Yuan and Huang *Boundary Value Problems* (2015) 2015:120
DOI 10.1186/s13661-015-0385-6

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Solutions for a degenerate $p(x)$-Laplacian equation with a nonsmooth potential

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Abstract

This paper is concerned with a degenerate $p(x)$-Laplacian equation with a nonsmooth potential. We establish a compact embedding $W^{1,p(x)}(\omega,\Omega) \hookrightarrow L^{q(x)}(\alpha(x),\Omega)$ under suitable conditions and obtain the existence and multiplicity of solutions to the degenerate $p(x)$-Laplacian equation by the theories of nonsmooth critical point and the variable exponent Lebesgue-Sobolev spaces. Some recent results in the literature are generalized and improved.

MSC: 35J20; 35R70; 35J25

Keywords: nonsmooth critical point; locally Lipschitz; nonsmooth dual fountain theorem; weighted variable exponent

1 Introduction

In this paper, we discuss the existence and multiplicity of solutions for the following degenerate $p(x)$-Laplacian equation with a nonsmooth potential (hemivariational inequality):

$$\begin{cases} -\operatorname{div}(\omega(x)|\nabla u|^{p(x)-2}\nabla u) \in \lambda\alpha_1(x)\partial j_1(x,u) + \mu\alpha_2(x)\partial j_2(x,u) & \text{for a.a. } x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2 boundary $\partial\Omega$, $j_1, j_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ are jointly measurable potential functions, which are locally Lipschitz and in general nonsmooth in $u \in \mathbb{R}$, and the following conditions are satisfied:

(P) $p(x) \in C(\bar{\Omega})$, $1 < p^- = \inf_\Omega p(x) \le p^+ = \sup_\Omega p(x) < +\infty$;

(W) ω is a measurable positive and a.a. finite function in Ω. $\omega \in L^1_{\mathrm{loc}}(\Omega)$ and $\omega^{-s(\cdot)} \in L^1(\Omega)$ for some $s \in C(\bar{\Omega})$ such that $s(x) \in (\frac{N}{p(x)}, \infty) \cap [\frac{1}{p(x)-1}, \infty)$ for all $x \in \bar{\Omega}$.

As is well known, the $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian (a constant), for example, it is inhomogeneous and, in general, it does not have the first eigenvalue. In other words, the infimum equals 0 (see [1]). $p(x)$-Laplacian can be found in the areas, the thermistor problem [2], electro-rheological fluids [3], or the problem of image recovery [4]. When ω is not bounded (or not separated from zero) $\omega(x)$ is called degenerate (or singular). A degenerate second order linear differential operator was basically due to Murthy and Stampacchia [5], and higher order linear degenerate elliptic operators were extended in the 1980s, and quasilinear elliptic equations including p-Laplacian were developed in the 1990s (see [6]). Degenerate phenomena appear in the area of oceanography, turbulent fluid flows, electrochemical problems and induction heating. These problems are interesting in applications and raise many difficult mathematical problems. The results can be found in [7–10] and the references therein.

If $\omega(x) = 1$ and $\mu = 0$, then problem (1.1) becomes

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \in \lambda\partial j(x,u), & x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.2}$$

There exist several existence results for problem (1.2). For example, Dai and Liu [11] obtained the existence of three solutions for problem (1.2) by a version of the nonsmooth three critical points theorem. Qian and Shen [12], using the theory of nonsmooth critical point theory, derived the existence and multiplicity of solutions for problem (1.2), where $\lambda = 1$. Ge *et al.* [13], employing a variational method combined with suitable truncation techniques based on nonsmooth critical point theory for locally Lipschitz function, proved the existence of at least five solutions under suitable conditions. It is well known that when $p(x) = p$ (a constant), p-Laplacian differential inclusion has been studied sufficiently (see, *e.g.*, [14–20] and the references therein).

Being influenced by the above results, we want to discuss problem (1.1). To the best of our knowledge, there exist few papers to study problem (1.1). Compared with the previous works, our framework presents new nontrivial difficulties. In particular, there is no compact embedding $W^{1,p(x)}(\omega,\Omega) \hookrightarrow L^{q(x)}(\alpha(x),\Omega)$. To deal with the difficulty, we borrow an idea from the compact embedding theorem $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ (see [21]) to prove the compactness $W^{1,p(x)}(\omega,\Omega) \hookrightarrow L^{q(x)}(\alpha(x),\Omega)$ under suitable assumptions.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on the weighted variable exponent Sobolev space and nonsmooth critical point theory. In Section 3, in order to discuss problem (1.1), we firstly prove a compact embedding theorem for the weighted variable exponent Sobolev space, which plays an important role in this section. Then, based on this theorem, combining the nonsmooth fountain theorem, nonsmooth dual fountain theorem, Weierstrass theorem and nonsmooth pass mountain theorem, we obtain the existence and multiplicity results for problem (1.1).

2 Preliminaries

In this section we state some definitions and lemmas, which will be used throughout this paper. First of all, we give some definitions: $(X, \|\cdot\|)$ will denote a (real) Banach space and $(X^*, \|\cdot\|_*)$ its topological dual. While $u_n \to u$ (respectively, $u_n \rightharpoonup u$) in X means that the sequence $\{u_n\}$ converges strongly (respectively, weakly) in X. $h^- = \inf_{x\in\Omega} h(x)$ and $h^+ = \inf_{x\in\Omega} h(x)$.

We define the weighted variable exponent Lebesgue-Sobolev spaces and list some properties of these spaces. Since the variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ were thoroughly studied in [22–25], we only introduce the weighted variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\alpha(x),\Omega)$ and $W^{1,p(x)}(\omega,\Omega)$.

Set

$$C_+(\bar{\Omega}) = \Big\{h \in C(\bar{\Omega}) : \min_{x\in\bar{\Omega}} h(x) > 1\Big\}.$$

Denote by $S(\Omega)$ the set of all measurable real functions defined on Ω. For any $p \in C_+(\bar{\Omega})$ and $\alpha(x) \in S(\Omega)$, we define the variable weighted exponent Lebesgue space by

$$L^{p(x)}(\alpha,\Omega) = \left\{ u \in S(\Omega) : \int_\Omega \alpha(x)\left|u(x)\right|^{p(x)}\,\mathrm{d}x < \infty \right\}$$

with the norm

$$|u|_{L^{p(x)}(\alpha,\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \alpha(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)}\,\mathrm{d}x \le 1 \right\},$$

then $L^{p(x)}(\alpha,\Omega)$ is a Banach space. When $\alpha(x) \equiv 1$, we have $L^{p(x)}(\alpha,\Omega) \equiv L^{p(x)}(\Omega)$. The weighted variable exponent Sobolev space $W^{1,p(x)}(\omega,\Omega)$ is defined by

$$W^{1,p(x)}(\omega,\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\omega,\Omega) \right\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\omega,\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\omega,\Omega)}$$

or equivalently

$$\|u\|_{W^{1,p(x)}(\omega,\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \left(\left|\frac{u(x)}{\lambda}\right|^{p(x)} + \omega(x)\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} \right)\mathrm{d}x \le 1 \right\}$$

for all $u \in W^{1,p(x)}(\omega,\Omega)$. $W_0^{1,p(x)}(\omega,\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\omega,\Omega)}$. The following Hölder type inequality is useful for the next section.

Proposition 2.1 ([23, 25]) *The space $L^{p(x)}(\Omega)$ is a separable, uniform Banach space, and its conjugate space is $L^{p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left|\int_\Omega uv\,\mathrm{d}x\right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{L^{p(x)}(\Omega)}|v|_{L^{p'(x)}(\Omega)} \le 2|u|_{L^{p(x)}(\Omega)}|v|_{L^{p'(x)}(\Omega)}.$$

Proposition 2.2 ([24, 25]) *Set $\rho(u) = \int_\Omega \alpha(x)|u(x)|^{p(x)}\,\mathrm{d}x$. For $u, u_k \in L^{p(x)}(\alpha(x),\Omega)$, we have*

(i) *For $u \neq 0$, $|u|_{L^{p(x)}(\alpha(x),\Omega)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$;*
(ii) $|u|_{L^{p(x)}(\alpha(x),\Omega)} < 1 (= 1, > 1) \Leftrightarrow \rho(u) < 1 (= 1, > 1)$;
(iii) *If $|u|_{L^{p(x)}(\alpha(x),\Omega)} > 1$, then $|u|^{p^-}_{L^{p(x)}(\alpha(x),\Omega)} \le \rho(u) \le |u|^{p^+}_{L^{p(x)}(\alpha(x),\Omega)}$;*
(iv) *If $|u|_{L^{p(x)}(\alpha(x),\Omega)} < 1$, then $|u|^{p^+}_{L^{p(x)}(\alpha(x),\Omega)} \le \rho(u) \le |u|^{p^-}_{L^{p(x)}(\alpha(x),\Omega)}$;*
(v) $\lim_{k\to\infty} |u_k|_{L^{p(x)}(\alpha(x),\Omega)} = 0 \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = 0$;
(vi) $|u_k|_{L^{p(x)}(\alpha(x),\Omega)} \to \infty \Leftrightarrow \rho(u_k) \to \infty$.

If (P) and (W) hold, from [10], we have the following propositions.

Proposition 2.3 *Assume that* (P) *and* (W) *hold, then $W^{1,p(x)}(\omega,\Omega)$ and $W_0^{1,p(x)}(\omega,\Omega)$ are reflexive Banach spaces.*

To obtain a crucial embedding result which will be used in the later section, let us denote

$$p_s = \frac{p(x)s(x)}{1+s(x)} < p(x),$$

where $s(x)$ is given in hypothesis (W) and

$$p_s^* = \begin{cases} \frac{p(x)s(x)N}{(s(x)+1)N-p(x)s(x)} & \text{if } p_s(x) < N, \\ +\infty & \text{if } p_s(x) \geq N, \end{cases}$$

for all $x \in \bar{\Omega}$.

The following compact embedding theorem is very important in this paper.

Proposition 2.4 ([10]) *Assume that hypotheses* (P) *and* (W) *hold*, $q \in C_+(\bar{\Omega})$ *and* $1 \leq q(x) < p_s^*(x)$ *for all* $x \in \bar{\Omega}$, *then we have the continuous compact embedding*

$$W^{1,p(x)}(\omega, \Omega) \hookrightarrow L^{q(x)}(\Omega).$$

Furthermore, we also have the following Poincaré inequality type.

Proposition 2.5 ([10]) *If* (P) *and* (W) *hold, then the estimate*

$$|u|_{L^{p(x)}(\Omega)} \leq C|\nabla u|_{L^{p(x)}(\omega,\Omega)}$$

holds for all $u \in C_0^\infty(\Omega)$ *with a positive constant* C *independent of* u.

Let $X = W_0^{1,p(x)}(\Omega)$. We say that u is a weak solution of problem (1.1) if $u \in X$ and

$$\int_\Omega \omega(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \, \mathrm{d}x - \lambda \int_\Omega \alpha_1(x)\xi_1 v \, \mathrm{d}x - \mu \int_\Omega \alpha_2(x)\xi_2 v \, \mathrm{d}x = 0$$

for all $v \in X$, $\xi_1 \in \partial j_1(x,u)$ and $\xi_2 \in \partial j_2(x,u)$ a.a. on Ω. We write $A : X \to X^*$

$$\langle A(u), v\rangle = \int_\Omega \omega(x)|\nabla u|^{p(x)-2}\nabla u \nabla v \, \mathrm{d}x, \quad \forall u, v \in X.$$

Lemma 2.1

(i) $A : X \to X^*$ *is a continuous, bounded and strict monotone operator.*

(ii) A *is a mapping of type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *in* X *and* $\overline{\lim}_{n\to\infty}\langle A(u_n) - A(u), u_n - u\rangle \leq 0$, *then* $u_n \to u$ *in* X.

Remark 2.1 The proof is similar to that in [26] (see Theorem 3.1). Here we omit its proof.

Seeking a weak solution of problem (1.1) is equivalent to finding a critical point of the energy function $I : X \to \mathbb{R}$ for problem (1.1) defined by

$$I(u) = \int_\Omega \frac{\omega(x)}{p(x)}|\nabla u|^{p(x)} \, \mathrm{d}x - \lambda \int_\Omega \alpha_1(x) j_1(x,u) \, \mathrm{d}x - \mu \int_\Omega \alpha_2(x) j_2(x,u) \, \mathrm{d}x, \quad \forall u \in X. \tag{2.1}$$

Since I is Lipschitz continuous on bounded sets, hence it is locally Lipschitz (see [21], p.83).

Definition 2.1 A function I: $X \to \mathbb{R}$ is locally Lipschitz if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that for every $v, \eta \in U$,

$$\bigl|I(v) - I(\eta)\bigr| \le L\|v - \eta\|.$$

Definition 2.2 Let $I : X \to \mathbb{R}$ be a locally Lipschitz functional, $u, v \in X$: the generalized derivative of I in u along the direction v,

$$I^0(u;v) = \limsup_{\eta \to u, \tau \to 0^+} \frac{I(\eta + \tau v) - I(\eta)}{\tau}.$$

It is easy to see that the function $v \mapsto I^0(u;v)$ is sublinear, continuous and so is the support function of a nonempty, convex and ω^*-compact set $\partial I(u) \subset X^*$ defined by

$$\partial I(u) = \bigl\{u^* \in X^* : \langle u^*, v\rangle_X \le I^0(u;v) \text{ for all } v \in X\bigr\}.$$

If $I \in C^1(X)$, then

$$\partial I(u) = \bigl\{I'(u)\bigr\}.$$

Clearly, these definitions extend those of the Gâteaux directional derivative and gradient.

A point $u \in X$ is a critical point of I if $0 \in \partial I(u)$. It is easy to see that if $u \in X$ is a local minimum of I, then $0 \in \partial I(u)$. For more on locally Lipschitz functionals and their subdifferential calculus, we refer the reader to Clarke [21].

Lemma 2.2 ([21])

(i) $(-h)^\circ(u;z) = h^\circ(u;-z)$ *for all* $u, z \in X$;

(ii) $h^\circ(u;z) = \max\{\langle u^*, z\rangle_X : u^* \in \partial h(u)\}$ *for all* $u, z \in X$;

(iii) *Let* $j : X \to \mathbb{R}$ *be a continuously differentiable function. Then* $\partial j(u) = \{j'(u)\}$, $j^\circ(u;z)$ *coincides with* $\langle j'(u), z\rangle_X$ *and* $(h + j)^\circ(u;z) = h^\circ(u;z) + \langle j'(u), z\rangle_X$ *for all* $u, z \in X$;

(iv) (*Lebourg's mean value theorem*) *Let* u *and* v *be two points in* X. *Then there exists a point* ξ *in the open segment between* u *and* v, *and* $u_\xi^* \in \partial h(\omega)$ *such that*

$$h(u) - h(v) = \langle u_\xi^*, u - v\rangle_X;$$

(v) (*Second chain rule*) *Let* Y *be a Banach space and* $j : Y \to X$ *be a continuously differentiable function. Then* $h \circ j$ *is locally Lipschitz and*

$$\partial(h \circ j)(y) \subseteq \partial h\bigl(j(y)\bigr) \circ j'(y) \quad \textit{for all } y \in Y;$$

(vi) $m^I(u) = \inf_{u^* \in \partial I(u)} \|u^*\|_{X^*}$ *is lower semicontinuous.*

In the following, we introduce a nonsmooth version of the fountain theorem which was proved by Dai in [27].

Definition 2.3 Assume that the compact group G acts diagonally on V^k

$$g(v_1,\dots,v_k) = (gv_1,\dots,gv_k),$$

where V is a finite dimensional space. The action of G is admissible if every continuous equivariant map $\partial U \to V^{k-1}$, where U is an open bounded invariant neighborhood of 0 in V^k, $k \geq 2$, has a zero.

Example 2.1 The antipodal action $G = \mathbb{Z}$ on $V = \mathbb{R}$ is admissible.

We consider the following situation:

(A_1) The compact group G acts isometrically on the Banach space $X = \overline{\bigoplus_{m\in\mathbb{N}} X_m}$, the space X_m is invariant and there exists a finite dimensional space V such that, for every $m \in \mathbb{N}$, $X_m \simeq V$ and the action of G on V is admissible.

In this paper, we will use the following notations:

$$Y_k = \bigoplus_{m=0}^{k} X_m, \qquad Z_k = \overline{\bigoplus_{m=k}^{\infty} X_m},$$
$$B_k = \{u \in Y_k : \|u\| \leq \rho_k\}, \qquad N_k = \{u \in Z_k : \|u\| = r_k\},$$

where $\rho_k > r_k > 0$.

Definition 2.4 (i) We say that I satisfies the nonsmooth $(\mathrm{PS})_c$ if any sequence $\{u_n\} \subset X$, such that

$$I(u_n) \to c \quad \text{and} \quad m^I(u_n) \to 0 \quad \text{as } n \to +\infty,$$

has a strongly convergent subsequence, where $m^I(u_n) = \inf_{u_n^* \in \partial I(u_n)} \|u_n^*\|_{X^*}$.

(ii) We say that I satisfies the nonsmooth C-condition if any sequence $\{u_n\} \subset X$, such that

$$I(u_n) \to c \quad \text{and} \quad \bigl(1 + \|u_n\|_X\bigr) m^I(u_n) \to 0,$$

has a strongly convergent subsequence.

(iii) We say that I satisfies the nonsmooth $(\mathrm{PS})_c^*$ means that any sequence $\{u_{n_j}\} \subset X$, such that

$$n_j \to \infty, \qquad u_{n_j} \in Y_{n_j}, \qquad I(u_{n_j}) \to c \quad \text{and} \quad m^{I|_{Y_{n_j}}}(u_{n_j}) \to 0,$$

has a strongly convergent subsequence converging to a critical point of I.

Remark 2.2 (i) The nonsmooth C-condition is slightly weaker than the nonsmooth $(\mathrm{PS})_c$, while it retains the most important implications of the nonsmooth $(\mathrm{PS})_c$.

(ii) The $(\mathrm{PS})_c^*$ means the $(\mathrm{PS})_c$. Assume that $\{u_j\} \subset X$ such that

$$I(u_j) \to c, \qquad m^I(u_j) \to 0.$$

Then there exist sequences $\{v_{n_j}\}$, $\{n_j\}$ such that

$$n_j \to \infty, \qquad v_{n_j} \in Y_{n_j}, \qquad v_{n_j} - u_j \to 0,$$
$$I(v_{n_j}) - I(u_j) \to 0, \qquad m^I(v_{n_j}) - m^I(u_j) \to 0.$$

From $(\mathrm{PS})_c^*$, the sequence $\{v_{n_j}\}$ contains a convergent subsequence and hence $\{u_j\}$ includes also a convergent subsequence.

Theorem 2.1 *Under hypothesis* (A_1), *let* $I : X \to \mathbb{R}$ *be an invariant locally Lipschitz functional. If for every* $k \in \mathbb{N}$ *there exists* $\rho_k > r_k > 0$ *such that*

(A_2) $a_k = \max_{u\in Y_k,\|u\|=\rho_k} I(u) \leq 0$;
(A_3) $b_k = \inf_{u\in Z_k,\|u\|=r_k} I(u) \to \infty$, $k \to \infty$;
(A_4) *I satisfies the nonsmooth* $(\mathrm{PS})_c$ *for every* $c > 0$,

then I has an unbounded sequence of critical values.

Now we will give the nonsmooth dual fountain theorem, which was firstly proved by Dai *et al.* in [28].

Theorem 2.2 *Under hypothesis* (A_1), *let* $I : X \to \mathbb{R}$ *be an invariant locally Lipschitz functional. If, for every* $k \geq k_0$, *there exist* $\rho_k > r_k > 0$ *such that*

(A_2') $a_k = \inf_{u\in Z_k,\|u\|=\rho_k} I(u) \geq 0$,
(A_3') $b_k = \max_{u\in Y_k,\|u\|=r_k} I(u) < 0$,
(A_4') $d_k = \inf_{u\in Z_k,\|u\|\leq\rho_k} I(u) \to 0$, $k \to \infty$,
(A_5) *I satisfies the nonsmooth* $(\mathrm{PS})_c^*$ *for every* $c \in [d_{k_0}, 0)$,

then I has a sequence of negative critical values converging to 0.

The next theorem is the nonsmooth version of the classical mountain pass theorem.

Theorem 2.3 ([29]) *If there exist* $u_1 \in X$ *and* $r > 0$ *such that* $\|u_1\| > r$,

$$\max\{I(0), I(u_1)\} \leq \inf_{\|u\|=r} I(u)$$

and I satisfies the nonsmooth C-condition with

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = 0, \gamma(1) = u_1\}$, *then* $c \geq \inf_{\|u\|=r} I(u)$ *and c is a critical value of I. Moreover, if* $c = \inf\{I(u) : \|u\| = r\}$, *then there exists a critical point* u_0 *of I with* $I(u_0) = c$ *and* $\|u_0\| = r$ *(i.e.,* $K_c^I \cap \partial B_r \neq \emptyset$*).*

3 Existence and multiplicity of solutions

In this section, we let $X = W_0^{1,p(x)}(\omega, \Omega)$. For $u \in X$, we define an equivalent norm $\|u\| = |\nabla u|_{L^{p(x)}(\omega,\Omega)}$ due to Proposition 2.5. In order to discuss problem (1.1), we need the following hypotheses:

(H_1) For $i = 1, 2$, $\alpha_i \in L^{r_i(x)}(\Omega)$, $\alpha_i(x) > 0$, $|\xi_i| \le c_1 + c_2|u|^{q_i(x)-1}$ for a.a. $x \in \Omega$, $\forall \xi_i \in \partial j_i(x, u)$ and $u \in \mathbb{R}$, where c_1, c_2 are positive constants, $r_i, q_i \in C(\bar{\Omega})$, $r_i^- > 1$, $q_i^- > 1$, and $q_i(x) < \frac{r_i(x)-1}{r_i(x)} p_s^*(x)$;

(H_2) $q_1^+ < p^-$;

(H_3) $q_2^- > p^+$;

(H_4) There exists $a_1 > 0$ such that $j_2(x, u) \ge -a_1$ for a.a. $x \in \Omega$ and $u \in \mathbb{R}$;

(H_5) There exist $\theta > p^+$, $M > 0$ and $|u| \ge M$ for a.a. $x \in \Omega$ such that $0 < \theta j_2 \le u\xi_2$, where $\xi_2 \in \partial j_2(x, u)$;

(H_6) There exist $\delta_1 > 0$, $a_2, a_3 > 0$, $1 \le q_3(x), q_4(x) < \frac{r_1(x)-1}{r_1(x)} p_s^*(x)$ and $q_3^+, q_4^- < p^-$ such that

$$\frac{a_2 u^{q_3(x)}}{q_3(x)} \le j_1(x, u) \le \frac{a_3 u^{q_4(x)}}{q_4(x)} \quad \text{for a.a. } x \in \Omega, \forall u \in (0, \delta_1);$$

(H_7) There exist $\delta_2 > 0$, $a_4 > 0$, $q_5(x) \in C(\bar{\Omega})$, $q_5^- > p^+$ and $q_5(x) < \frac{r_2(x)-1}{r_2(x)} p_s^*(x)$ such that

$$\left|j_2(x, u)\right| \le a_4 |u|^{q_5(x)} \quad \text{for a.a. } x \in \Omega, \forall |u| \le \delta_2;$$

(H_8) There exist $\delta_3 > 0$, $a_5 > 0$, $q_6(x) \in C(\bar{\Omega})$, $1 \le q_6^- \le q_6^+ < p^-$ and $q_6(x) < \frac{r_1(x)-1}{r_1(x)} p_s^*(x)$ such that

$$j_1(x, u) \ge a_5 u^{q_6(x)} \quad \text{for a.a. } x \in \Omega, \forall u \in (0, \delta_3);$$

(H_9) For a.a. $x \in \Omega$, $i = 1, 2$, all $u \in \mathbb{R}$, $j_i(x, -u) = j_i(x, u)$.

In order to discuss the existence and multiplicity solutions for problem (1.1), we need the following lemma.

Lemma 3.1 *If* $p(x) \in C_+(\bar{\Omega})$, $\alpha(x) \in L^{r(x)}(\Omega)$, $\alpha(x) > 0$ *for a.a.* $x \in \Omega$, $r \in C(\bar{\Omega})$ *and* $r^- > 1$, $q(x) \in C(\bar{\Omega})$ *and*

$$1 \le q(x) < \frac{r(x) - 1}{r(x)} p_s^*(x) \quad \textit{for a.a. } x \in \bar{\Omega}, \tag{3.1}$$

then there exists a compact embedding $W^{1,p(x)}(\omega, \Omega) \hookrightarrow L^{q(x)}(\alpha(x), \Omega)$.

Proof Set $u(x) \in W^{1,p(x)}(\omega, \Omega)$, $h(x) = \frac{r(x)}{r(x)-1}$ and $\beta(x) = h(x)q(x)$. From (3.1), it is easy to see $\beta(x) < p_s^*(x)$. By virtue of Proposition 2.4, we have $W^{1,p(x)}(\omega, \Omega) \hookrightarrow L^{\beta(x)}(\Omega)$. For $u \in W^{1,p(x)}(\omega, \Omega)$, we have $|u(x)|^{q(x)} \in L^{h(x)}(\Omega)$ and from Proposition 2.1,

$$\int_\Omega \alpha(x)\left|u(x)\right|^{q(x)} \,\mathrm{d}x \le 2|\alpha|_{r(x)}\left|\left|u(x)\right|^{q(x)}\right|_{h(x)} < \infty.$$

This means that $W^{1,p(x)}(\omega, \Omega) \subset L^{q(x)}(\alpha(x), \Omega)$. Now set $\{u_n\} \subset W^{1,p(x)}(\omega, \Omega)$ and $u_n \rightharpoonup 0$ in $W^{1,p(x)}(\omega, \Omega)$. Then $u_n \to 0$ in $L^{\beta(x)}(\Omega)$ and from this we have $||u(x)|^{q(x)}|_{h(x)} \to 0$. Hence, we obtain

$$\int_\Omega \alpha(x)\left|u_n(x)\right|^{q(x)} \,\mathrm{d}x \le 2\left|\alpha(x)\right|_{r(x)}\left|\left|u_n(x)\right|^{q(x)}\right|_{h(x)} \to 0,$$

which means $|u_n(x)|_{L^{q(x)}(\alpha(x),\Omega)} \to 0$. This means that the embedding $W^{1,p(x)}(\omega,\Omega) \hookrightarrow L^{q(x)}(\alpha(x),\Omega)$ is compact. Thus, we complete the proof. □

The following lemma is very important when we use the nonsmooth fountain and dual fountain theorems to prove infinite solutions for problem (1.1).

Lemma 3.2 *If* $1 \le q(x) < \frac{r(x)-1}{r(x)} p_s^*(x)$, $\alpha(x) \in L^{r(x)}(\Omega)$, $\alpha(x) > 0$, $r \in C(\bar{\Omega})$ *and* $r^- > 1$, *then we have*

$$\beta_k = \sup_{u \in Z_k, \|u\|=1} |u|_{L^{q(x)}(\alpha(x),\Omega)} \to 0 \quad \text{as } k \to \infty.$$

Proof It is clear that $0 < \beta_{k+1} \le \beta_k$, so there exists $\beta \ge 0$ such that $\beta_k \to \beta$ as $k \to \infty$. We will show $\beta = 0$. From the definition of β_k, for every $k \ge 0$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $0 \le \beta - |u|_{L^{q(x)}(\alpha(x),\Omega)} \le \frac{1}{k}$. Then there exists a subsequence of $\{u_k\}$, for convenience we still denote it by u_k, such that $u_k \rightharpoonup u$ in $W^{1,p(x)}(\omega,\Omega)$, and

$$\langle e_j^*, u \rangle = \lim_{k\to\infty} \langle e_j^*, u_k \rangle = 0, \quad j = 1, 2, \ldots,$$

which means that $u = 0$ and $u_k \rightharpoonup 0$ in $W^{1,p(x)}(\omega,\Omega)$. Note that $W^{1,p(x)}(\omega,\Omega) \hookrightarrow L^{q(x)}(\alpha(x),\Omega)$ is compact, then $u_k \to 0$ in $L^{q(x)}(\alpha(x),\Omega)$. Hence we obtain that $\beta = 0$. □

Next, we will use the nonsmooth fountain theorem to prove the existence of infinitely many large energy solutions for problem (1.1).

Theorem 3.1 *If hypotheses* (P), (W), (H_1)-(H_3), (H_5) *and* (H_9) *are satisfied, for all* $\mu > 0$ *and* $\lambda \in \mathbb{R}$, *problem* (1.1) *has a sequence of solutions* $\{\pm u_k\}$ *such that* $I(\pm u_k) \to \infty$ *as* $k \to \infty$.

Proof We choose an orthonormal basis (e_j) of X and set $X_j = \mathbb{R}e_j$. On X we consider the antipodal action of $\mathbb{Z}_2$. We have known that I is locally Lipschitz on X. Considering (H_9), we can employ the nonsmooth version of fountain theorem to prove Theorem 3.1.

Claim 1. I satisfies the nonsmooth $(\mathrm{PS})_c$.

Let $\{u_n\}_{n\ge1} \subset X$ be a sequence such that

$$\bigl|I(u_n)\bigr| \le c_3 \quad \text{for all } n \ge 1 \quad \text{and} \quad m^I(u_n) \to 0 \quad \text{as } n \to \infty \tag{3.2}$$

for some $c_3 > 0$. We assume that $\{u_n\}_{n\ge1}$ is unbounded in X, then up to a subsequence $\|u_n\| \to \infty$ as $n \to \infty$. From (3.2), for n large enough, we have

$$\begin{aligned} -\langle u_n^*, u_n \rangle &= -\int_\Omega \omega(x) \bigl|\nabla u_n(x)\bigr|^{p(x)} \,\mathrm{d}x + \lambda \int_\Omega \alpha_1(x) \xi_{1,n}(x) u_n(x) \,\mathrm{d}x \\ &\quad + \mu \int_\Omega \alpha_2(x) \xi_{2,n}(x) u_n(x) \,\mathrm{d}x \\ &\le \varepsilon_n \|u_n\|, \end{aligned} \tag{3.3}$$

and

$$\theta \int_\Omega \frac{\omega(x)}{p(x)} \left|\nabla u_n(x)\right|^{p(x)} \,\mathrm{d}x - \theta\lambda \int_\Omega \alpha_1(x) j_1(x,u_n)\,\mathrm{d}x \\ - \theta\mu \int_\Omega \alpha_2(x) j_2(x,u_n)\,\mathrm{d}x \le \theta c_3, \tag{3.4}$$

where $\varepsilon_n \to 0$, $u_n^* \in \partial I(u_n)$, $\xi_{1,n} \in \partial j_1(x,u_n)$, $\xi_{2,n} \in \partial j_2(x,u_n)$ for a.a. $x \in \Omega$. Adding (3.3) and (3.4), from Proposition 2.2, (H_1), (H_5), Lemma 3.1 and Lebourg's mean value theorem, we have

$$\begin{aligned}
&\varepsilon_n \|u_n\| + \theta c_3 \\
&\quad \ge \int_\Omega \left(\frac{\theta}{p(x)} - 1\right) \omega(x) |\nabla u_n|^{p(x)} \,\mathrm{d}x + \lambda \int_\Omega \alpha_1(x)\bigl(\xi_{1,n} u_n - \theta j_1(x,u_n)\bigr)\,\mathrm{d}x \\
&\qquad + \mu \int_\Omega \alpha_2(x)\bigl(\xi_{2,n} u_n - \theta j_2(x,u_n)\bigr)\,\mathrm{d}x \\
&\quad \ge \int_\Omega \left(\frac{\theta}{p^+} - 1\right) \omega(x) |\nabla u_n|^{p(x)} \,\mathrm{d}x + \lambda \int_\Omega \alpha_1(x)\bigl(\xi_{1,n} u_n - \theta j_1(x,u_n)\bigr)\,\mathrm{d}x \\
&\qquad + \mu \int_{\{x\in\Omega:|u_n|\le M\}} \alpha_2(x)\bigl(\xi_{2,n} u_n - \theta j_2(x,u_n)\bigr)\,\mathrm{d}x \\
&\qquad + \mu \int_{\{x\in\Omega:|u_n|> M\}} \alpha_2(x)\bigl(\xi_{2,n} u_n - \theta j_2(x,u_n)\bigr)\,\mathrm{d}x \\
&\quad \ge \left(\frac{\theta}{p^+} - 1\right) \|u_n\|^{p^-} + \lambda \int_\Omega \alpha_1(x)\bigl(\xi_{1,n} u_n - \theta j_1(x,u_n)\bigr)\,\mathrm{d}x \\
&\qquad + \mu \int_{\{x\in\Omega:|u_n|\le M\}} \alpha_2(x)\bigl(\xi_{2,n} u_n - \theta j_2(x,u_n)\bigr)\,\mathrm{d}x \\
&\quad \ge \left(\frac{\theta}{p^+} - 1\right) \|u_n\|^{p^-} - \lambda \int_\Omega \alpha_1(x)\bigl(b_1 + b_2 |u_n|^{q_1(x)}\bigr)\,\mathrm{d}x - c_5 \\
&\quad \ge \left(\frac{\theta}{p^+} - 1\right) \|u_n\|^{p^-} - \lambda b_1 \int_\Omega \alpha_1(x)\,\mathrm{d}x - \lambda b_2 |u_n|_{L^{q_1^+}(\alpha(x),\Omega)} - c_5 \\
&\quad \ge \left(\frac{\theta}{p^+} - 1\right) \|u_n\|^{p^-} - \lambda c_4 \|u_n\|^{q_1^+} - b_3
\end{aligned} \tag{3.5}$$

for some $b_1, b_2, b_3, c_4, c_5 > 0$. Note that $1 < q_1^+ < p^-$, taking $n \to \infty$ in the inequality above, we derive a contradiction. Therefore $\{u_n\}_{n\ge1}$ is bounded in X. Thus, by passing to a subsequence if necessary, we assume that

$$u_n \rightharpoonup u \quad \text{in } X, \qquad u_n \to u \quad \text{in } L^{q(x)}\bigl(\alpha(x),\Omega\bigr) \tag{3.6}$$

$\alpha(x)$ and $q(x)$ are defined by Lemma 3.1. So we have

$$\left|\bigl\langle A(u_n), u_n - u\bigr\rangle - \lambda \int_\Omega \alpha_1(x)\xi_{1,n}(u_n - u)\,\mathrm{d}x - \mu \int_\Omega \alpha_2(x)\xi_{2,n}(u_n - u)\,\mathrm{d}x\right| \\ \le \varepsilon_n \|u_n - u\|, \tag{3.7}$$

where $\xi_{i,n} \in \partial j_i(x, u_n)$ $(i = 1, 2)$, $\varepsilon_n \to 0$. From (H_1), Propositions 2.1, 2.2 and the definition of $L^{p(x)}(\alpha(x), \Omega)$, we obtain

$$\begin{aligned}
&\left|\int_\Omega \alpha_1(x)\xi_{1,n}(u_n - u)\,\mathrm{d}x\right| \\
&\quad\le \int_\Omega \alpha_1(x)\big(c_1 + c_2|u_n|^{q_1(x)-1}\big)|u_n - u|\,\mathrm{d}x \\
&\quad\le c_1|u_n - u|_{L^1(\alpha_1(x),\Omega)} + c_2\int_\Omega \big(\alpha_1^{\frac{1}{q_1(x)}}(x)|u_n|\big)^{q_1(x)-1}\big(\alpha_1^{\frac{1}{q_1(x)}}(x)|u_n - u|\big)\,\mathrm{d}x \\
&\quad\le c_1|u_n - u|_{L^1(\alpha_1(x),\Omega)} + 2c_2\left(\int_\Omega \alpha_1(x)|u_n|^{q_1(x)}\,\mathrm{d}x\right)^{\frac{q_1(x)-1}{q_1(x)}} \\
&\qquad\times\left(\int_\Omega \alpha_1(x)|u_n - u|^{q_1(x)}\,\mathrm{d}x\right)^{\frac{1}{q_1(x)}} \\
&\quad\le c_1|u_n - u|_{L^1(\alpha_1(x),\Omega)} + 2c_2\big(|u_n|^{q_1^+-1}_{L^{q_1(x)}(\alpha_1(x),\Omega)} + 1\big)|u_n - u|_{L^{q_1(x)}(\alpha_1(x),\Omega)}.
\end{aligned}$$

Noting that $q_1^- > 1$, from Lemma 3.1 and (3.6) we infer that

$$\lim_{n\to\infty}\int_\Omega \alpha_1(x)\xi_{1,n}(x)(u_n - u)\,\mathrm{d}x = 0. \tag{3.8}$$

In a similar way, we have

$$\lim_{n\to\infty}\int_\Omega \alpha_2(x)\xi_{2,n}(x)(u_n - u)\,\mathrm{d}x = 0. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we have

$$\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \le 0.$$

From Lemma 2.1, we know that A is a mapping of type (S_+). Hence we have

$$u_n \to u \quad \text{in } X.$$

In what follows, let us verify conditions (A_2) and (A_3) of the nonsmooth fountain theorem (see Theorem 2.1). From hypotheses (H_1) and (H_5), we can obtain

$$j_2(x, u) \ge c_6|u|^\theta - c_7 \quad \text{for some } c_6, c_7 > 0. \tag{3.10}$$

The proof of (3.10) can be found in [20] (Theorem 10). For $\forall u \in Y_k$, by virtue of (H_1), Lemma 3.1, Lebourg's mean value theorem and (3.10), we obtain

$$\begin{aligned}
I(u) &\le \int_\Omega \frac{\omega(x)}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x - \lambda\int_\Omega \alpha_1(x)j_1(x,u)\,\mathrm{d}x - \mu c_6\int_\Omega \alpha_2(x)|u|^\theta\,\mathrm{d}x \\
&\quad + \mu c_7\int_\Omega \alpha_2(x)\,\mathrm{d}x \\
&\le \frac{1}{p^-}\int_\Omega \omega(x)|\nabla u|^{p(x)}\,\mathrm{d}x + |\lambda|\int_\Omega \alpha_1(x)d_1|u|^{q_1(x)}\,\mathrm{d}x \\
&\quad - \mu c_6\int_\Omega \alpha_2(x)|u|^\theta\,\mathrm{d}x + c_8
\end{aligned} \tag{3.11}$$

for some $c_8, d_1 > 0$. Since $q_1^+ < p^- \leq p^+ < \theta$ and all norms on a finite dimensional space Y_k are equivalent, from (3.11), we can choose $\rho_k > 0$ large enough such that $a_k = \max_{u \in Y_k, \|u\| = \rho_k} I(u) \leq 0$, *i.e.*, relation (A_2) is satisfied.

In view of (H_1)-(H_3), Lemma 3.1, Lebourg's mean value theorem and Lemma 3.2, for $\forall u \in Z_k$, $|u| \geq 1$ large enough, we have

$$\begin{aligned} I(u) &\geq \int_\Omega \frac{\omega(x)}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x - \lambda \int_\Omega \alpha_1(x)\bigl(\nu_1 |u|^{q_1(x)} + c_9\bigr)\,\mathrm{d}x \\ &\quad - \mu \int_\Omega \alpha_2(x)\bigl(\nu_2 |u|^{q_2(x)} + c_{10}\bigr)\,\mathrm{d}x \\ &\geq \int_\Omega \frac{\omega(x)}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x - \int_\Omega \bigl(|\lambda| \nu_1 \alpha_1(x) + \mu \nu_2 \alpha_2(x)\bigr) |u|^{q_2(x)}\,\mathrm{d}x - c_{11} \\ &\geq \frac{1}{p^+} \|u\|^{p^-} - c_{12} \beta_k^{q_2^-} \|u\|^{q_2^+} - c_{11}, \end{aligned}$$

where ν_1, ν_2, c_{11} and c_{12} are some positive constants. Choosing $r_k = (c_{12} q_2^- \beta_k^{q_2^-})^{\frac{1}{p^- - q_2^+}}$, for $u \in Z_k$ and $\|u\| = r_k$, then

$$I(u) \geq \left(\frac{1}{p^+} - \frac{1}{q_2^-}\right) \bigl(c_{12} q_2^- \beta_k^{q_2^-}\bigr)^{\frac{p^-}{p^- - q_2^+}} - c_{11}.$$

Since $1 < p^- \leq p^+ < q_2^-$ and $\beta_k \to 0$ as $k \to \infty$, we obtain

$$b_k = \inf_{u \in Z_k, \|u\| = r_k} I(u) \to +\infty.$$

Hence, from the nonsmooth fountain theorem, we obtain that problem (1.1) has a sequence of solutions $\{\pm u_k\}$ such that $I(\pm u_k) \to \infty$ as $k \to \infty$. The proof of Theorem 3.1 is completed. □

In the following, we will use the nonsmooth dual fountain theorem to prove the existence of infinitely small energy solutions for problem (1.1).

Theorem 3.2 *If hypotheses* (P), (W) (H_1)-(H_3), (H_5)-(H_7) *and* (H_9) *hold, then for every* $\lambda > 0$ *and* $\mu \in \mathbb{R}$, *problem* (1.1) *has a sequence of solutions* $\{\pm v_k\}$ *such that* $I(\pm v_k) < 0$ *and* $I(\pm v_k) \to 0$ *as* $k \to \infty$.

Proof Let us verify all the conditions of the nonsmooth dual fountain theorem. From Theorem 3.1, we know that I is locally Lipschitz and even functional. Choosing an orthonormal basis (e_j) of X and setting $X_j = \mathbb{R} e_j$ on X, we consider the antipodal action of $\mathbb{Z}_2$ on X.

In order to verify (A_2'), set $1 > R > 0$ such that

$$\begin{aligned} &\|u\| \leq R, \qquad |u|_{L^{q_4(x)}(\alpha_1(x), \Omega)} < 1, \\ &|u|_{L^{q_5(x)}(\alpha_2(x), \Omega)} < 1 \quad \text{and} \quad |\mu| a_4 c_{13} \|u\|^{q_5^-} \leq \frac{1}{2p^+} \|u\|^{p^+}, \end{aligned}$$

c_{13} is some positive constant. Hence, from (H_6) and (H_7), for $u \in Z_k$, $u \in (0, \min\{\delta_1, \delta_2\})$, $\|u\| \le R$ and k large enough, we derive

$$\begin{aligned}
I(u) &= \int_\Omega \frac{\omega(x)}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x - \lambda \int_\Omega \alpha_1(x) j_1(x,u)\,\mathrm{d}x - \mu \int_\Omega \alpha_2(x) j_2(x,u)\,\mathrm{d}x \\
&\ge \frac{1}{p^+}\|u\|^{p^+} - \lambda a_3 \int_\Omega \frac{\alpha_1(x)}{q_4(x)} u^{q_4(x)}\,\mathrm{d}x - |\mu| a_4 \int_\Omega \alpha_2(x) u^{q_5(x)}\,\mathrm{d}x \\
&\ge \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda a_3 \beta_k^{q_4^-}}{q_4^-}\|u\|^{q_4^-} - |\mu| a_4 c_{13} \|u\|^{q_5^-} \\
&\ge \frac{1}{2p^+}\|u\|^{p^+} - \frac{\lambda a_3 \beta_k^{q_4^-}}{q_4^-}\|u\|^{q_4^-}.
\end{aligned} \tag{3.12}$$

We set $\rho_k = (\frac{2p^+ \lambda a_3 \beta_k^{q_4^-}}{q_4^-})^{\frac{1}{p^+ - q_4^-}}$, $\|u\| = \rho_k$. From Lemma 3.2 and noting that $q_4^- < p^+$, we deduce that $\rho_k \to 0$ as $k \to \infty$. There exists $k_0 > 0$ such that $\rho_k \le R$ when $k \ge k_0$. Thus, for $k \ge k_0$, $u \in Z_k$ and $\|u\| = \rho_k$, we have $a_k = \inf_{u \in Z_k, \|u\| = \rho_k} I(u) \ge 0$ and (A_2') is proved.

For $u \in Y_k$, there exists $\varepsilon \in (0,1)$ such that for all $u \in Y_k \cap B_\varepsilon$, $|u| \le \min\{\delta_1, \delta_2\}$, $|u|_{L^{q_3(x)}(\alpha_1(x),\Omega)} \le 1$ and $|u|_{L^{q_5(x)}(\alpha_2(x),\Omega)} \le 1$. By virtue of hypotheses (H_6), (H_7) and Proposition 2.2, we have

$$\begin{aligned}
I(u) &= \int_\Omega \frac{\omega(x)}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x - \lambda \int_\Omega \alpha_1(x) j_1(x,u)\,\mathrm{d}x - \mu \int_\Omega \alpha_2(x) j_2(x,u)\,\mathrm{d}x \\
&\le \frac{1}{p^-}\|u\|^{p^-} - \frac{\lambda a_2}{q_3^+} \int_\Omega \alpha_1(x) |u|^{q_3(x)}\,\mathrm{d}x + |\mu| a_4 \int_\Omega \alpha_2(x) |u|^{q_5(x)}\,\mathrm{d}x \\
&\le \frac{1}{p^-}\|u\|^{p^-} - \frac{\lambda a_2}{q_3^+} |u|^{q_3^+}_{L^{q_3(x)}(\alpha_1(x),\Omega)} + |\mu| a_4 |u|^{q_5^-}_{L^{q_5(x)}(\alpha_2(x),\Omega)}.
\end{aligned}$$

Since $q_3^+ < p^- < q_5^-$ and all norms on Y_k are equivalent, there exists $r_k \in (0, \varepsilon)$ small enough such that

$$b_k = \max_{u \in Y_k, \|u\| = r_k} I(u) < 0.$$

Hence relation (A_3') of Theorem 2.2 is satisfied. Since $Y_k \cap Z_k \ne \varnothing$ and $r_k < \rho_k$, we have

$$d_k = \inf_{u \in Z_k, \|u\| \le \rho_k} I(u) \le b_k = \max_{u \in Y_k, \|u\| = r_k} I(u) < 0.$$

On the other hand, from (3.12), for $k \ge k_0$, $u \in Z_k$, $\|u\| \le \rho_k$,

$$I(u) \ge -\frac{\lambda a_3 \beta_k^{q_4^-}}{q_4^-}\|u\|^{q_4^-} \ge -\frac{\lambda a_3 \beta_k^{q_4^-}}{q_4^-}\rho_k^{q_4^-}.$$

Since $\beta_k \to 0$ and $\rho_k \to 0$ as $k \to \infty$, we have

$$d_k = \inf_{u \in Z_k, \|u\| \le \rho_k} I(u) \to 0 \quad \text{as } k \to \infty.$$

Relation (A_4') of Theorem 2.2 is verified.

Finally, let us prove that I satisfies the nonsmooth $(\mathrm{PS})_c^*$ for all $c \in \mathbb{R}$. Consider a sequence $\{u_{n_j}\} \subset X$ such that

$$n_j \to \infty, \qquad u_{n_j} \in Y_{n_j}, \qquad I(u_{n_j}) \to c, \qquad m^{I|_{Y_{n_j}}}(u_{n_j}) \to 0,$$

where $m^{I|_{Y_{n_j}}}(u_{n_j}) = \inf_{u_{n_j}^* \in \partial I(u_{n_j})} \|u_{n_j}^*\|_{X^*}$. Similar to the process of verifying the $(\mathrm{PS})_c$ in the proof of Theorem 3.1, we can prove that $u_{n_j} \to u$ in X. So it only remains to show $0 \in \partial I(u)$. Note that

$$\begin{aligned} 0 \le m^I(u) &= m^I(u) - m^I(u_{n_j}) + m^I(u_{n_j}) \\ &= m^I(u) - m^I(u_{n_j}) + m^{I|_{Y_{n_j}}}(u_{n_j}). \end{aligned}$$

By virtue of Lemma 2.2, we obtain $m^I(u) \le 0$ as $j \to \infty$. Hence $m^I(u) = 0$, *i.e.*, $0 \in \partial I(u)$, which means that I satisfies the $(\mathrm{PS})_c^*$ for $c \in R$. So all the conditions of Theorem 2.2 are verified. We complete the proof of Theorem 3.2. □

Theorem 3.3 *If hypotheses* (P), (W), (H_1)-(H_4), (H_7) *and* (H_8) *hold, and* $j_1(x,0) = 0$, *then for all* $\lambda > 0$ *and* $\mu \le 0$, *problem* (1.1) *has at least one nontrivial solution.*

Proof By virtue of hypotheses (H_1), (H_4), Lebourg's mean value theorem and Lemma 3.1, for u large enough, we have

$$\begin{aligned} I(u) &= \int_\Omega \frac{\omega(x)}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x - \lambda \int_\Omega \alpha_1(x) j_1(x,u)\,\mathrm{d}x - \mu \int_\Omega \alpha_2(x) j_2(x,u)\,\mathrm{d}x \\ &\ge \frac{1}{p^+} \|u\|^{p^-} - \lambda \int_\Omega \alpha_1(x)\bigl(d_2 \bigl|u(x)\bigr|^{q_1(x)} + c_1\bigr)\,\mathrm{d}x + \mu \int_\Omega a_1 \alpha_2(x)\,\mathrm{d}x \\ &\ge \frac{1}{p^+} \|u\|^{p^-} - \lambda c_{14} \|u\|^{q_1^+} - c_{15} \quad (d_2, c_{14}, c_{15} \text{ are some positive constants}). \end{aligned}$$

Note that $p^- > q_1^+$ and $\lambda > 0$, then we have

$$I(u) \to +\infty \quad \text{as } \|u\| \to +\infty.$$

This means that I has a minimizer solution u_0 for problem (1.1).

In the following, we prove $u_0 \ne 0$. Choosing $v_0 \in C_0^\infty(\Omega)$ such that $0 \le v_0(x) \le \min\{\delta_2, \delta_3\}$, $\int_\Omega \alpha_1(x) v_0^{q_6(x)}(x)\,\mathrm{d}x = b_4 > 0$ and $\int_\Omega \alpha_2(x) v_0^{q_5(x)}(x)\,\mathrm{d}x = b_5 > 0$. From ($H_7$) and ($H_8$), for $t \in (0,1)$ small enough, we have

$$\begin{aligned} I(tv_0) &= \int_\Omega \frac{\omega(x)}{p(x)} |\nabla tv_0|^{p(x)}\,\mathrm{d}x - \lambda \int_\Omega \alpha_1(x) j_1(x,tv_0)\,\mathrm{d}x - \mu \int_\Omega \alpha_2(x) j_2(x,tv_0)\,\mathrm{d}x \\ &\le t^{p^-} \int_\Omega \frac{\omega(x)}{p(x)} |\nabla v_0|^{p(x)}\,\mathrm{d}x - \lambda \int_\Omega \alpha_1(x) a_5 \bigl(tv_0(x)\bigr)^{q_6(x)}\,\mathrm{d}x \\ &\quad - \mu \int_\Omega \alpha_2(x) a_4 \bigl(tv_0(x)\bigr)^{q_5(x)}\,\mathrm{d}x \\ &\le t^{p^-} \int_\Omega \frac{\omega(x)}{p(x)} |\nabla v_0|^{p(x)}\,\mathrm{d}x - t^{q_6^+} \lambda a_5 b_4 - t^{q_5^-} \mu a_4 b_5. \end{aligned}$$

Since $q_6^+ < p^- < q_5^-$, we can find $t_0 \in (0,1)$ such that $I(t_0 v_0) < 0$. This implies that $I(u_0) = \inf_{u \in X} I(u) < 0$. Hence $u_1 \neq 0$ ($I(0) = 0$). So we complete the proof of Theorem 3.3. □

Theorem 3.4 *If hypotheses* (P), (W), (H$_1$)-(H$_3$), (H$_5$), (H$_7$) *and* (H$_8$) *hold, and* $j_1(x,0) = 0$, *for* $\mu > 0$, *there exists* $\lambda_0(\mu) > 0$ *such that when* $|\lambda| \leq \lambda_0(\mu)$, *problem* (1.1) *has at least one nontrivial solution.*

Proof From the proof of Claim 1 in Theorem 3.1, we can obtain that I satisfies the nonsmooth C-condition. By hypotheses (H$_1$) and (H$_7$), we have

$$j_2(x,u) \leq a_4 |u|^{q_5(x)} + d_3 |u|^{q_2(x)} \tag{3.13}$$

for a.a. $x \in \Omega$, $d_3 > 0$. Then, for sufficiently small u,

$$\begin{aligned} I(u) &= \int_\Omega \frac{\omega(x)}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x - \lambda \int_\Omega \alpha_1(x) j_1(x,u)\,\mathrm{d}x - \mu \int_\Omega \alpha_2(x) j_2(x,u)\,\mathrm{d}x \\ &\geq \frac{1}{p^+} \|u\|^{p^+} - \lambda \int_\Omega \alpha_1(x) j_1(x,u)\,\mathrm{d}x - \mu a_4 \int_\Omega \alpha_2(x)|u|^{q_5(x)}\,\mathrm{d}x \\ &\quad - \mu d_3 \int_\Omega \alpha_2(x)|u|^{q_2(x)}\,\mathrm{d}x \\ &\geq \frac{1}{p^+} \|u\|^{p^+} - c_{16}\|u\|^{q_5^-} - c_{17}\|u\|^{q_2^-} - \lambda \int_\Omega \alpha_1(x) j_1(x,u)\,\mathrm{d}x \end{aligned}$$

for some $c_{16}, c_{17} > 0$. Note that $p^+ < q_5^-$ and $p^+ < q_2^-$. So there exist $r > 0$ and $\theta_1 > 0$ such that

$$\frac{1}{p^+} \|u\|^{p^+} - c_{16}\|u\|^{q_5^-} - c_{17}\|u\|^{q_2^-} > \theta_1 \quad \text{for a.a. } x \in \Omega, \|u\| \leq r.$$

We can find $\lambda_0(\mu) > 0$ such that $\lambda \int_\Omega \alpha_1(x) j_1(x,u)\,\mathrm{d}x \leq \frac{\theta_1}{2}$ when $|\lambda| \leq \lambda_0(\mu)$ for a.a. $x \in \Omega$, $\|u\| \leq r$. That is to say, when $|\lambda| \leq \lambda_0(\mu)$, we obtain

$$I(u) \geq \frac{\theta_1}{2} > 0 \quad \text{for a.a. } x \in \Omega, \|u\| < r.$$

Then we have

$$\inf\{I(u) : \|u\| = r\} > 0.$$

By virtue of (3.11) in Theorem 3.1, we can find $h \in X$, $\|h\| > r$ such that

$$I(h) < 0.$$

Hence, from the nonsmooth mountain pass theorem, we can deduce that problem (1.1) has at least one nontrivial solution. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Acknowledgements
The authors would like to thank the editor-in-chief, the associate editor and the anonymous reviewers for their valuable comments and constructive suggestions, which helped to improve the presentation of this paper. Research is supported by the National Natural Science Foundation of China (11371127).

Received: 26 November 2014 Accepted: 1 July 2015 Published online: 18 July 2015

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