i=1}^{m} \frac{\|\psi_i\| T^{\beta_i}}{\Gamma(\beta_i+1)}$.

Step 2. *The operator $\mathcal{B}$ is completely continuous on S.*

We first show that the operator $\mathcal{B}$ is continuous on E. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then by the Lebesgue dominated convergence theorem, for all $t \in J$, we obtain

$$\begin{aligned} \lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x_n(s)\bigr)\, ds \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \lim_{n\to\infty} g\bigl(s, x_n(s)\bigr)\, ds \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s, x(s)\bigr)\, ds. \end{aligned}$$

This implies that $\mathcal{B}$ is continuous on S.

Next we will prove that the set $\mathcal{B}(S)$ is a uniformly bounded in S. For any $x \in S$, we have

$$\begin{aligned} \left|\mathcal{B}x(t)\right| &= \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s,x(s)\bigr)\,ds\right| \\ &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)\Psi(r)\,ds \\ &\le \|p\|\Psi(r)\frac{T^\alpha}{\Gamma(\alpha+1)} = K_1, \end{aligned}$$

for all $t \in J$. Therefore, $\|\mathcal{B}\| \le K_1$, which shows that $\mathcal{B}$ is uniformly bounded on S.

Now, we will show that $\mathcal{B}(S)$ is an equicontinuous set in E. Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $x \in S$. Then we have

$$\begin{aligned} \left|\mathcal{B}x(\tau_2) - \mathcal{B}x(\tau_1)\right| &= \left|\int_0^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s,x(s)\bigr)\,ds - \int_0^{\tau_1} \frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)} g\bigl(s,x(s)\bigr)\,ds\right| \\ &\le \int_0^{\tau_1} \frac{|(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}|}{\Gamma(\alpha)} \left|g\bigl(s,x(s)\bigr)\right|\,ds \\ &\quad + \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} \left|g\bigl(s,x(s)\bigr)\right|\,ds \\ &\le \int_0^{\tau_1} \frac{|(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}|}{\Gamma(\alpha)} \|p\|\Psi(r)\,ds \\ &\quad + \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} \|p\|\Psi(r)\,ds, \end{aligned}$$

which is independent of $x \in S$. As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. Therefore, it follows from the Arzelá-Ascoli theorem that $\mathcal{B}$ is a completely continuous operator on S.

Step 3. *The hypothesis* (c_1) *of Lemma* 3.1 *is satisfied.*

Let $x \in E$ and $y \in S$ be arbitrary elements such that $x = \mathcal{A}x\mathcal{B}y + \mathcal{C}x$. Then we have

$$\begin{aligned} \left|x(t)\right| &\le \left|\mathcal{A}x(t)\right|\left|\mathcal{B}y(t)\right| + \left|\mathcal{C}x(t)\right| \\ &\le \left|f\bigl(t,x(t)\bigr)\right| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|g\bigl(s,x(s)\bigr)\right|\,ds + \sum_{i=1}^m \int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} \left|h_i\bigl(s,x(s)\bigr)\right|\,ds \\ &\le \bigl(\left|f\bigl(t,x(t)\bigr) - f(t,0)\right| + \left|f(t,0)\right|\bigr) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|p\|\Psi(r)s \\ &\quad + \sum_{i=1}^m \int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)} \bigl(\left|h_i\bigl(s,x(s)\bigr) - h_i(s,0)\right| + \left|h_i(s,0)\right|\bigr)\,ds \\ &\le \bigl(r\|\phi\| + F_0\bigr)\left(\|p\|\Psi(r)\frac{T^\alpha}{\Gamma(\alpha+1)}\right) + \sum_{i=1}^m \frac{(r\|\psi_i\| + K_0)T^{\beta_i}}{\Gamma(\beta_i+1)}, \end{aligned}$$

which leads to

$$\|x\| \le \bigl(r\|\phi\| + F_0\bigr)\left(\|p\|\Psi(r)\frac{T^\alpha}{\Gamma(\alpha+1)}\right) + \sum_{i=1}^m \frac{(r\|\psi_i\| + K_0)T^{\beta_i}}{\Gamma(\beta_i+1)} \le r.$$

Therefore, $x \in S$.

Step 4. *Finally we show that* $\delta M + \rho < 1$, *that is*, (d_1) *of Lemma* 3.1 *holds*.

Since

$$\begin{aligned} M &= \|\mathcal{B}(S)\| \\ &= \sup_{x\in S}\Big\{\sup_{t\in J}\big|\mathcal{B}x(t)\big|\Big\} \\ &\le \|p\|\Psi(r)\frac{T^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned} \tag{3.10}$$

and by (H_3) we have

$$\|\phi\|M + \sum_{i=1}^{m}\frac{\|\psi_i\|T^{\beta_i}}{\Gamma(\beta_i+1)} < 1,$$

with $\delta = \|\phi\|$ and $\rho = \sum_{i=1}^{m}\frac{\|\psi_i\|T^{\beta_i}}{\Gamma(\beta_i+1)}$.

Thus all the conditions of Lemma 3.1 are satisfied and hence the operator equation $x = \mathcal{A}x\mathcal{B}x + \mathcal{C}x$ has a solution in S. In consequence, problem (1.1) has a solution on J. This completes the proof. □

4 Hybrid fractional sequential integro-differential equations

In this section we consider the initial value problem (1.2). An existence result will be proved by using the following fixed point theorem due to Dhage.

Lemma 4.1 [28] *Let M be a nonempty, closed, convex and bounded subset of the Banach space X and let $A : X \to X$ and $B : M \to X$ be two operators such that*

(i) *A is a contraction,*

(ii) *B is completely continuous, and*

(iii) *$x = Ax + By$ for all $y \in M \Rightarrow x \in M$.*

Then the operator equation $Ax + Bx = x$ has a solution.

Lemma 4.2 *Suppose that* $0 < \alpha, \omega \le 1$, $1 < \alpha + \omega \le 2$, $\gamma > 0$, *and the functions* f, g, h_i, $i = 1, 2, \dots, m$ *satisfy problem* (1.2). *Then the unique solution of the hybrid fractional sequential integro-differential problem* (1.2) *is given by*

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)} f\big(s,x(s)\big)\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} g\big(u,x(u),I^{\gamma}x(u)\big)\,du\,ds \\ &\quad + \sum_{i=1}^{m} I^{\beta_i+\omega}h_i\big(t,x(t)\big), \quad t \in J. \end{aligned} \tag{4.1}$$

Proof By using the method of proving of Lemma 3.2 and applying the semigroup property, *i.e.*, $I^{\omega}I^{\beta_i}h_i = I^{\beta_i+\omega}h_i$, $i = 1, 2, \dots, m$, we obtain the desired integral equation in (4.1). □

Theorem 4.1 *Assume that*:

(A_1) *The functions $f : J \times \mathbb{R} \to \mathbb{R}\setminus\{0\}$ and $g : J \times \mathbb{R}^2 \to \mathbb{R}$, are continuous and there exist two positive functions ϕ, χ with bound $\|\phi\|$ and $\|\chi\|$, respectively, such that*

$$\big|f\big(t,x(t)\big) - f\big(t,y(t)\big)\big| \le \phi(t)\big|x(t) - y(t)\big| \tag{4.2}$$

and

$$\big|g\big(t,x(t),y(t)\big)-g\big(t,\bar{x}(t),\bar{y}(t)\big)\big| \leq \chi(t)\big(\big|x(t)-y(t)\big|+\big|\bar{x}(t)-\bar{y}(t)\big|\big), \tag{4.3}$$

for $t \in J$ *and* $x,y,\bar{x},\bar{y} \in \mathbb{R}$.

(A$_2$) $|f(t,x)| \leq \mu(t)$, $\forall(t,x) \in J\times\mathbb{R}$, $\mu \in C(J,\mathbb{R}^+)$, $|g(t,x,y)| \leq \nu(t)$, $\forall(t,x,y) \in J\times\mathbb{R}\times\mathbb{R}$, $\nu \in C(J,\mathbb{R}^+)$, *and* $|h_i(t,x)| \leq \theta_i(t)$, $\forall(t,x)\in J\times\mathbb{R}$, $\theta_i \in C(J,\mathbb{R}^+)$, $i=1,2,\ldots,m$.

If

$$\frac{T^\alpha}{\Gamma(\alpha+1)}\frac{T^\omega}{\Gamma(\omega+1)}\|\nu\|\|\phi\|+\|\mu\|\|\chi\|\left[\frac{T^\omega}{\Gamma(\omega+1)}+\frac{T^{\omega+\gamma}}{\Gamma(\omega+\gamma+1)}\right]<1, \tag{4.4}$$

then problem (1.2) *has at least one solution on* J.

Proof Setting $\sup_{t\in J}|\mu(t)|=\|\mu\|$, $\sup_{t\in J}|\nu(t)|=\|\nu\|$, $\sup_{t\in J}|\theta_i(t)|=\|\theta_i\|$, $i=1,2,\ldots,m$, and choosing

$$R \geq \sum_{i=1}^m \frac{T^{\beta_i+\omega}}{\Gamma(\beta_i+\omega+1)}\|\theta_i\|+\frac{T^{\omega+\alpha}}{\Gamma(\omega+\alpha+1)}\|\mu\|\|\nu\|, \tag{4.5}$$

we consider $B_R=\{x\in C(J,\mathbb{R}):\|x\|\leq R\}$. We define the operators $\mathcal{A}:E\to E$ as in (3.7), $\mathcal{D}:B_R\to E$ by

$$\mathcal{D}x(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g\big(s,x(s),I^\gamma x(s)\big)\,ds, \quad t\in J \tag{4.6}$$

and

$$\mathcal{Q}x(t)=\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+\omega)}\int_0^t(t-s)^{\beta_i+\omega-1}h_i\big(s,x(s)\big)\,ds, \quad t\in J, \tag{4.7}$$

and

$$\mathcal{T}x(t)=\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}\mathcal{A}x(s)\mathcal{D}x(s)\,ds, \quad t\in J. \tag{4.8}$$

For any $y\in B_R$, we have

$$\begin{aligned}
\big|x(t)\big| &= \big|\mathcal{Q}x(t)+\mathcal{T}y(t)\big| \\
&\leq \sum_{i=1}^m\frac{1}{\Gamma(\beta_i+\omega)}\int_0^t(t-s)^{\beta_i+\omega-1}\big|h_i\big(s,x(s)\big)\big|\,ds \\
&\quad+\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}\big|\mathcal{A}y(s)\big|\big|\mathcal{D}y(s)\big|\,ds \\
&\leq \sum_{i=1}^m\frac{1}{\Gamma(\beta_i+\omega)}\int_0^t(t-s)^{\beta_i+\omega-1}\big|\theta_i(s)\big|\,ds \\
&\quad+\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}\big|\mu(s)\big|\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\big|\nu(\tau)\big|\,d\tau\,ds
\end{aligned}$$

$$\leq \sum_{i=1}^{m} \frac{T^{\beta_i+\omega}}{\Gamma(\beta_i+\omega+1)}\|\theta_i\| + \frac{T^{\omega+\alpha}}{\Gamma(\omega+\alpha+1)}\|\mu\|\|\nu\|$$
$$\leq R,$$

and therefore $\|x\| \leq R$, which means that $x \in B_R$. Hence, the condition (iii) of Lemma 4.1 holds.

Next we will show that $\mathcal{Q}$ satisfy the condition (ii) of Lemma 4.1. The operator $\mathcal{Q}$ is obviously continuous. Also, $\mathcal{Q}$ is uniformly bounded on B_R as

$$\|\mathcal{Q}x\| \leq \sum_{i=1}^{m} \frac{T^{\beta_i+\omega}}{\Gamma(\beta_i+\omega+1)}\|\theta_i\|.$$

Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $x \in B_R$. We define $\sup_{(t,x)\in J\times B_R} |h_i(t,x)| = \bar{h}_i < \infty$, $i = 1, 2, \ldots, m$. Then we have

$$\begin{aligned}
\left|\mathcal{Q}x(\tau_2) - \mathcal{Q}x(\tau_1)\right| &= \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+\omega)}\left|\int_0^{\tau_2} (\tau_2 - s)^{\beta_i+\omega-1} h_i\bigl(s, x_i(s)\bigr)\,ds \right.\\
&\quad \left. - \int_0^{\tau_1} (\tau_1 - s)^{\beta_i+\omega-1} h_i\bigl(s, x_i(s)\bigr)\,ds\right|\\
&\leq \sum_{i=1}^{m} \frac{\bar{h}_i}{\Gamma(\beta_i+\omega)}\left|\int_0^{\tau_1} \bigl[(\tau_2 - s)^{\beta_i+\omega-1} - (\tau_1 - s)^{\beta_i+\omega-1}\bigr]\,ds \right.\\
&\quad \left. + \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\beta_i+\omega-1}\,ds\right|\\
&\leq \sum_{i=1}^{m} \frac{\bar{h}_i}{\Gamma(\beta_i+\omega+1)}\left|\tau_2^{\beta_i+\omega} - \tau_1^{\beta_i+\omega}\right|,
\end{aligned}$$

which is independent of x and tends to zero as $\tau_2 - \tau_1 \to 0$. Thus, $\mathcal{Q}$ is equicontinuous. So $\mathcal{Q}$ is relatively compact on B_R. Hence, by the Arzelá-Ascoli theorem, $\mathcal{Q}$ is compact on B_R.

Now we show that $\mathcal{T}$ is a contraction mapping. Let $x, y \in B_R$. Then for $t \in J$ we have

$$\begin{aligned}
&\left|\mathcal{T}x(t) - \mathcal{T}y(t)\right|\\
&\quad = \left|\int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)}\bigl[\mathcal{A}x(s)\mathcal{D}x(s) - \mathcal{A}y(s)\mathcal{D}y(s)\bigr]\,ds\right|\\
&\quad = \left|\int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)}\bigl[\mathcal{A}x(s)\mathcal{D}x(s) - \mathcal{A}y(s)\mathcal{D}x(s) + \mathcal{A}y(s)\mathcal{D}x(s) - \mathcal{A}y(s)\mathcal{D}y(s)\bigr]\,ds\right|\\
&\quad \leq \int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)}\bigl\{\left|\mathcal{D}x(s)\right|\left|\mathcal{A}x(s) - \mathcal{A}y(s)\right| + \left|\mathcal{A}y(s)\right|\left|\mathcal{D}x(s) - \mathcal{D}y(s)\right|\bigr\}\,ds\\
&\quad \leq \int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)}\left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|\nu\|\|\phi\|\|x-y\|\right.\\
&\qquad \left. + \|\mu\|\|\chi\|\left[\|x-y\| + \|x-y\|\int_0^s \frac{(s-u)^{\gamma-1}}{\Gamma(\gamma)}\,du\right]\right\}ds\\
&\quad \leq \left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)}\frac{T^{\omega}}{\Gamma(\omega+1)}\|\nu\|\|\phi\| + \|\mu\|\|\chi\|\left[\frac{T^{\omega}}{\Gamma(\omega+1)} + \frac{T^{\omega+\gamma}}{\Gamma(\omega+\gamma+1)}\right]\right\}\|x-y\|.
\end{aligned}$$

Hence, by (4.4), $\mathcal{T}$ is a contraction mapping, and thus the condition (i) of Lemma 4.1 is satisfied.

Thus all the assumptions of Lemma 4.1 are satisfied. Therefore, the conclusion of Lemma 4.1 implies that problem (1.2) has at least one solution on J. □

5 Examples

In this section, we present two examples to illustrate our results.

Example 5.1 Consider the following hybrid fractional integro-differential equation:

$$\begin{cases} D^{\frac{3}{4}}\left[\frac{x(t)-\sum_{i=1}^{4} I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right] = \frac{(t-1)^2+3}{35(13-t^2)}(7|x(t)|+15), \\ x(0)=0, \quad t\in[0,1], \end{cases} \tag{5.1}$$

where

$$\begin{aligned} \sum_{i=1}^{4} I^{\beta_i}h_i\bigl(t,x(t)\bigr) = {}& I^{1/3}\frac{2te^{-3t}}{15(3+t)}\left(\frac{x^2(t)+9|x(t)|}{|x(t)|+5}+\frac{12e^{3t}}{5}\right) \\ &+ I^{7/4}\frac{t\sin t}{7(4+e^t)}\left(\frac{x^2(t)+4|x(t)|}{|x(t)|+3}+\cos t\right) \\ &+ I^{10/3}\frac{2\sin\pi t}{9+(5+t)^2}\left(\frac{x^2(t)+8|x(t)|}{|x(t)|+5}+\frac{4}{3}\right) \\ &+ I^{29/6}\frac{3t\cos t+4t\sin t}{10(4-t)^2}\left(\frac{x^2(t)+5|x(t)|}{|x(t)|+4}+\frac{t}{t+2}\right) \end{aligned}$$

and

$$f\bigl(t,x(t)\bigr) = \frac{3(\cos\pi t+2t)}{5(2+10t)^2}\left(\frac{x^2(t)+5|x(t)|}{|x(t)|+3}\right)+\frac{8-2^{2-t}}{27}.$$

Here $\alpha=3/4$, $T=1$, $m=4$, $\beta_1=1/3$, $\beta_2=7/4$, $\beta_3=10/3$, and $\beta_4=29/6$. We can show that

$$\bigl|f(t,x)-f(t,y)\bigr| \le \left(\frac{1+2t}{(2+10t)^2}\right)|x-y|$$

and

$$\begin{aligned} \bigl|h_1(t,x)-h_1(t,y)\bigr| &\le \left(\frac{18t}{75(3+t)}\right)|x-y|, \\ \bigl|h_2(t,x)-h_2(t,y)\bigr| &\le \left(\frac{4t}{21(4+e^t)}\right)|x-y|, \\ \bigl|h_3(t,x)-h_3(t,y)\bigr| &\le \left(\frac{16}{45+5(5+t)^2}\right)|x-y|, \\ \bigl|h_4(t,x)-h_4(t,y)\bigr| &\le \left(\frac{5t}{8(4-t)^2}\right)|x-y|. \end{aligned}$$

It follows that $\phi(t)=(1+2t)(2+10t)^{-2}$, $\psi_1(t)=18t(3+t)^{-1}/75$, $\psi_2(t)=4t(4+e^t)^{-1}/21$, $\psi_3(t)=16(45+5(5+t)^2)^{-1}$, and $\psi_4(t)=5t(4-t)^{-2}/8$, which give norms $\|\phi\|=1/4$, $\|\psi_1\|=3/50$, $\|\psi_2\|=4(28+7e)^{-1}/3$, $\|\psi_3\|=8/85$, and $\|\psi_4\|=5/72$.

Since

$$\left|g(t,x,y)\right| = \left|\frac{(t-1)^2+3}{35(13-t^2)}(7|x|+15)\right| \le \left(\frac{(t-1)^2+3}{13-t^2}\right)\left(\frac{|x|}{5}+\frac{3}{7}\right),$$

we set $p(t) = ((t-1)^2+3)/(13-t^2)$ and also the function Ψ as

$$\Psi(x) = \frac{x}{5} + \frac{3}{7}.$$

It is easy to verify that $\|p\| = 4/13$, $F_0 = \sup_{t\in[0,2]} |f(t,0)| = 4/27$, and $K_0 = \sup_{t\in[0,2]} |h_i(t,0)| = 2/25$, $i = 1,2,3,4$. We see that condition (H_3) is followed with a number $r \in [0.187454, 131.292851]$. Consequently all conditions in Theorem 3.1 are satisfied. Therefore, problem (5.1) has at least one solution on $[0,1]$.

Example 5.2 Consider the following hybrid fractional sequential integro-differential equation:

$$\begin{cases} D^{4/5}\left[\frac{D^{2/3}x(t)-\sum_{i=1}^{5} I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right] \\ \quad = \frac{1}{2}\arctan\left(\frac{2|x(t)|}{3+|x(t)|}\cos\frac{\pi t}{6}\right) - \frac{1}{2}\arctan\left(\frac{2|I^{8/3}x(t)|}{3+|I^{8/3}x(t)|}\cos\frac{\pi t}{6}\right), \\ x(0) = 0, \qquad D^{2/3}x(0) = 0, \quad t \in [0,1], \end{cases} \tag{5.2}$$

where

$$\begin{aligned} \sum_{i=1}^{5} I^{\beta_i}h_i\big(t,x(t)\big) = {} & I^{1/6}\log\left(1+\big|x(t)\big|\frac{e^t\sin t}{1+|x(t)|}\right) \\ & + I^{3/5}\left(\frac{\cos t\sin x(t)}{\sqrt{2+t}} + \frac{\cos x(t)\sin t}{\sqrt{2-t}}\right) \\ & + I^{5/4}\big(te^t\cos\big(x(t)+\pi t\big) + e^{-t}\sin\big(x(t)+\pi t\big)\big) \\ & + I^{7/3}\left(\frac{2t^2+e^{-|x(t)|}\sin t}{1+|x(t)|+t^3}\right) \\ & + I^{9/2}\left(\frac{te^t}{1+e^t} + \frac{t^2|x(t)|}{1+t|x(t)|}\right) \end{aligned}$$

and

$$f\big(t,x(t)\big) = \left(\frac{|x(t)|+1}{|x(t)|+2}\right)\frac{(7-e^t)}{2\sqrt{25-t^2}} + \frac{2-t}{10}.$$

Here $\alpha = 4/5$, $\omega = 2/3$, $m = 5$, $T = 1$, $\beta_1 = 1/6$, $\beta_2 = 3/5$, $\beta_3 = 5/4$, $\beta_4 = 7/3$, $\beta_5 = 9/2$, $\gamma = 8/3$. We can show that

$$\left|f(t,x) - f(t,y)\right| \le \left(\frac{7-e^t}{8\sqrt{25-t^2}}\right)|x-y|$$

and

$$\left|g(t,x,y) - g(t,\bar{x},\bar{y})\right| \le \frac{1}{3}\cos\frac{\pi t}{6}\big(|x-y| + |\bar{x}-\bar{y}|\big).$$

Therefore, we choose

$$\phi(t) = \frac{7-e^t}{8\sqrt{25-t^2}} \quad \text{and} \quad \chi(t) = \frac{1}{3}\cos\frac{\pi t}{6}.$$

It is easy to see that

$$\begin{aligned} \left|f(t,x)\right| &\leq \frac{7-e^t}{2\sqrt{25-t^2}} + \frac{2-t}{10} = \mu(t), \\ \left|g(t,x,y)\right| &\leq 2\cos\frac{\pi t}{6} = \nu(t). \end{aligned}$$

The functions $h_i(t,x(t))$, $i = 1,2,\ldots,5$, are bounded by the corresponding positive functions $\theta_i(t)$, $i = 1,2,\ldots,5$, as follows:

$$\begin{aligned} \left|h_1(t,x)\right| &\leq 1 + e^t \sin t = \theta_1(t), \\ \left|h_2(t,x)\right| &\leq \sqrt{\frac{2-t\cos 2t}{4-t^2}} + \frac{1}{2} = \theta_2(t), \\ \left|h_3(t,x)\right| &\leq \sqrt{\frac{e^{4t}+1}{e^{2t}}} = \theta_3(t), \\ \left|h_4(t,x)\right| &\leq \frac{2t^2+\sin t}{1+t^3} = \theta_4(t), \\ \left|h_5(t,x)\right| &\leq \frac{t(1+2e^t)}{1+e^t} = \theta_5(t). \end{aligned}$$

Hence the conditions (A_1) and (A_2) are satisfied. Putting $\|\phi\| = 3/20$, $\|\chi\| = 1/3$, $\|\mu\| = 4/5$, and $\|\nu\| = 2$, we obtain

$$\frac{T^\alpha}{\Gamma(\alpha+1)}\frac{T^\omega}{\Gamma(\omega+1)}\|\nu\|\|\phi\| + \|\mu\|\|\chi\|\left[\frac{T^\omega}{\Gamma(\omega+1)} + \frac{T^{\omega+\gamma}}{\Gamma(\omega+\gamma+1)}\right] \approx 0.680993 < 1.$$

By Theorem 4.1, problem (5.2) has a solution on $[0,1]$.

6 Conclusions

In this paper, we have studied the existence of solutions for initial value problems of first and second order for hybrid fractional integro-differential equations. The first result has been obtained by using a hybrid fixed point theorem for three operators in a Banach algebra due to Dhage [27], while the second has been obtained by using a useful generalization of Krasnoselskii's fixed point theorem due to Dhage [28]. The main results are well illustrated with the help of examples.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in this article. They read and approved the final manuscript.

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References

1. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
2. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
3. Lakshmikantham, V, Leela, S, Vasundhara Devi, J: Theory of Fractional Dynamic Systems. Cambridge Academic Publishers, Cambridge (2009)
4. Lakshmikantham, V, Vatsala, AS: Basic theory of fractional differential equations. Nonlinear Anal. **69**(8), 2677-2682 (2008)
5. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
6. Sabatier, J, Agrawal, OP, Machado, JAT (eds.): Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Springer, Dordrecht (2007)
7. Ahmad, B: Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations. Appl. Math. Lett. **23**, 390-394 (2010)
8. Ahmad, B, Nieto, JJ: Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. Comput. Math. Appl. **58**, 1838-1843 (2009)
9. Thiramanus, P, Ntouyas, SK, Tariboon, J: Existence and uniqueness results for Hadamard-type fractional differential equations with nonlocal fractional integral boundary conditions. Abstr. Appl. Anal. **2014**, Article ID 902054 (2014)
10. Tariboon, J, Ntouyas, SK, Sudsutad, W: Fractional integral problems for fractional differential equations via Caputo derivative. Adv. Differ. Equ. **2014**, 181 (2014)
11. Ahmad, B, Ntouyas, SK, Alsaedi, A: New existence results for nonlinear fractional differential equations with three-point integral boundary conditions. Adv. Differ. Equ. **2011**, Article ID 107384 (2011)
12. Ahmad, B, Ntouyas, SK: A four-point nonlocal integral boundary value problem for fractional differential equations of arbitrary order. Electron. J. Qual. Theory Differ. Equ. **2011**, 22 (2011)
13. Ahmad, B, Sivasundaram, S: Existence and uniqueness results for nonlinear boundary value problems of fractional differential equations with separated boundary conditions. Commun. Appl. Anal. **13**, 121-228 (2009)
14. Ahmad, B, Sivasundaram, S: On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order. Appl. Math. Comput. **217**, 480-487 (2010)
15. Ahmad, B, Ntouyas, SK, Alsaedi, A: Existence theorems for nonlocal multi-valued Hadamard fractional integro-differential boundary value problems. J. Inequal. Appl. **2014**, 454 (2014)
16. Chen, W, Zhao, Y: Solvability of boundary value problems of nonlinear fractional differential equations. Adv. Differ. Equ. **2015**, 36 (2015)
17. Zhao, Y, Sun, S, Han, Z, Li, Q: The existence of multiple positive solutions for boundary value problems of nonlinear fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **16**, 2086-2097 (2011)
18. Zhao, Y, Sun, S, Han, Z, Li, Q: Theory of fractional hybrid differential equations. Comput. Math. Appl. **62**, 1312-1324 (2011)
19. Sun, S, Zhao, Y, Han, Z, Li, Y: The existence of solutions for boundary value problem of fractional hybrid differential equations. Commun. Nonlinear Sci. Numer. Simul. **17**, 4961-4967 (2012)
20. Ahmad, B, Ntouyas, SK: An existence theorem for fractional hybrid differential inclusions of Hadamard type with Dirichlet boundary conditions. Abstr. Appl. Anal. **2014**, Article ID 705809 (2014)
21. Dhage, BC, Ntouyas, SK: Existence results for boundary value problems for fractional hybrid differential inclusions. Topol. Methods Nonlinear Anal. **44**, 229-238 (2014)
22. Zhao, Y, Wang, Y: Existence of solutions to boundary value problem of a class of nonlinear fractional differential equations. Adv. Differ. Equ. **2014**, 174 (2014)
23. Ahmad, B, Ntouyas, SK, Alsaedi, A: Existence results for a system of coupled hybrid fractional differential equations. Sci. World J. **2014**, Article ID 426438 (2014)
24. Dhage, BC: Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations. Differ. Equ. Appl. **2**, 465-486 (2010)
25. Dhage, BC: Periodic boundary value problems of first order Carathéodory and discontinuous differential equations. Nonlinear Funct. Anal. Appl. **13**(2), 323-352 (2008)
26. Dhage, BC: Basic results in the theory of hybrid differential equations with mixed perturbations of second type. Funct. Differ. Equ. **19**, 1-20 (2012)
27. Dhage, BC: A fixed point theorem in Banach algebras with applications to functional integral equations. Kyungpook Math. J. **44**, 145-155 (2004)
28. Dhage, BC: A nonlinear alternative with applications to nonlinear perturbed differential equations. Nonlinear Stud. **13**(4), 343-354 (2006)
29. Coffey, WT, Kalmykov, YP, Waldron, JT: The Langevin Equation, 2nd edn. World Scientific, Singapore (2004)