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Global classical solution for a 3D viscous liquid-gas two-fluid flow model in a half-space

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Abstract

In this paper, we consider the three-dimensional viscous liquid-gas two phase flow model in a half-space $\Omega = \{x \in \mathbb{R}^3 : x_3 > 0\}$, where the initial vacuum is allowed. We prove the existence of the global classical solutions when the energy of the initial data is small enough.

MSC: 76T10; 76N10; 35L65

Keywords: viscous liquid-gas two-fluid flow model; classical solution; vacuum; a half-space

1 Introduction

We consider a 3D viscous liquid-gas two-fluid flow model in the following form:

$$\begin{cases} m_t + \operatorname{div}(mu) = 0, \\ n_t + \operatorname{div}(nu) = 0, \\ (mu)_t + \operatorname{div}(mu \otimes u) + \nabla P(m,n) = \mu \Delta u + (\lambda + \mu)\nabla \operatorname{div} u, \quad x \in \Omega, t > 0, \end{cases} \tag{1.1}$$

the half-space $\Omega = \{x \in \mathbb{R}^3 : x_3 > 0\}$ and the initial boundary conditions are

$$(m,n,u)|_{t=0} = (m_0, n_0, u_0), \quad x \in \Omega, \tag{1.2}$$

$$\bigl(u^1(x), u^2(x), u^3(x)\bigr) = \beta\bigl(u^1_{x_3}, u^2_{x_3}, 0\bigr), \quad \beta > 0, \text{on } \partial\Omega, \tag{1.3}$$

$$(m,n,u)(x,t) \to (\tilde{m}, \tilde{n}, 0), \quad \text{as } |x| \to \infty, (x,t) \in \Omega \times (0,T). \tag{1.4}$$

Here $\tilde{m}$, $\tilde{n}$ are positive constants. The variables $m = \alpha_l \rho_l$, $n = \alpha_g \rho_g$, $u = (u^1, u^2, u^3)$, and $P = P(m,n)$ denote the liquid mass, the gas mass, the velocity of the liquid and the gas, and the common pressure for both fluids, respectively; μ and λ denote the two Lamé coefficients, which are assumed to satisfy $\mu > 0$, $2\mu + 3\lambda \geq 0$. The other unknown variables ρ_l and ρ_g denote the liquid and gas densities, satisfying the equations of state $\rho_l = \rho_{l,0} + \frac{P - P_{l,0}}{a_l^2}$ and $\rho_g = \frac{P}{a_g^2}$, where a_l, a_g are known constants denoting the sonic speeds in the liquid and in the gas, respectively; $P_{l,0}$ and $\rho_{l,0}$ are the reference pressure and density given as constants; $\alpha_l, \alpha_g \in [0,1]$ denote the liquid and gas volume fractions, satisfying $\alpha_l + \alpha_g = 1$. We note from the expression of ρ_l and ρ_g that the pressure satisfies

$$P(m,n) = C^0\left(-b(m,n) + \sqrt{b(m,n)^2 + c(m,n)}\right), \tag{1.5}$$

with $C^0 = \frac{1}{2}a_l^2$, $k_0 = \rho_{l,0} - \frac{P_{l,0}}{a_l^2} > 0$, $a_0 = (\frac{a_g}{a_l})^2$, and

$$b(m,n) = k_0 - m - \left(\frac{a_g}{a_l}\right)^2 n = k_0 - m - a_0 n, \qquad c(m,n) = 4k_0\left(\frac{a_g}{a_l}\right)^2 n = 4k_0 a_0 n.$$

For more information as regards the above models, we can refer to [1–4].

The investigation of the above models has been a topic during the last decade. There has been much progress on the numerical properties of this model or other relevant models recently. It was [5, 6] who initiated some work in this direction. For the one-dimensional case, when the liquid is incompressible and the gas is polytropic, the global existence and uniqueness of a weak solution to the free boundary value problem were studied in [6–9]. When both of the two fluids are compressible, for their results one can consult [10], where the existence of the global weak solution was obtained. Moreover, for the results as regards the 1D case of the two-flow model, more interesting phenomena are described, such as the unequal fluid velocity, the well-reservoir interaction by allowing two kinds of gas between well and formation (surrounding reservoir), external forces, and the general pressure law [11–13]. In [10], Evje and Karlsen gave the global existence of a weak solution Cauchy problem for model (1.1) in 1D. For a multi-dimensional result as regards the two-fluid flow model, Yao *et al.* in [14] obtained the existence of the global weak solution to the 2D model when the initial energy is small and there is no initial vacuum. Later on, Wen *et al.* [15, 16] proved the blow-up criterion in terms of the upper bound of the liquid mass for a local strong solution to the 3D (or 2D) viscous liquid-gas two-phase flow model in the two different cases when there is an initial vacuum and no initial vacuum in a bounded domain. Yao *et al.* in [17] gave the blow-up criterion in terms of the $L^1(0,T;L^\infty)$-norm of the velocity with Dirichlet boundary condition and Navier-slip boundary condition without the restriction on viscosity coefficients. In [18] the authors investigated the Cauchy problem of model (1.1) in the framework of Besov space and obtained the global existence and uniqueness of the strong solution for the initial data close to a stable equilibrium. For the non-conservative viscous compressible two-fluid system (1.1), we can see [5, 19] for the global existence of weak solution. Guo *et al.* [20] have proved the global existence of strong solution to Cauchy problem of system (1.1), where the initial data may vanish in an open set. Recently, in [21] one proved the global existence of the classical solution, which is a continuation of [20].

In the present paper, we consider a three-dimensional viscous liquid-gas two-phase flow model in a half-space $\Omega = \{x \in \mathbb{R}^3 : x_3 > 0\}$, and we obtain the global existence and uniqueness of classical solutions with vacuum subject to some smallness assumptions on the initial energy. Before this work, there have been some results on the global existence and uniqueness of strong and classical solutions with vacuum in $\mathbb{R}^3$. When the boundary conditions, such as the half-space problem, are involved, things will become more complicated. For the boundary problems, there is no clarity as to the boundary value of the effective viscous u_x and vorticity so that some elliptic estimates are not easy to get. We extend the results from single-phase flow to two-phase flow. The technique in [22] and in some other references on liquid-gas two-phase flow plays a very important role in the proof.

Before stating the main results, we explain the notations used throughout this paper. We will use the following conventions throughout this paper:

$$\int f\,dx = \int_\Omega f\,dx.$$

We denote the standard homogeneous and inhomogeneous Sobolev spaces as follows:

$$\begin{aligned} &L^r = L^r(\Omega), \qquad H^k = W^{k,2}, \qquad D^{k,r} = \left\{u \in L^1_{\mathrm{loc}}(\Omega) \mid \left\|\nabla^k u\right\|_{L^r} < \infty\right\}, \\ &\|u\|_{D^{k,r}} = \left\|\nabla^k u\right\|_{L^r}, \qquad W^{k,r} = L^r \cap D^{k,r}, \qquad D^k = D^{k,2}, \\ &D^1 = \left\{u \in L^6 \mid \|\nabla u\|_{L^2} < \infty, \text{and } u = 0 \text{ as } |x| \to \infty\right\}. \end{aligned}$$

The potential energy function G takes the following form:

$$G\left(m, \frac{n}{m}\right) = m \int_{\tilde{m}}^{m} \frac{P(s, \frac{n}{m}) - P(\tilde{m}, \tilde{n})}{s^2}\,ds + \frac{m}{\tilde{m}} P(\tilde{m}, \tilde{n}) - \frac{m}{\tilde{m}} P\left(\tilde{m}, \frac{n}{m}\tilde{m}\right), \tag{1.6}$$

and the initial energy is given by

$$C_0 = \int \left(\frac{1}{2} m_0 |u_0|^2 + G\left(m_0, \frac{n_0}{m_0}\right)\right) dx. \tag{1.7}$$

Let

$$\int |\nabla u_0|^2\,dx \le M, \tag{1.8}$$

where M is a positive constant. It follows that there is a $\delta_1 \in (0, 1]$ small enough, which will be fixed throughout, such that

$$\tilde{m}\tilde{n} \ge \frac{2}{\delta_1(2 + \delta_1)}. \tag{1.9}$$

We define the convective derivative $\frac{D}{Dt}$ by $\frac{Df}{Dt} = \dot{f} = f_t + u \cdot \nabla f$.

Then we give the main result of this paper.

Theorem 1.1 *Assume that the conditions* (1.6)-(1.9) *hold. For given positive constants* $\bar{m}$, $\bar{n}$, M (*not necessarily small*) *with* $\bar{m} > 2\tilde{m}$, *the initial data* (m_0, n_0, u_0) *satisfies*

$$0 \le \inf_x m_0 \le \sup_x m_0 \le \bar{m}, \qquad 0 \le \inf_x n_0 \le \sup_x n_0 \le \bar{n}, \tag{1.10}$$

$$u_0 \in D^1 \cap D^3, \qquad (m_0 - \tilde{m}, n_0 - \tilde{n}) \in H^3, \tag{1.11}$$

and the compatibility condition

$$-\mu \Delta u_0 - (\lambda + \mu)\nabla \operatorname{div} u_0 + \nabla P(m_0, n_0) = m_0 g, \tag{1.12}$$

for some $g \in D^1$ *with* $m_0^{\frac{1}{2}} g \in L^2$. *Furthermore, assume that*

$$0 \le \underline{s}_0 m_0 \le n_0 \le \frac{\tilde{n}}{\tilde{m}} m_0, \tag{1.13}$$

where $\underline{s}_0$ *is a positive constant, satisfying*

$$\frac{\tilde{n}}{2\tilde{m}} \le \underline{s}_0 \le \frac{\tilde{n}}{\tilde{m}}. \tag{1.14}$$

Then there exists a positive constant ε *depending on* $\bar{m}$, M, $\tilde{m}$, $\tilde{n}$, C^0, a_0, $\underline{s}_0$, μ, *and* λ, *such that, if* $C_0 \le \varepsilon$, *then the Cauchy problem* (1.1)-(1.3) *has a unique global classical solution* (m, n, u) *satisfying for any* $0 < \tau < T < \infty$,

$$0 \le m \le 2\bar{m}, \qquad 0 \le \underline{s}_0 m \le n \le \frac{\tilde{n}}{\tilde{m}} m, \qquad x \in \Omega, \qquad t \ge 0, \tag{1.15}$$

$$\begin{aligned} &(m - \tilde{m}, n - \tilde{n}) \in C\bigl(0, T; H^3\bigr), \\ &u \in C\bigl(0, T; D^1 \cap D^3\bigr) \cap L^2\bigl(0, T; D^4\bigr) \cap L^\infty\bigl(\tau, T; D^4\bigr), \end{aligned} \tag{1.16}$$

$$(m_t, n_t) \in C\bigl(0, T; H^2\bigr), \qquad \sqrt{m} u_t \in L^\infty\bigl(0, T; L^2\bigr), \qquad \sqrt{m} u_{tt} \in L^2\bigl(0, T; L^2\bigr), \tag{1.17}$$

$$u_t \in L^\infty\bigl(0, T; D^1\bigr) \cap L^2(0, T; D_2) \cap L^\infty\bigl(\tau, T; D^2\bigr) \cap H^1\bigl(\tau, T; D^1\bigr). \tag{1.18}$$

We should mention that the ideas introduced by Wen *et al.* in [15], in Yao *et al.* in [14, 16], Yao *et al.* in [20, 21] for the two-fluid flow model, Cho and Kim in [23], Huang *et al.* in [24], Duan in [22], Hoff in [25], and Perepelitsa in [26] for the single-fluid Navier-Stokes equations play crucial roles in our proof here.

2 *A priori* estimates

Firstly, using similar methods to [27] and the references therein, we can obtain the local existence and the uniqueness of the solutions to (1.1)-(1.3) with the regularities as in Theorem 1.1. We omit it here for brevity. In the following of this section, we derive some *a priori* estimates for a local classical solution of problem (1.1)-(1.3) on $\Omega \times [0, T]$ for some $T > 0$, with the initial data (m_0, n_0, u_0) satisfying (1.10)-(1.14).

Next, we give the well-known Gagliardo-Nirenberg inequality, which will be used frequently later.

Lemma 2.1 (Gagliardo-Nirenberg inequality) [14] *For any* $p, b', r' \in [1, \infty]$ *and any integer* l *and* j, *there exist some generic constant* $\alpha \in [0, 1]$ *and* $C > 0$, *for every function* $u \in C_0^\infty$, *such that*

$$\bigl\| \nabla^j u \bigr\|_{L^p(\mathbb{R}^N)} \le C \bigl\| \nabla^l u \bigr\|_{L^{b'}(\mathbb{R}^N)}^\alpha \|u\|_{L^{r'}(\mathbb{R}^N)}^{1-\alpha}, \tag{2.1}$$

where

$$\frac{1}{p} = \frac{j}{N} + \alpha \left(\frac{1}{b'} - \frac{l}{N} \right) + (1 - \alpha) \frac{1}{r'}, \quad \frac{j}{l} \le \alpha \le 1. \tag{2.2}$$

$\alpha \neq 1$, *when* $l - \frac{N}{b'} = j$ *and* $1 < p < \infty$.

Next, the following Zlotnik inequality will be used to get the uniform (in time) upper bound of the m and n.

Lemma 2.2 ([28]) *Let the function y satisfy*

$$y'(t) = g(y) + b'(t), \quad on\ [0,T], \qquad y(0) = y^0,$$

with $g \in C(\mathbb{R})$ and $y, b \in W^{1,1}(0,T)$. If $g(\infty) = -\infty$ and

$$b(t_2) - b(t_1) \le N_0 + N_1(t_2 - t_1) \tag{2.3}$$

for all $0 \le t_1 < t_2 \le T$ with some $N_0 \ge 0$ and $N_1 \ge 0$, then

$$y(t) \le \max\{y^0, \overline{\zeta}\} + N_0 < \infty, \quad on\ [0,T],$$

where $\overline{\zeta}$ is a constant such that

$$g(\zeta) \le -N_1 \quad for\ \zeta \ge \overline{\zeta}. \tag{2.4}$$

Lemma 2.3 ([29]) *Assume $X \subset E \subset Y$ are Banach spaces and $X \hookrightarrow\hookrightarrow E$. Then the following embeddings are compact*:

$$\text{(i)} \quad \left\{\varphi : \varphi \in L^q(0,T;X), \frac{\partial \varphi}{\partial t} \in L^1(0,T;Y)\right\} \hookrightarrow\hookrightarrow L^q(0,T;E), \quad if\ 1 \le q \le \infty;$$

$$\text{(ii)} \quad \left\{\varphi : \varphi \in L^\infty(0,T;X), \frac{\partial \varphi}{\partial t} \in L^r(0,T;Y)\right\} \hookrightarrow\hookrightarrow C([0,T];E), \quad if\ 1 < r \le \infty.$$

Firstly, we denote

$$A_1(T) = \sup_{0<t\le T} \sigma \int |\nabla u|^2\,dx + \int_0^T \int \sigma m|\dot{u}|^2\,dx\,ds,$$

$$A_2(T) = \sup_{0<t\le T} \sigma^3 \int m|\dot{u}|^2\,dx + \int_0^T \int \sigma^3 |\nabla \dot{u}|^2\,dx\,ds,$$

here $\sigma(t) = \min\{1,t\}$. For any $(x,t) \in \Omega \times [0,T]$, we make the following *a priori* assumptions:

$$0 \le m(x,t) \le 2\bar{m} \tag{2.5}$$

and

$$A_1(T) + A_2(T) \le 2C_0^{\frac{1}{2}}. \tag{2.6}$$

Under the assumption (2.5), we make the following remark.

Remark 2.1 Under the conditions of Theorem 1.1, for any $0 \le t \le T$, we have

$$0 \le \underline{s}_0 m \le n \le \frac{\tilde{n}}{\tilde{m}} m, \quad x \in \Omega. \tag{2.7}$$

Proof In fact, define the particle trajectories $x = X(t,y)$ given by

$$\begin{cases} \frac{d}{dt}X(t,y) = u(X(t,y),t), \\ X(0,y) = y. \end{cases} \tag{2.8}$$

From $(1.1)_1$ and $(1.1)_2$, we have

$$\left(\frac{n}{m}\right)_t + u \cdot \nabla\left(\frac{n}{m}\right) = 0, \tag{2.9}$$

which implies

$$\frac{n(x,t)}{m(x,t)} = \frac{n_0}{m_0}\big(X^{-1}(t,x)\big) := s_0 = s_0(x,t), \tag{2.10}$$

where X^{-1} denotes the inverse of X. We obtain (2.7) together with (1.13). □

In the following, C denotes a generic constant depending only on $\bar{m}$, $\tilde{m}$, $\tilde{n}$, a_0, C^0, $\underline{s}_0$, μ, λ, and the initial data may vary in different estimates; we write $C(\alpha)$ to emphasize that C depends on α.

Remark 2.2 Under the conditions of Theorem 1.1, we have

$$0 \le P_m \le C\big(C^0\big), \quad \text{in } \Omega \times [0,T], \tag{2.11}$$

$$2a_0C^0 \le P_n \le C\big(C^0, a_0, \underline{s}_0\big), \quad \text{in } \Omega \times [0,T]. \tag{2.12}$$

The above remark plays a key role in the proof of the upper bound of the density (m,n). Now, the standard energy estimate is given as follows.

Proposition 2.1 *If* $(m,n,u)(x,t)$ *is a classical solution of* (1.1)-(1.3) *in* $[0,T]$, *satisfying the assumptions* (2.5)-(2.6), *then we have*

$$\begin{aligned} &\sup_{0\le t\le T}\int \left(\frac{1}{2}m|u|^2 + G(m,n)\right)dx + \int_0^T \int \big(\mu|\nabla u|^2 + (\lambda+\mu)|\operatorname{div} u|^2\big)\,dx\,dt \\ &\quad + \beta^{-1}\int_0^T\int_{\partial\Omega}|u|^2\,dS_x\,dt \le C_0. \end{aligned} \tag{2.13}$$

Proof In [14, 20], one obtained this standard energy estimate in $\mathbb{R}^N$ ($N = 2,3$). Using a similar argument to [14, 20], we can easily obtain this energy estimate and omit the details. □

Notice that the function $G(m,n)$ is equivalent to $(m-\tilde{m})^2 + (n-\tilde{n})^2$, which implies

$$\begin{aligned} &\sup_{0\le t\le T}\int \big(m|u|^2 + (m-\tilde{m})^2 + (n-\tilde{n})^2\big)\,dx \\ &\quad + \int_0^T\int |\nabla u|^2\,dx\,dt + \beta^{-1}\int_0^T\int_{\partial\Omega}|u|^2\,dS_x\,dt \le CC_0. \end{aligned} \tag{2.14}$$

Now, we use ω and F, denoting the vorticity matrix and the effective viscous flux, defined in the following form:

$$F \triangleq (2\mu + \lambda)\operatorname{div} u - P(m,n) + P(\tilde{m},\tilde{n}), \qquad \omega \triangleq \nabla \times u,$$

then we can rewrite $(1.1)_3$ in the form

$$m\dot{u}^j = \partial_j F + \mu \partial_k \omega^{j,k}, \tag{2.15}$$

which implies

$$\Delta F = \operatorname{div}(m\dot{u}), \qquad \mu \Delta \omega^{j,k} = \partial_k\big(m\dot{u}^j\big) - \partial_j\big(m\dot{u}^k\big) \tag{2.16}$$

and

$$(\mu + \lambda)\Delta u^j = \partial_j F + (\mu + \lambda)\partial_k w^{j,k} + \partial_j\big(P(m,n) - P(\tilde{m},\tilde{n})\big). \tag{2.17}$$

Together with the energy estimate, we can get the following lemma.

Lemma 2.4 *If* (m,n,u) *is a classical solution of* (1.1)-(1.3), *we have for any* $p \in [2,6]$

$$\|u\|_{L^p} \le C(\bar{m}) C_0^{\frac{6-p}{4p}} \|\nabla u\|_{L^2}^{\frac{3p-6}{2p}} + C_0^{\frac{6-p}{6p}} \|\nabla u\|_{L^2}, \tag{2.18}$$

$$\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} \le C\big(\|m\dot{u}\|_{L^p} + \|\nabla u\|_{L^p}\big), \tag{2.19}$$

$$\begin{aligned} \|F\|_{L^p} + \|\omega\|_{L^p} \le{}& C\|m\dot{u}\|_{L^2}^{\frac{3p-6}{2p}} \big(\|\nabla u\|_{L^2} + \big\|P(m,n) - P(\tilde{m},\tilde{n})\big\|_{L^2}\big)^{\frac{6-p}{2p}} \\ &+ C\big(\|\nabla u\|_{L^2} + \big\|P(m,n) - P(\tilde{m},\tilde{n})\big\|_{L^2}^{\frac{6-p}{2p}} \|\nabla u\|_{L^2}^{\frac{3p-6}{2p}}\big), \end{aligned} \tag{2.20}$$

$$\|\nabla u\|_{L^p} \le C\big(\|F\|_{L^p} + \|\omega\|_{L^p}\big) + C\big\|P(m,n) - P(\tilde{m},\tilde{n})\big\|_{L^p}, \tag{2.21}$$

$$\begin{aligned} \|\nabla u\|_{L^p} \le{}& C\big(\|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \big(\|m\dot{u}\|_{L^2} + \|\nabla u\|_{L^2}\big)\big) \\ &+ C\big(\big\|P(m,n) - P(\tilde{m},\tilde{n})\big\|_{L^2} + \big\|P(m,n) - P(\tilde{m},\tilde{n})\big\|_{L^6}\big)^{\frac{3p-6}{2p}}. \end{aligned} \tag{2.22}$$

Also, for $0 \le t_1 \le t_2 \le T$, *we have for any* $p \ge 2$ *and* $r \ge 0$

$$\int_{t_1}^{t_2} \int \sigma^r \big|P(m,n) - P(\tilde{m},\tilde{n})\big|^p \,dx\,ds \le C\bigg(\int_{t_1}^{t_2} \int \sigma^r |F|^p \,dx\,ds + C_0\bigg). \tag{2.23}$$

Proof Using a similar argument to that in [22] (Lemma 4.3) and [30] (Lemma 3.3), we can obtain this lemma and omit the details. □

The proofs of the following estimates are similar to [14, 24, 25, 30, 31].

Lemma 2.5 *If* (m,n,u) *is the classical solution of* (1.1)-(1.3) *in* $[0,T]$ *as in Proposition* 2.1, *then we have*

$$\begin{aligned} A_1(T) ={}& \sup_{0<t\le T} \sigma \int |\nabla u|^2 \,dx + \int_0^T \int \sigma m|\dot{u}|^2 \,dx\,dt \\ \le{}& CC_0 + C\int_0^T \int \sigma |\nabla u|^3 \,dx\,dt + C\int_0^T \int \sigma\big(|u|^2|\nabla u| + |u||\nabla u|^2\big) \,dx\,dt \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} A_2(T) &= \sup_{0<t\le T} \sigma^3 \int m|\dot u|^2\,dx + \int_0^T \int \sigma^3 |\nabla u|^2\,dx\,dt \\ &\le CC_0 + CA_1(T) \\ &\quad + C\int_0^T \int \sigma^3\big[|u|^4 + |\nabla u|^4 + |\dot u||\nabla u||u| + |\dot u||\nabla u|^2\big]\,dx\,dt. \end{aligned} \tag{2.25}$$

Proof Multiplying $(1.1)_3$ by $\sigma\dot u$, and integrating over Ω, we obtain

$$\begin{aligned} \int \sigma m|\dot u|^2\,dx &= \int \big(-\sigma\dot u\cdot\nabla P + \mu\sigma\Delta u\cdot\dot u + (\lambda+\mu)\sigma\nabla\operatorname{div} u\cdot\dot u\big)\,dx \\ &= \sum_{i=1}^3 J_i. \end{aligned} \tag{2.26}$$

Integrating by parts and using $(1.1)_1$, $(1.1)_2$, and the Cauchy inequality, we have

$$\begin{aligned} J_1 &= \int -\sigma\dot u\cdot\nabla P\,dx \\ &= \left(\int \sigma\operatorname{div} u\big(P(m,n) - P(\tilde m,\tilde n)\big)\,dx\right)_t \\ &\quad - \int \big\{\sigma'\operatorname{div} u\big(P(m,n) - P(\tilde m,\tilde n)\big) + \sigma P(m,n)_t\operatorname{div} u + \sigma(u\cdot\nabla u)\cdot\nabla P\big\}\,dx \\ &= \left(\int \sigma\operatorname{div} u\big(P(m,n) - P(\tilde m,\tilde n)\big)\,dx\right)_t - \int \sigma'\operatorname{div} u\big(P(m,n) - P(\tilde m,\tilde n)\big)\,dx \\ &\quad + \int \sigma\big\{(mP_m + nP_n)(\operatorname{div} u)^2 - P(m,n)(\operatorname{div} u)^2 + P(m,n)\partial_i u^j\partial_j u^i\big\}\,dx \\ &\le \left(\int \sigma\operatorname{div} u\big(P(m,n) - P(\tilde m,\tilde n)\big)\,dx\right)_t + C\|\nabla u\|_{L^2}^2 + C\sigma' C_0, \end{aligned} \tag{2.27}$$

and noting the boundary condition (2.13) and the outer normal vector $N = (N^1, N^2, N^3) = (0,0,-1)$, we get

$$\begin{aligned} J_2 &= \mu\sigma\int \Delta u\cdot\dot u\,dx \\ &= \mu\sigma\int \Delta u\cdot(u_t + u\cdot\nabla u)\,dx \\ &= -\frac{\mu}{2}\left(\int \sigma|\nabla u|^2\,dx\right)_t + \frac{\mu}{2}\int \sigma'|\nabla u|^2\,dx - \mu\sigma\int \partial_i u^j\partial_i u^k\partial_k u^j\,dx \\ &\quad + \frac{\mu}{2}\int \sigma\partial_k u^k\big(\partial_i u^j\big)^2\,dx + \mu\sigma\int_{\partial\Omega} \partial_i u^j\dot u^j N^i\,dS_x \\ &= -\frac{\mu}{2}\left(\int \sigma|\nabla u|^2\,dx\right)_t + \frac{\mu}{2}\int \sigma'|\nabla u|^2\,dx - \mu\sigma\int \partial_i u^j\partial_i u^k\partial_k u^j\,dx \\ &\quad + \frac{\mu}{2}\int \sigma\partial_k u^k\big(\partial_i u^j\big)^2\,dx - \frac{\mu}{2}\left(\sigma\int_{\partial\Omega} \beta^{-1}|u|^2\,dS_x\right)_t \\ &\quad - \mu\sigma\int_{\partial\Omega} \beta^{-1}u^j u^i u^j_i\,dS_x + \frac{\mu}{2}\sigma'\int_{\partial\Omega} \beta^{-1}|u|^2\,dS_x \end{aligned}$$

$$\begin{aligned}
&\triangleq -\frac{\mu}{2}\left(\int \sigma|\nabla u|^2\,dx\right)_t + \frac{\mu}{2}\int \sigma'|\nabla u|^2\,dx - \mu\sigma\int \partial_i u^j \partial_i u^k \partial_k u^j\,dx \\
&\quad + \frac{\mu}{2}\int \sigma\,\partial_k u^k\left(\partial_i u^j\right)^2 dx - \frac{\mu}{2}\left(\sigma\int_{\partial\Omega}\beta^{-1}|u|^2\,dS_x\right)_t + J_2^1 + J_2^2. \qquad (2.28)
\end{aligned}$$

Now we need to estimate the boundary term J_2^1 and J_2^2. We apply the fact that for $h \in (C^1 \cap W^{1,1})(\overline{\Omega})$,

$$\int_{\partial\Omega} h(x)\,dS = \int_{\Omega\bigcap\{0\le x_3\le 1\}} \left[h(x) + (x_3-1)h_{x_3}(x)\right] dx. \qquad (2.29)$$

Since $j,k \in \{1,2\}$, using the fact (2.29), and integrating by parts in the x_1 and x_2 directions imply

$$\begin{aligned}
J_2^1 &= -\mu\sigma\int_{\partial\Omega}\beta^{-1}u^j u^i u_i^j\,dS_x \\
&= -\mu\sigma\int_{\Omega\bigcap\{0\le x_3\le 1\}}\beta^{-1}\left[u^j u^i u_i^j + (x_3-1)\left(u^j u^i u_i^j\right)_{x_3}\right] dx \\
&= -\mu\sigma\int_{\Omega\bigcap\{0\le x_3\le 1\}}\beta^{-1}\left\{u^j u^i u_i^j + (x_3-1)\left(u_{x_3}^j u^i u_i^j + u^j u_{x_3}^i u_i^j\right)\right\} \\
&\quad + \mu\sigma\int_{\Omega\bigcap\{0\le x_3\le 1\}}\beta^{-1}(x_3-1)\left(u_i^j u^i u_{x_3}^j + u^j u_i^i u_{x_3}^j\right) dx \\
&\le C\sigma\int_{\Omega}\left(|u|^2|\nabla u| + |u||\nabla u|^2\right) dx, \qquad (2.30)\\
J_2^2 &= \frac{\mu}{2}\sigma'\int_{\partial\Omega}\beta^{-1}|u|^2\,dS_x \\
&= \frac{\mu}{2}\sigma'\int_{\Omega\bigcap\{0\le x_3\le 1\}}\beta^{-1}\left[|u|^2 + 2(x_3-1)u\cdot u_{x_3}\right] dx \\
&\le C\sigma'\int |u|^2\,dx + C\|\nabla u\|_{L^2}^2 \le C\sigma'\left(C_0 + C_0^{\frac{2}{3}}\int |\nabla u|^2\,dx\right) + C\|\nabla u\|_{L^2}^2 \\
&\le CC_0\sigma' + C\|\nabla u\|_{L^2}^2, \qquad (2.31)
\end{aligned}$$

here we have used (2.18), then we can estimate J_2 as

$$\begin{aligned}
J_2 &\le -\frac{\mu}{2}\left(\int \sigma|\nabla u|^2\,dx + \sigma\int_{\partial\Omega}\beta^{-1}|u|^2\,dS_x\right)_t + C\|\nabla u\|_{L^2}^2 \\
&\quad + C\sigma\int |\nabla u|^3\,dx + C\sigma\int_{\Omega}\left(|u|^2|\nabla u| + |u||\nabla u|^2\right) dx + CC_0\sigma'. \qquad (2.32)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
J_3 &= (\lambda+\mu)\int \sigma\nabla(\operatorname{div} u)\cdot\dot{u}\,dx \\
&\le -\frac{(\lambda+\mu)}{2}\left(\int \sigma|\operatorname{div} u|^2\,dx\right)_t + C\|\nabla u\|_{L^2}^2 + \int \sigma|\nabla u|^3\,dx \\
&\le -\frac{(\lambda+\mu)}{2}\left(\int \sigma|\operatorname{div} u|^2\,dx\right)_t + \int \sigma|\nabla u|^3\,dx + C\|\nabla u\|_{L^2}^2. \qquad (2.33)
\end{aligned}$$

Combing (2.28)-(2.33), then integrating the resulting inequality over $[0,T]$ and using Young's inequality, we can obtain

$$\begin{aligned}
&\sup_{0\le t\le T}\sigma\|\nabla u\|_{L^2}^2+\int_0^T\int\sigma m|\dot u|^2\,dx\,dt+\int_0^T\int\sigma|\operatorname{div}u|^2\,dx\,dt\\
&\quad+\int_0^T\int_{\partial\Omega}\sigma\beta^{-1}|u|^2\,dS_x\,dt\\
&\quad\le CC_0+C\int_0^T\int\sigma|\nabla u|^3\,dx\,dt+C\int_0^T\int\sigma\bigl(|u|^2|\nabla u|+|\nabla u|^2|u|\bigr)\,dx\,dt.
\end{aligned}\tag{2.34}$$

Multiplying $\sigma^3\dot u^j(\frac{\partial}{\partial t}+\operatorname{div}(u\cdot))$ to $(1.1)_3^j$, summing with respect to j, and integrating the resulting equation over Ω, we have

$$\begin{aligned}
\left(\frac{\sigma^3}{2}\int m|\dot u|^2\,dx\right)_t&=\frac{3}{2}\int\sigma^2\sigma_t m|\dot u|^2\,dx+\mu\sigma^3\int\dot u^j\bigl[\Delta u_t^j+\operatorname{div}\bigl(u\Delta u^j\bigr)\bigr]\,dx\\
&\quad-\sigma^3\int\dot u^j\bigl[\partial_jP_t+\operatorname{div}(\partial_jPu)\bigr]\,dx\\
&\quad+(\lambda+\mu)\sigma^3\int\dot u^j\bigl[\partial_t\partial_j\operatorname{div}u+\operatorname{div}(u\partial_j\operatorname{div}u)\bigr]\,dx=\sum_{i=1}^4H_i.
\end{aligned}\tag{2.35}$$

Integrating by parts and using Young's inequality again, we have

$$\begin{aligned}
H_2&=\mu\int\sigma^3\dot u^j\bigl[\Delta u_t^j+\operatorname{div}\bigl(u\Delta u^j\bigr)\bigr]\,dx\\
&=-\mu\int\sigma^3\bigl[|\nabla\dot u|^2+\partial_i\dot u^j\partial_ku^k\partial_iu^j-\partial_i\dot u^j\partial_iu^k\partial_ku^j-\partial_iu^j\partial_iu^k\partial_k\dot u^j\bigr]\,dx\\
&\quad-\mu\int_{\partial\Omega}\sigma^3\beta^{-1}|\dot u|^2\,dS_x+\mu\int_{\partial\Omega}\sigma^3\dot u^ju_k^ju^k\,dS_x-\mu\int_{\partial\Omega}\sigma^3\partial_k\dot u^ju^k\partial_3u^j\,dS_x\\
&\le-\frac{\mu}{2}\int\sigma^3|\nabla\dot u|^2\,dx+C\int\sigma^3|\nabla u|^4\,dx-\mu\int_{\partial\Omega}\sigma^3\beta^{-1}|\dot u|^2\,dS_x\\
&\quad+C\mu\int\sigma^3\bigl[|u||\nabla u||\dot u|+|u||\nabla u||\nabla\dot u|+|\nabla u|^2|\dot u|\bigr]\,dx\\
&\le-\frac{\mu}{4}\int\sigma^3|\nabla\dot u|^2\,dx+C\int\sigma^3|\nabla u|^4\,dx-\mu\int_{\partial\Omega}\sigma^3\beta^{-1}|\dot u|^2\,dS_x\\
&\quad+C\int\sigma^3\bigl(|\nabla u|^4+|u|^4\bigr)\,dx+C\mu\int\sigma^3\bigl[|u||\nabla u||\dot u|+|\nabla u|^2|\dot u|\bigr]\,dx,
\end{aligned}\tag{2.36}$$

where we have used

$$\int_{\partial\Omega}\sigma^3\beta^{-1}\dot u^ju_k^ju^k\partial_3u^j\,dS_x\le C\int_\Omega\sigma^3|\nabla u||u||\nabla\dot u|\,dx\tag{2.37}$$

and

$$\int_{\partial\Omega}\sigma^3\partial_k\dot u^ju^k\,dS_x\le C\int_\Omega\sigma^3\bigl[|u||\nabla u||\dot u|+|u||\nabla u||\nabla\dot u|+|\nabla u|^2|\dot u|\bigr]\,dx.\tag{2.38}$$

From $(1.1)_1$ and $(1.1)_2$, we have

$$\begin{aligned}
H_3 &= -\int \sigma^3 \dot{u}^j \big[\partial_j P_t + \operatorname{div}(\partial_j P u)\big]\,dx \\
&= -\int \sigma^3 \big[P_m(m \operatorname{div} u + u\cdot\nabla m)\partial_j \dot{u}^j + P_n(n \operatorname{div} u + u\cdot\nabla n)\partial_j\dot{u}^j\big]\,dx \\
&\quad - \int \sigma^3 P(m,n)\partial_j\big(\partial_k \dot{u}^j u^k\big)\,dx \\
&= \int \sigma^3\big[-P_m m \operatorname{div} u \partial_j \dot{u}^j - P_n n \operatorname{div} u\partial_j\dot{u}^j + \partial_k\big(\partial_j \dot{u}^j u^k\big)P - P\big(\partial_j\big(\partial_k \dot{u}^j u^k\big)\big)\big]\,dx \\
&= \int \sigma^3\big[-P_m m \operatorname{div} u\partial_j\dot{u}^j - P_n n\operatorname{div} u\partial_j \dot{u}^j + \partial_j \dot{u}^j \operatorname{div} uP - \partial_k \dot{u}^j \partial_j u^k P\big]\,dx \\
&\le C\int \sigma^3 |\nabla u||\nabla \dot{u}|\,dx \le \frac{\mu}{8}\int \sigma^3|\nabla\dot{u}|^2\,dx + C\int \sigma^3|\nabla u|^2\,dx.
\end{aligned} \tag{2.39}$$

Similarly,

$$H_4 \le -\frac{\lambda+\mu}{2}\int \sigma|\operatorname{div}\dot{u}|^2\,dx + C\int \sigma|\nabla u|^4\,dx. \tag{2.40}$$

Combining (2.35)-(2.40) and integrating the result inequality over $(0,T)$, noting that $\int_0^T\int \sigma' m|\dot{u}|^2\,dx\,dt \le \int m|\dot{u}|^2\,dx \le A_1(T)$, we can obtain (2.25), thus we complete the proof of Lemma 2.5. □

The following lemma will be applied to bound the higher-order terms occurring on the right hand side of (2.24) and (2.25).

Lemma 2.6 *If* (m,n,u) *is the classical solution of* (1.1)-(1.3) *in* $[0,T]$ *as in Proposition* 2.1, *then there is a positive constant* T_1 *such that*

$$\sup_{t\in[0,T_1\wedge T]}\int |\nabla u|^2\,dx + \int_0^{T_1\wedge T}\int m|\dot{u}^2|\,dx\,dt \le C(1+M). \tag{2.41}$$

Proof In fact, multiplying $(1.1)_3$ by $\dot{u}$, then integrating the resulting equality over $\Omega\times[0,t]$, we have

$$\int_0^t\int m|\dot{u}|^2\,dx\,ds = \int_0^t\int\big\{-\dot{u}\cdot\nabla P(m,n) + \mu\Delta u\cdot u + (\mu+\lambda)\nabla(\operatorname{div} u)\cdot\dot{u}\big\}\,dx\,ds.$$

Using a similar argument to that in the proof of (2.24), we have

$$\begin{aligned}
&\int|\nabla u|^2\,dx + \int_0^t\int m|\dot{u}|^2\,dx\,ds \\
&\quad\le C(C_0+M) + C\int_0^t\int|\nabla u|^3\,dx\,ds + \int_0^t\int |u|^2|\nabla u| + |u||\nabla u|^2\,dx\,ds.
\end{aligned}$$

From (2.5), (2.6), (2.14), (2.19)-(2.23), using Young's inequality and the Hölder inequality, we obtain

$$
\begin{aligned}
\int_0^t \int |\nabla u|^3\,dx\,ds &\le C\int_0^t \int \big(|F|^3 + |\omega|^3\big)\,dx\,ds + C\int_0^t \int \big[P(m,n) - P(\tilde{m},\tilde{n})\big]^3\,dx\,ds \\
&\le C\int_0^t \big(\|F\|_{L^2}^{\frac{3}{2}}\|\nabla F\|_{L^2}^{\frac{3}{2}} + \|\omega\|_{L^2}^{\frac{3}{2}}\|\nabla\omega\|_{L^2}^{\frac{3}{2}}\big)\,ds + CC_0 \\
&\le C\int_0^t \big(\|\nabla u\|_{L^2} + \big\|P(m,n) - P(\tilde{m},\tilde{n})\big\|_{L^2}\big)^{\frac{3}{2}} \\
&\quad \times \big(\|m\dot{u}\|_{L^2} + \|\nabla u\|_{L^2}\big)^{\frac{3}{2}}\,ds + CC_0 \\
&\le \frac{1}{4}\int_0^t \int m|\dot{u}|^2\,dx\,ds + C\int_0^t \|\nabla u\|_{L^2}^6\,ds + CC_0
\end{aligned}
\tag{2.42}
$$

and

$$
\begin{aligned}
&\int_0^t \int |u|^2|\nabla u| + |u||\nabla u|^2\,dx\,ds \\
&\quad\le C\int_0^t \int \big(|\nabla u|^2 + |u|^4\big)\,dx\,ds + C\int_0^t \|\nabla u\|_{L^3}\|\nabla u\|_{L^2}^2\,ds \\
&\quad\le C\int_0^t \big[C_0^{\frac{1}{2}}\|\nabla u\|_{L^2}^3 + C_0^{\frac{1}{3}}\|\nabla u\|_{L^2}^4\big]\,ds + C\int_0^t \big(\|\nabla u\|_{L^3}^3 + \|\nabla u\|_{L^2}^3\big)\,ds \\
&\quad\le \frac{1}{4}\int_0^t \int m\dot{u}\,dx\,ds + C\int_0^t \big(\|\nabla u\|_{L^2}^3 + \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^2}^6\big)\,ds + CC_0,
\end{aligned}
\tag{2.43}
$$

which implies

$$
\begin{aligned}
&\int |\nabla u|^2\,dx + \int_0^t \int m|\dot{u}|^2\,dx\,ds \\
&\quad\le C(1+M) + Ct\Big(1 + \sup_{s\in[0,t]}\big\|\nabla u(\cdot,t)\big\|_{L^2}^6\Big) \\
&\quad\le C(1+M) + Ct\sup_{s\in[0,t]}\big\|\nabla u(\cdot,t)\big\|_{L^2}^6 \le C(1+M),
\end{aligned}
$$

we can easily obtain (2.41) when we choose $T_1 = \min\{1, \frac{1}{8C^3(1+M)^2}\}$. □

Proposition 2.2 *If (m,n,u) is the classical solution of* (1.1)-(1.3) *in* $[0,T]$ *as in Proposition* 2.1, *then there is a positive constant C such that*

$$
A_1(T) + A_2(T) \le C_0^{\frac{1}{2}},
\tag{2.44}
$$

provided $C_0 \le \varepsilon_1$.

Proof From Lemma 2.5, we get

$$
\begin{aligned}
A_1(T) + A_2(T) &\le CC_0 + C\int_0^T \int \sigma^3|\nabla u|^4\,dx\,dt + C\int_0^T \int \sigma|\nabla u|^3\,dx\,dt \\
&\quad + C\int_0^T \sigma^3\|u\|_{L^4}^4\,dt + C\int_0^T \int \sigma^3\big[|u||\nabla u||\dot{u}| + |\nabla u|^2|\dot{u}|\big]\,dx\,dt \\
&\quad + C\int_0^T \int \sigma\big(|\nabla u||u|^2 + |\nabla u||u|\big)\,dx\,dt.
\end{aligned}
\tag{2.45}
$$

We now estimate the term of the right hand sides of (2.45). Using (2.21), we have

$$\int_0^T\int\sigma^3|\nabla u|^4\,dx\,dt\le C\int_0^T\int\sigma^3\big[|F|^4+|\omega|^4+\big|P(m,n)-P(\tilde m,\tilde n)\big|^4\big]\,dx\,dt. \tag{2.46}$$

From (2.5)-(2.6), (2.14), and (2.20)-(2.22), we have

$$\begin{aligned}
&\int_0^T\int\sigma^3\big(|F|^4+|\omega|^4\big)\,dx\,dt\\
&\quad\le C\int_0^T\sigma^3\big(\|\nabla u\|_{L^2}+\big\|P(m,n)-P(\tilde m,\tilde n)\big\|_{L^2}\big)\|m\dot u\|_{L^2}^3\,dt\\
&\qquad+C\int_0^T\sigma^3\|\nabla u\|_{L^2}^4\,dt+C\int_0^T\sigma^3\big\|P(m,n)-P(\tilde m,\tilde n)\big\|_{L^2}\|\nabla u\|_{L^2}^3\,dt\\
&\quad\le C\sup_{0<t\le T}\big[\sigma^{\frac32}\|\sqrt{m}\dot u\|_{L^2}\big(\sigma^{\frac12}\|\nabla u\|_{L^2}+C_0^{\frac12}\big)\big]\int_0^T\int\sigma m|\dot u|^2\,dx\,dt\\
&\qquad+CC_0^{\frac32}\sup_{0<t\le T}\big(\sigma^{\frac12}\|\nabla u\|_{L^2}\big)+CC_0\sup_{0<t\le T}\big(\sigma\|\nabla u\|_{L^2}^2\big)\\
&\quad\le C\big(A_1^{\frac12}(T)+C_0^{\frac12}\big)A_2^{\frac12}(T)A_1(T)+CC_0^{\frac32}A_1(T)^{\frac12}+CC_0A_1(T)\\
&\quad\le CC_0,
\end{aligned} \tag{2.47}$$

using (2.23) and (2.47), we get

$$\int_0^T\int\sigma^3\big|P(m,n)-P(\tilde m,\tilde n)\big|^4\,ds\le C\bigg(\int_0^T\int\sigma^3|F|^4\,dx\,ds+C_0\bigg)\le CC_0. \tag{2.48}$$

Notice that

$$\int_0^T\int\sigma|\nabla u|^3\,dx\,ds=\int_0^{T_1\wedge T}\int\sigma|\nabla u|^3\,dx\,ds+\int_{T_1\wedge T}^T\int\sigma|\nabla u|^3\,dx\,ds. \tag{2.49}$$

By Young's inequality, (2.14) and (2.46)-(2.47), we obtain

$$\begin{aligned}
\int_{T_1\wedge T}^T\int\sigma|\nabla u|^3\,dx\,ds&\le\int_{T_1\wedge T}^T\int\sigma\big(|\nabla u|^4+|\nabla u|^2\big)\,dx\,ds\\
&\le C\int_{T_1\wedge T}^T\int\sigma^3|\nabla u|^4\,dx\,ds\\
&\qquad+\int_{T_1\wedge T}^T\int|\nabla u|^2\,dx\,ds\le CC_0,
\end{aligned} \tag{2.50}$$

and by (2.14), (2.22), and Lemma 2.6, we get

$$\begin{aligned}
\int_0^{T_1\wedge T}\sigma\|\nabla u\|_{L^3}^3\,dt&\le C\int_0^{T_1\wedge T}\sigma\|\nabla u\|_{L^2}^{\frac32}\big(\|m\dot u\|_{L^2}^{\frac32}+\|\nabla u\|_{L^2}^{\frac32}+C_0^{\frac14}+C_0^{\frac34}\big)\,dt\\
&\le C\int_0^{T_1\wedge T}\big(\sigma^{\frac14}\|\nabla u\|_{L^2}^{\frac32}\big)\bigg(\sigma\int m|\dot u|^2\,dx\bigg)^{\frac34}\,dt\\
&\qquad+C\int_0^{T_1\wedge T}\sigma\|\nabla u\|_{L^2}^3\,ds+CC_0
\end{aligned}$$

$$\begin{aligned}
&\le C \sup_{t\in(0,T_1\wedge T]} \left(\left(\sigma \|\nabla u\|_{L^2}^2 \right)^{\frac14} \|\nabla u\|_{L^2}^{\frac12} \right) \\
&\quad \times \int_0^{T_1\wedge T} \|\nabla u\|_{L^2}^{\frac12} \left(\sigma \int m|\dot u|^2\,dx \right)^{\frac34} dt \\
&\quad + \sup_{t\in(0,T_1\wedge T]} \left(\sigma \|\nabla u\|_{L^2}^2 \right)^{\frac12} \int_0^{T_1\wedge T} \sigma^{\frac12} \|\nabla u\|_{L^2}^2\,dt + CC_0 \\
&\le CA_1(T) C_0^{\frac14} + CC_0 \le CC_0^{\frac34}. \qquad (2.51)
\end{aligned}$$

Using (2.14), (2.18)-(2.22) we get

$$\int_0^T \int \sigma^3 |u|^4\,dx\,dt \le C \int_0^T \sigma^3 \left(C_0^{\frac12} \|\nabla u\|_{L^2}^3 + C_0^{\frac13} \|\nabla u\|_{L^2}^4 \right) dt \le CC_0, \qquad (2.52)$$

$$\begin{aligned}
&\int_0^T \int \sigma \left(|\nabla u||u|^2 + |\nabla u|^2|u| \right) dx \\
&\quad \le C \int_0^T \int |\nabla u|^2\,dx\,dt + \int_0^T \int \sigma^2 |u|^4\,dx\,dt + \int_0^T \sigma \|\nabla u\|_{L^3} \|\nabla u\|_{L^2}^2\,dt \\
&\quad \le C \int_0^T \int |\nabla u|^2\,dx\,dt + \int_0^T \int \sigma^2 |u|^4\,dx\,dt + C \int \sigma \left(\|\nabla u\|_{L^3}^3 + \|\nabla u\|_{L^2}^3 \right) dt \\
&\quad \le CC_0^{\frac34}, \qquad (2.53)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \int \sigma \left(|u||\nabla u|\dot u| + |\nabla u|^2|\dot u| \right) dx\,dt \\
&\quad \le \int_0^T \sigma^3 \|u\|_{L^3} \|\nabla u\|_{L^2} \|\nabla \dot u\|_{L^2}\,dt + C \int_0^T \sigma^3 \|\nabla u\|_{L^3} \|\nabla u\|_{L^2} \|\nabla \dot u\|_{L^2}\,dt \\
&\quad \le C \int_0^T \sigma^3 \left[C_0^{\frac14} \|\nabla u\|_{L^2}^{\frac12} + C_0^{\frac16} \|\nabla u\|_{L^2} \right] \|\nabla u\|_{L^2} \|\nabla \dot u\|_{L^2}\,dt \\
&\qquad + C \int_0^T \sigma^3 \|\nabla u\|_{L^3} \|\nabla u\|_{L^2} \|\nabla \dot u\|_{L^2}\,dt \\
&\quad \le CC_0^{\frac14} \int_0^T \sigma^3 \|\nabla u\|_{L^2}^{\frac32} \|\nabla \dot u\|_{L^2}\,dt + CC_0^{\frac16} \int_0^T \sigma^3 \|\nabla u\|_{L^2}^2 \|\nabla \dot u\|_{L^2}\,dt \\
&\qquad + C \int_0^T \sigma^3 \|\nabla u\|_{L^3} \|\nabla u\|_{L^2} \|\nabla \dot u\|_{L^2}\,dt \\
&\quad \le CC_0^{\frac34}, \qquad (2.54)
\end{aligned}$$

here we have used the following fact:

$$\begin{aligned}
&\int_0^T \sigma^3 \|\nabla u\|_{L^3} \|\nabla u\|_{L^2} \|\nabla \dot u\|_{L^2}\,dt \\
&\quad \le C \int_0^T \sigma^3 \|\nabla u\|_{L^3}^3\,ds + C \int_0^T \sigma^3 \|\nabla u\|_{L^2} \|\nabla u\|_{L^2}^{\frac12} \|\nabla \dot u\|_{L^2}^{\frac32}\,ds \\
&\quad \le CC_0^{\frac34} + CA_1(T)^{\frac12} \left(\int_0^T \sigma \|\nabla u\|_{L^2}^2\,ds \right)^{\frac14} \left(\int_0^T \sigma^3 \|\nabla \dot u\|_{L^2}^2\,ds \right)^{\frac34} \le CC_0^{\frac34}, \qquad (2.55)
\end{aligned}$$

which together with (2.45)-(2.54) give

$$A_1(T) + A_2(T) \le CC_0^{\frac{3}{4}} \le C_0^{\frac{1}{2}}, \tag{2.56}$$

when we choose $\varepsilon_1 \le C(\bar{m}, M)^{-4}$. This completes the proof of Proposition 2.2. □

From Proposition 2.2, we can obtain the following corollary.

Corollary 2.1 *If* (m,n,u) *is the classical solution of* (1.1)-(1.3) *in* $[0,T]$ *as in Proposition* 2.1, *then there is a positive constant C such that*

$$\sup_{0<t\le T} \|\nabla u\|_{L^2}^2 + \int_0^T \int m|\dot{u}|^2\,dx\,dt \le C(\bar{m}, M), \tag{2.57}$$

$$\sup_{0<t\le T} \int \sigma m|\dot{u}|^2\,dx + \int_0^T \int \sigma|\nabla\dot{u}|^2\,dx\,dt \le C(\bar{m}, M), \tag{2.58}$$

provided $C_0 < \varepsilon_1$.

Proof Obviously, we can get (2.57) by Lemma 2.6 and Proposition 2.2.

Next, we prove (2.58). Applying the operator $\sigma\dot{u}(\frac{\partial}{\partial t} + \operatorname{div}(u\cdot))$ to $(1.1)_3$ and integrating the resulting equality over $\Omega \times [0,T]$ and by using integration by parts, $(1.1)_1$, $(1.1)_2$, Young's inequality, (2.14), (2.22), and (2.57), we have

$$\begin{aligned}
&\sup_{0<t\le T} \int \sigma m|\dot{u}|^2\,dx + \int_0^T \int \sigma|\nabla\dot{u}|^2\,dx\,dt \\
&\quad\le \int_0^T \int \sigma_t m|\dot{u}|^2\,dx\,dt + C\int_0^T \int \sigma|\nabla u|^4\,dx\,dt + C(\bar{m})C_0 \\
&\qquad + C\int_0^T \int \sigma|u|^4\,dx + C\int_0^T \int \sigma\big[|u||\nabla u||\dot{u}| + |\nabla u|^2|\dot{u}|\big]\,dx\,dt \\
&\quad\le \int_0^{\sigma(T)} \int m|\dot{u}|^2\,dx\,dt + C\int_{T_1\wedge T}^T \int \sigma|\nabla u|^4\,dx\,dt + C\int_0^{T_1\wedge T} \int \sigma|\nabla u|^4\,dx\,dt \\
&\qquad + C\int_0^T \int \sigma^3|u|^4\,dx\,dt + C\int_0^T \int \sigma\big[|u||\nabla u||\dot{u}| + |\nabla u|^2|\dot{u}|\big]\,dx\,dt + C(\bar{m})C_0 \\
&\quad\le C(\bar{m}, M) + C\int_{T_1\wedge T}^T \int \sigma^3|\nabla u|^4\,dx\,dt + C\int_0^{T_1\wedge T} \int \sigma|\nabla u|^4\,dx\,dt \\
&\qquad + C\int_0^T \int \sigma\big[|u||\nabla u||\dot{u}| + |\nabla u|^2|\dot{u}|\big]\,dx\,dt \\
&\quad\le C(\bar{m}, M) + C\int_0^{T_1\wedge T} \sigma\|\nabla u\|_{L^2}\big(\|m\dot{u}\|_{L^2}^3 + \big\|P(m,n) - P(\tilde{m},\tilde{n})\big\|_{L^6}^3 \\
&\qquad + \|\nabla u\|_{L^2}^3 + \big\|P(m,n) - P(\tilde{m},\tilde{n})\big\|_{L^2}^3\big)\,dt \\
&\qquad + C\int_0^T \int \sigma\big[|u||\nabla u||\dot{u}| + |\nabla u|^2|\dot{u}|\big]\,dx\,dt \\
&\quad\le C(\bar{m}, M) + C(\bar{m}) \sup_{t\in(0,T_1\wedge T]} \big[\big(\sigma^{\frac{1}{2}}\|\nabla u\|_{L^2}\big)\big(\sigma^{\frac{1}{2}}\|m\dot{u}\|_{L^2}\big)\big]
\end{aligned}$$

$$\begin{aligned}
&\times \int_0^{T_1\wedge T} \|m\dot{u}\|_{L^2}^2\,dt + \sup_{t\in(0,T)}\big(\sigma\|\nabla u\|_{L^2}^2\big)\int_0^{T_1\wedge T}\|\nabla u\|_{L^2}^2\,dt + \frac{1}{2}\int_0^T \sigma\|\nabla\dot{u}\|_{L^2}^2\,dt \\
&\leq C(\bar{m},M) + C(\bar{m},M)\sup_{t\in(0,T]}\sigma^{\frac{1}{2}}\|m\dot{u}\|_{L^2} + \frac{1}{2}\int_0^T \sigma\|\nabla\dot{u}\|_{L^2}^2\,dt.
\end{aligned} \tag{2.59}$$

This completes the proof of Corollary 2.1. □

Next, we give the time-independent upper bound of m and n by using similar arguments to [22, 24]. It is noted that (2.11) plays a key role here.

Proposition 2.3 *If* (m,n,u) *is the classical solution of* (1.1)-(1.3) *in* $[0,T]$ *as in Proposition* 2.1, *then there is a positive constant* C *such that*

$$\sup_{0\leq t\leq T}\|m(t)\|_{L^\infty} \leq \frac{7\bar{m}}{4}, \qquad \sup_{0\leq t\leq T}\|n(t)\|_{L^\infty} \leq \frac{7\bar{m}}{4}\frac{\tilde{n}}{\tilde{m}}, \quad (x,t)\in\Omega\times[0,T], \tag{2.60}$$

provided that $C_0 \leq \varepsilon$.

Proof Rewrite the equation of the mass conservation $(1.1)_1$ as

$$D_t m = g(m) + b'(t),$$

where

$$\begin{aligned}
&D_t m \triangleq m_t + u\cdot\nabla m, \qquad g(m) \triangleq -\frac{m}{2\mu+\lambda}\big(P(m,n) - P(\tilde{m},\tilde{n})\big), \\
&b(t) \triangleq -\frac{1}{2\mu+\lambda}\int_0^t mF\,dt.
\end{aligned}$$

For $t\in[0,\sigma(T)]$, by Lemma 2.1, (2.14), (2.19), and (2.20), we have, for all $0\leq t_1<t_2\leq \sigma(T)$,

$$\begin{aligned}
&\big|b(t_2)-b(t_1)\big| \\
&\quad\leq C(\bar{m})\int_{t_1}^{t_2}\big\|F(\cdot,t)\big\|_{L^\infty}\,dt \\
&\quad\leq C(\bar{m})\int_0^{\sigma(T)}\big\|F(\cdot,t)\big\|_{L^2}^{1/4}\big\|\nabla F(\cdot,t)\big\|_{L^6}^{3/4}\,dt \\
&\quad\leq C(\bar{m})\int_0^{\sigma(T)}\big(\|\nabla u\|_{L^2}^{\frac{1}{4}} + \big\|P(m,n)-P(\tilde{m},\tilde{n})\big\|_{L^2}^{\frac{1}{4}}\big) \\
&\qquad\times\big(\|\nabla\dot{u}\|_{L^2}^{\frac{3}{4}} + \|m\dot{u}\|_{L^2}^{\frac{3}{4}} + \|\nabla u\|_{L^2}^{\frac{3}{4}} + \big\|P(m,n)-P(\tilde{m},\tilde{n})\big\|_{L^2}^{\frac{3}{4}} \\
&\qquad+ \big\|P(m,n)-P(\tilde{m},\tilde{n})\big\|_{L^6}^{\frac{3}{4}}\big)\,dt \\
&\quad\leq C(\bar{m})\int_0^{\sigma(T)}\big(\sigma^{-\frac{1}{2}}\big(\sigma^{\frac{1}{2}}\|\nabla u\|_{L^2}\big)^{\frac{1}{4}} + C_0^{\frac{1}{8}}\sigma^{-\frac{3}{8}}\big)\big[\big(\sigma\|\nabla\dot{u}\|_{L^2}^2\big)^{\frac{3}{8}} + \big(\big(\sigma\|m\dot{u}\|_{L^2}^2\big)^{\frac{3}{8}}\big)\big]\,dt \\
&\qquad+ C(\bar{m})\int_0^{\sigma(T)}\big(\sigma^{\frac{1}{2}}\|\nabla u\|_{L^2}\big)^{\frac{1}{4}}\sigma^{-\frac{1}{8}}\big(\big\|P(m,n)-P(\tilde{m},\tilde{n})\big\|_{L^2}^2\big)^{\frac{3}{8}}\,dt + C(\bar{m})C_0^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\le C(\bar{m})C_0^{\frac{1}{16}} \int_0^{\sigma(T)} \left(\sigma^{-\frac{1}{2}}+1\right)\left[\left(\sigma\|\nabla\dot{u}\|_{L^2}^2\right)^{\frac{3}{8}}+\left(\sigma\|m\dot{u}\|_{L^2}^2\right)^{\frac{3}{8}}\right]dt \\
&\quad + C(\bar{m})C_0^{\frac{1}{16}} \int_0^{\sigma(T)} \sigma^{-\frac{1}{8}}\left(\left\|P(m,n)-P(\tilde{m},\tilde{n})\right\|_{L^2}^2\right)^{\frac{3}{8}}dt + C(\tilde{m})C_0^{\frac{1}{2}} \\
&\le C(\bar{m})C_0^{\frac{1}{16}}\left(1+\int_0^1 \sigma^{-\frac{4}{5}}\,dt\right)^{\frac{5}{8}}\left[\left(\int_0^{\sigma(T)}\sigma\|\nabla\dot{u}\|_{L^2}^2\,dt\right)^{\frac{3}{8}}+\left(\int_0^{\sigma(T)}\sigma\|m\dot{u}\|_{L^2}^2\,dt\right)^{\frac{3}{8}}\right] \\
&\quad + C(\bar{m})C_0^{\frac{1}{16}}\left(\int_0^{\sigma(T)}\sigma^{-\frac{1}{5}}\,dt\right)^{\frac{5}{8}}\left(\int_0^{\sigma(T)}\left\|P(m,n)-P(\tilde{m},\tilde{n})\right\|_{L^2}^2\,dt\right)^{\frac{3}{8}} + C(\tilde{m})C_0^{\frac{1}{2}} \\
&\le C(\bar{m},M)C_0^{\frac{1}{16}},
\end{aligned}
$$

provided that $C_0 \le \varepsilon_1$. Therefore, for $t \in [0,\sigma(T)]$, choose $N_1 = 0$ and $N_0 = C(\bar{m},M)C_0^{\frac{1}{16}}$ and $\bar{\xi} = 2\tilde{m}$. Then

$$
\begin{aligned}
g(\xi) &= -\frac{\xi}{2\mu+\lambda}\left(P(\xi,\xi s_0)-P(\tilde{m},\tilde{n})\right) \\
&= -\frac{\xi}{2\mu+\lambda}\left(P(\xi,s_0\xi)-P(\xi,\tilde{n})+P(\xi,\tilde{n})-P(\tilde{m},\tilde{n})\right) \\
&= -\frac{\xi}{2\mu+\lambda}\left(P_m\left(\tilde{m}+\theta_1(\xi-\tilde{m}),\tilde{n}\right)(\xi-\tilde{m})+P_n\left(\xi,\tilde{n}+(s_0\xi-\tilde{n})\theta_2\right)(s_0\xi-\tilde{n})\right) \\
&\triangleq -\frac{1}{2\mu+\lambda}z(\xi),
\end{aligned}
\tag{2.61}
$$

where $\theta_1,\theta_2 \in (0,1)$ are constants. From Remark 2.2 and (1.14), we obtain

$$
z(\xi) \ge 2a_0C^0\xi(\underline{s}_0\xi-\tilde{n}) \ge 4a_0C^0\tilde{m}(2\underline{s}_0\tilde{m}-\tilde{n}) \ge 0, \quad \text{for all } \xi \ge \bar{\xi} = 2\tilde{m},
$$

and Lemma 2.2 yields

$$
\sup_{t\in[0,\sigma(T)]}\|m\|_{L^\infty} \le \max\{\bar{m},2\tilde{m}\} + C(\bar{m},M)C_0^{\frac{1}{16}} \le \bar{m} + C(\bar{m},M)C_0^{\frac{1}{16}} \le \frac{3}{2}\bar{m}, \tag{2.62}
$$

provided that

$$
C_0 \le \min\{\varepsilon_1,\varepsilon_2\}, \quad \text{where } \varepsilon_2 = \left(\frac{\bar{m}}{2C(\bar{m},M)}\right)^{16}.
$$

When $t \in [\sigma(T),T]$, by using Lemma 2.1, Proposition 2.2, (2.14), (2.19), and (2.58), we have

$$
\begin{aligned}
\left|b(t_2)-b(t_1)\right| &\le C(\bar{m})\int_{t_1}^{t_2}\left\|F(\cdot,t)\right\|_{L^\infty}dt \\
&\le \frac{a_0C^0}{2\mu+\lambda}(t_2-t_1) + C(\bar{m})\int_{t_2}^{t_1}\left\|F(\cdot,t)\right\|_{L^\infty}^{\frac{8}{3}}dt \\
&\le \frac{a_0C^0}{2\mu+\lambda}(t_2-t_1) + C(\bar{m})\int_{t_2}^{t_1}\left\|F(\cdot,t)\right\|_{L^2}^{2/3}\left\|\nabla F(\cdot,t)\right\|_{L^6}^2\,dt \\
&\le \frac{a_0C^0}{2\mu+\lambda}(t_2-t_1) + C(\bar{m})\int_{t_2}^{t_1}\left(\|\nabla\dot{u}\|_{L^2}^{\frac{2}{3}} + \left\|P(m,n)-P(\tilde{m},\tilde{n})\right\|_{L^2}^{\frac{2}{3}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(\|m\dot{u}\|_{L^6}^2 + \|\nabla u\|_{L^2}^2 + \left\|P(m,n) - P(\tilde{m},\tilde{n})\right\|_{L^6}^2\right) dt \\
\leq{}& \frac{a_0 C^0}{2\mu+\lambda}(t_2 - t_1) + C(\bar{m}) C_0^{\frac{1}{6}} \int_{\sigma(T)}^T \|\nabla \dot{u}\|_{L^2}^2 \, dt \\
&+ C(\bar{m}) C_0^{\frac{1}{6}} + C(\bar{m}) C_0^{\frac{1}{2}} (t_2 - t_1) + C_0^{\frac{1}{6}} \int_{\sigma(T)}^T \|m\dot{u}\|_{L^2}^2 \, dt \\
\leq{}& \left(\frac{a_0 C^0}{2\mu+\lambda} + C(\bar{m}) C_0^{\frac{1}{2}}\right)(t_2 - t_1) + C(\bar{m}) C_0^{\frac{2}{3}} \\
\leq{}& \frac{2a_0 C^0}{2\mu+\lambda}(t_2 - t_1) + C(\bar{m}) C_0^{\frac{2}{3}},
\end{aligned}
$$

provided that $C_0 \leq \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, where $\varepsilon_3 = (\frac{a_0 C^0}{C(\bar{m})(2\mu+\lambda)})^2$. Therefore, for $t \in [\sigma(T), T]$, choose $N_1 = \frac{2a_0 C^0}{2\mu+\lambda}$ and $N_0 = C(\bar{m}) C_0^{\frac{2}{3}}$ and $\bar{\xi} = (2+\delta_1)\tilde{m}$. From Remark 2.2, (1.9), and (1.14), we obtain

$$
\begin{aligned}
&z(\xi) \geq 2a_0 C^0 (2+\delta_1)\tilde{m}\big((2+\delta_1)\underline{s}_0 \tilde{m} - \tilde{n}\big) \geq 2a_0 C^0 (2+\delta_1)\tilde{m}\tilde{n}\frac{\delta_1}{2} \geq 2a_0 C^0, \\
&\quad \text{for all } \xi \geq \bar{\xi} = (2+\delta_1)\tilde{m},
\end{aligned}
$$

where $\delta_1 \in (0,1]$ is a small constant. Then Lemma 2.3 yields

$$
\begin{aligned}
\sup_{t\in[\sigma(T),T]} \|m\|_{L^\infty} &\leq \max\left\{\frac{3}{2}\bar{m}, (2+\delta_1)\tilde{m}\right\} + C\big(\bar{m}, \tilde{m}, \tilde{n}, C^0, a_0, \underline{s}_0, \mu, \lambda\big) C_0^{\frac{2}{3}} \\
&\leq \frac{3}{2}\bar{m} + C(\bar{m}) C_0^{\frac{2}{3}} \leq \frac{7}{4}\bar{m},
\end{aligned}
\tag{2.63}
$$

provided that

$$
C_0 \leq \varepsilon \triangleq \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}, \quad \text{where } \varepsilon_4 = \left(\frac{\bar{m}}{4C(\bar{m})}\right)^{\frac{3}{2}}.
$$

Then (2.62) and (2.63) complete the proof of Proposition 2.3. □

From now on, we will assume that the initial energy $C_0 < \varepsilon$ and the constant C may depend on T, $\|\sqrt{m}g\|_{L^2}$, $\|\nabla g\|_{L^2}$, $\|(m_0 - \tilde{m}, n_0 - \tilde{n})\|_{H^3}$, $\|u_0\|_{D^1\cap D^3}$, besides μ, λ, $\tilde{m}$, $\tilde{n}$, $\bar{m}$, a_0, C^0, $\underline{s}_0$, and M, where g is the same as in (1.12).

Finally, we give the proof of the high-order regularity estimates of (m,n,u), which is due to [22–24, 30] for the single-fluid Navier-Stokes equations.

Proposition 2.4 ([20, 22]) *If* (m,n,u) *is a classical solution of* (1.1)-(1.3) *in* $[0,T]$, *we have the following estimates*:

$$
\sup_{0<t\leq T} \int m|\dot{u}|^2 \, dx + \int_0^T \int |\nabla \dot{u}|^2 \, dx \, dt \leq C,
\tag{2.64}
$$

$$
\sup_{0<t\leq T} \left(\|\nabla m\|_{L^2\cap L^6} + \|\nabla n\|_{L^2\cap L^6} + \|\nabla u\|_{H^1}\right) + \int_0^T \|\nabla u\|_{L^\infty} \, dt \leq C.
\tag{2.65}
$$

Proof Applying the operator $\dot{u}(\frac{\partial}{\partial t} + \operatorname{div}(u\cdot))$ to $(1.1)_3$ and integrating the resulting equality over $[0, T]$, we have

$$\left(\frac{1}{2}\int m|\dot{u}|^2\,dx\right)_t = -\int \dot{u}^j\left[\partial_j P_t + \operatorname{div}(\partial_j P u)\right]dx + \int \mu\dot{u}^j\left(\Delta u_t^j + \operatorname{div}\left(u\Delta u^j\right)\right)dx$$
$$+ \int(\lambda+\mu)\dot{u}^j\left(\partial_j\partial_t(\operatorname{div}u) + \operatorname{div}\left(u\partial_j(\operatorname{div}u)\right)\right)dx.$$

Using integration by parts, $(1.1)_1$, $(1.1)_2$, and Young's inequality, we have

$$\begin{aligned}
&\left(\int m|\dot{u}|^2\,dx\right)_t + \int|\nabla\dot{u}|^2\,dx \\
&\quad\le C\left(\|\nabla u\|_{L^4}^4 + \|\nabla u\|_{L^2}^2 + \|u\|_{L^4}^4\right) + C\int\left(|u||\nabla u||\dot{u}| + |\nabla u|^2|\dot{u}|\right)dx \\
&\quad\le C\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}^3 + C\|\nabla u\|_{L^2}^3 + C\int\left(|u||\nabla u||\dot{u}| + |\nabla u|^2|\dot{u}|\right)dx + C \\
&\quad\le C\left(\|F\|_{L^6}^3 + \|\omega\|_{L^6}^3 + \left\|P(m,n) - P(\tilde{m},\tilde{n})\right\|_{L^6}^3\right) + \delta\|\nabla\dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^3}^3 + C \\
&\quad\le C\left(\|\nabla F\|_{L^2}^3 + \|\nabla\omega\|_{L^2}^3\right) + \frac{1}{2}\|\nabla\dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^3}^3 + C \\
&\quad\le C\|\sqrt{m}\dot{u}\|_{L^2}^4 + C\|\nabla u\|_{L^3}^3 + \frac{1}{2}\|\nabla\dot{u}\|_{L^2}^2 + C,
\end{aligned}$$

using (2.22) and

$$\begin{aligned}
\int_0^T\|\nabla u\|_{L^3}^3\,ds &\le \int_0^{T\wedge T_1}\|\nabla u\|_{L^3}^3\,ds + \int_{T\wedge T_1}^T\sigma^3\|\nabla u\|_{L^3}^3\,ds \\
&\le C + C\int_0^{T\wedge T_1}\|\nabla u\|_{L^2}^{\frac{3}{2}}\left(\|m\dot{u}\|_{L^2}^{\frac{3}{2}} + \left\|P(m,n) - P(\tilde{m},\tilde{n})\right\|_{L^2}^{\frac{3}{2}}\right. \\
&\quad\left.\times\left\|P(m,n) - P(\tilde{m},\tilde{n})\right\|_{L^6}^{\frac{3}{2}} + \|\nabla u\|_{L^2}^{\frac{3}{2}}\right)ds \\
&\le C
\end{aligned}$$

and the compatibility condition, we can define

$$\sqrt{m}\dot{u}|_{t=0} = -\sqrt{m_0}g.$$

Then Gronwall's inequality gives (2.64).

Next, we prove (2.65). For $p \in [2, 6]$, differentiating $(1.1)_1$ with respect to x_i, and then multiplying both sides of the result equation by $p|\partial_i m|^{p-2}\partial_i m$, we get

$$\begin{aligned}
&\left(|\nabla m|^p\right)_t + \operatorname{div}\left(|\nabla m|^p u\right) + (p-1)|\nabla m|^p\operatorname{div}u \\
&\quad + p|\nabla m|^{p-2}(\nabla m)^T\nabla u(\nabla m) + pm|\nabla m|^{p-2}\nabla m\cdot\nabla\operatorname{div}u = 0.
\end{aligned}\tag{2.66}$$

Similarly, we can obtain

$$\begin{aligned}
&\left(|\nabla n|^p\right)_t + \operatorname{div}\left(|\nabla n|^p u\right) + (p-1)|\nabla n|^p\operatorname{div}u \\
&\quad + p|\nabla n|^{p-2}(\nabla n)^T\nabla u(\nabla n) + pn|\nabla n|^{p-2}\nabla n\cdot\nabla\operatorname{div}u = 0.
\end{aligned}\tag{2.67}$$

By the standard L^p-estimate for an elliptic system,

$$-\mu\Delta u-(\lambda+\mu)\nabla\operatorname{div}u=m\dot{u}+\nabla P,\quad \text{in } \Omega,$$
$$(u_1,u_2,u_3)=\beta\left(u^1_{x_3},u^2_{x_3},0\right)\quad \text{in } \partial\Omega, \tag{2.68}$$

we obtain

$$\left\|\nabla^2 u\right\|_{L^p}\le C\left(\|m\dot{u}\|_{L^p}+\|\nabla P\|_{L^p}\right). \tag{2.69}$$

Now, we give the estimate for $\|\nabla u\|_{L^\infty}$, which is crucial to obtain the estimate of $\|(\nabla m,\nabla n)\|_{L^p}$. As in [22], we set $w=u-v$; w and v satisfy

$$\begin{cases}-\mu\Delta v-(\mu+\lambda)\nabla\operatorname{div}v=-\nabla(P(m,n)-P(\tilde{m},\tilde{n})),\quad \text{in } \Omega,\\ (v^1(x),v^2(x),v^3(x))=\beta(v^1_{x_3}(x),v^2_{x_3}(x),0)\quad \text{on } \partial\Omega, t>0,\end{cases}$$

then, by the standard regularity estimate for elliptic systems, we have

$$\begin{aligned}&\|\nabla v\|_{L^q}\le C\left(\left\|P(m,n)-P(\tilde{m},\tilde{n})\right\|_{L^q}\right),\\ &\left\|\nabla^2 v\right\|_{L^q}\le C\left\|\nabla\left(P(m,n)-P(\tilde{m},\tilde{n})\right)\right\|_{L^q},\quad \text{for } q\in[2,\infty)\end{aligned} \tag{2.70}$$

and w satisfies

$$\begin{cases}-\mu\Delta w-(\mu+\lambda)\nabla\operatorname{div}w=m\dot{u},\quad \text{in } \Omega,\\ (w^1(x),w^2(x),w^3(x))=\beta(w^1_{x_3}(x),w^2_{x_3}(x),0)\quad \text{on } \partial\Omega, t>0.\end{cases}$$

Similarly,

$$\left\|\nabla^2 w\right\|_{L^q}\le C\|m\dot{u}\|_{L^q},\qquad \|\nabla w\|_{L^\infty}\le\left(\|m\cdot u\|_{L^2}+\|m\dot{u}\|_{L^6}\right)\quad \text{for } q\in(1,\infty). \tag{2.71}$$

In order to obtain $\|\nabla v\|_{L^\infty}$, we give the following fact.

Remark 2.3 Let $\Omega=\{x\in\mathbb{R}^3\mid x_3>0\}$ and $\nabla v\in W^{1,q}(\Omega)$ with $q\in(3,\infty)$. There exists a constant C depending only on q such that

$$\|\nabla v\|_{L^\infty}\le C\left(1+\ln\left(e+\left\|\nabla^2 v\right\|_{L^q}\right)\right)\|\nabla v\|_{BMO},\quad \text{with } q\in(3,\infty); \tag{2.72}$$

here

$$\|\nabla v\|_{BMO}=\|\nabla v\|_{L^2}+[\nabla v]_{BMO},\qquad [\nabla v]_{BMO}=\sup_{r>0,x\in\Omega}\frac{1}{\Omega_r(x)}\int_{\Omega_r(x)}\left|\nabla v(y)-\nabla v_r(x)\right|dy,$$
$$\nabla v_r(x)=\frac{1}{|\Omega_r(x)|}\int_{\Omega_r(x)}\nabla v(y)\,dy.$$

Then, by using the classical theory for elliptic systems, we have

$$\|\nabla v\|_{BMO}\le C\left(\left\|P(m,n)-P(\tilde{m},\tilde{n})\right\|_{L^2}+\left\|P(m,n)-P(\tilde{m},\tilde{n})\right\|_{L^\infty}\right)\le C(\bar{m}),$$

which together with (2.72) implies

$$\|\nabla v\|_{L^\infty} \le C\bigl(1 + \ln\bigl(e + \|\nabla^2 v\|_{L^q}\bigr)\bigr). \tag{2.73}$$

From (2.66) and (2.67) we have

$$\begin{aligned} \partial_t\bigl(\|\nabla m\|_{L^p} + \|\nabla n\|_{L^p}\bigr) &\le C\|\nabla u\|_{L^\infty}\bigl(\|\nabla m\|_{L^p} + \|\nabla n\|_{L^p}\bigr) + C\|\nabla^2 u\|_{L^p} \\ &\le C\bigl(1 + \|\nabla u\|_{L^\infty}\bigr)\bigl(\|\nabla m\|_{L^p} + \|\nabla n\|_{L^p}\bigr) + C\|m\dot{u}\|_{L^p}. \end{aligned} \tag{2.74}$$

Then, using Remark 2.3 and (2.70), we obtain

$$\begin{aligned} \|\nabla u\|_{L^\infty} &\le C\bigl(\|\nabla w\|_{L^\infty} + \|\nabla v\|_{L^\infty}\bigr) \\ &\le C\|\nabla w\|_{L^\infty} + \bigl(1 + \ln\bigl(e + \|\nabla^2 v\|_{L^q}\bigr)\bigr) \\ &\le C\bigl(1 + \|m\dot{u}\|_{L^6} + \ln\bigl(e + \|\nabla m\|_q + \|\nabla n\|_{L^q}\bigr)\bigr). \end{aligned} \tag{2.75}$$

We set

$$f(t) = e + \|\nabla m\|_{L^6} + \|\nabla n\|_{L^6}, \qquad g(t) = 1 + \|m\dot{u}\|_{L^6}.$$

Substituting (2.74) into (2.75) with $p = 6$, we have

$$f'(t) \le Cg(t)f(t) + Cf(t)\ln f(t) + Cg(t),$$

which yields

$$\bigl(\ln f(t)\bigr)' \le Cg(t) + C\ln f(t). \tag{2.76}$$

Then Lemma 2.4, (2.60), and (2.64) imply

$$\int_0^T g(t)\,dt \le C\int_0^T \bigl(1 + \|m\dot{u}\|_{L^6}\bigr)\,dt \le C\int_0^T \bigl(1 + \|\nabla\dot{u}\|_{L^2}\bigr)\,dt \le C, \tag{2.77}$$

and we get by using Gronwall's inequality and (2.76)

$$\sup_{0\le t\le T} f(t) \le C,$$

i.e.,

$$\sup_{0\le t\le T}\bigl(\|\nabla m\|_{L^6} + \|\nabla n\|_{L^6}\bigr) \le C. \tag{2.78}$$

From (2.75), (2.77), and (2.78), we obtain

$$\int_0^T \|\nabla u\|_{L^\infty}\,dt \le C. \tag{2.79}$$

Setting $p = 2$ in (2.74), we have

$$\sup_{0\le t\le T}\left(\|\nabla m\|_{L^2} + \|\nabla n\|_{L^2}\right) \le C. \tag{2.80}$$

This completes the proof of Proposition 2.4. □

Corollary 2.2 *If* (m, n, u) *is a classical solution of* (1.1)-(1.3) *in* $[0, T]$, *we have the following estimates*:

$$\sup_{0\le t\le T}\int m|u_t|^2\,dx + \int_0^T\int |\nabla u_t|^2\,dx\,ds \le C. \tag{2.81}$$

In order to get the higher-order regularity estimates of the solution, we need to bound P_{mm}, P_{mn}, P_{nn}, P_{mmm}, P_{mmn}, P_{mnn}, and P_{nnn}.

Lemma 2.7 *Under the conditions of Theorem* 1.1, *we have*

$$|P_{mm}| \le C, \qquad |P_{mn}| \le C, \qquad |P_{nn}| \le C, \tag{2.82}$$

$$\begin{aligned} &|P_{mmm}| \le C, \qquad |P_{mmn}| \le C, \qquad |P_{nnm}| \le C, \\ &|P_{nnn}| \le C, \quad \text{in } \mathbb{R}^3 \times [0, T]. \end{aligned} \tag{2.83}$$

Proof Since

$$\begin{aligned}
&P_{mm} = C^0\frac{c}{(b^2+c)^{\frac{3}{2}}}, \qquad P_{nn} = -C^0\frac{4k_0a_0^2m}{(b^2+c)^{\frac{3}{2}}}, \qquad P_{mn} = C^0 2k_0a_0\frac{(k_0-m)+a_0n}{(b^2+c)^{\frac{3}{2}}}, \\
&P_{mmn} = \frac{4k_0a_0C^0}{(b^2+c)^{\frac{3}{2}}} - \frac{12C^0k_0a_0n(k_0a_0+a_0m+a_0^2n)}{(b^2+c)^{\frac{5}{2}}}, \qquad P_{mmm} = \frac{12C^0k_0a_0nb}{(b^2+c)^{\frac{5}{2}}}, \\
&P_{nnn} = \frac{12C^0k_0a_0^2m(k_0a_0+a_0m+a_0^2n)}{(b^2+c)^{\frac{5}{2}}}, \\
&P_{mnn} = \frac{2k_0a_0^2C^0}{(b^2+c)^{\frac{3}{2}}} - \frac{6C^0k_0a_0^2(k_0^2-m^2+2a_0k_0n+a_0^2n)}{(b^2+c)^{\frac{5}{2}}},
\end{aligned}$$

by (2.7) and (2.60), we can obtain (2.82) and (2.83) easily. □

Proposition 2.5 *If* (m, n, u) *is a classical solution of* (1.1)-(1.3) *in* $[0, T]$, *we have the following estimates*:

$$\sup_{0\le t\le T}\left(\|m-\tilde{m}\|_{H^2} + \|n-\tilde{n}\|_{H^2} + \left\|P(m,n)-P(\tilde{m},\tilde{n})\right\|_{H^2}\right) \le C. \tag{2.84}$$

Proof At first, we give the elliptic estimate as follows:

$$\|\nabla u\|_{H^2} \le C\left(\|F\|_{H^2} + \|\omega\|_{H^2} + \left\|P(m,n)-P(\tilde{m},\tilde{n})\right\|_{H^2}\right). \tag{2.85}$$

Then from $(1.1)_1$, $(1.1)_2$, and the above estimate, we have

$$\begin{aligned} &\frac{d}{dt}\big(\|\nabla^2 m\|_{L^2}^2 + \|\nabla^2 n\|_{L^2}^2\big) \\ &\quad\le C\big(1 + \|\nabla u\|_{L^\infty}\big)\big(\|\nabla^2 m\|_{L^2}^2 + \|\nabla^2 n\|_{L^2}^2\big) + C\|\nabla u\|_{H^2}^2 + C. \end{aligned} \tag{2.86}$$

Using Proposition 2.4 and the same idea as the proof in [22] (Lemma 4.3), we have

$$\begin{aligned} &\|F\|_{H^2} + \|\omega\|_{H^2} + \|P(m,n) - P(\tilde{m},\tilde{n})\|_{H^2} \\ &\quad\le C\big(\|F\|_{H^1} + \|\omega\|_{H^1} + \|m\dot{u}\|_{H^1} + \|P(m,n) - P(\tilde{m},\tilde{n})\|_{H^1}\big) \\ &\qquad + C\big(\|\nabla^2 m\|_{L^2}\|\nabla^2 n\|_{L^2}\big) \\ &\quad\le C\big(1 + \|m\dot{u}\|_{L^2} + \|\nabla(m\dot{u})\|_{L^2}\big) + C\big(\|\nabla^2 m\|_{L^2} + \|\nabla^2 n\|_{L^2}\big) \\ &\quad\le C\big(1 + \|\nabla\dot{u}\|_{L^2} + \|\nabla m\|_{L^3}\|\dot{u}\|_{L^6}\big) + C\big(\|\nabla^2 m\|_{L^2} + \|\nabla^2 n\|_{L^2}\big), \end{aligned}$$

which together with Proposition 2.4 and Gronwall's inequality yields

$$\sup_{0\le t\le T}\int\big(\|\nabla^2 m\|_{L^2} + \|\nabla^2 n\|_{L^2}\big)\,dx \le C.$$

From the lemma, we have $\sup_{0\le t\le T}\|P(m,n) - P(\tilde{m},\tilde{n})\|_{H^2} \le C$. Thus we finish the proof of Proposition 2.5. □

Proposition 2.6 *If* (m,n,u) *is a classical solution of* (1.1)-(1.3) *in* $[0,T]$, *we have the following estimates*:

$$\sup_{0\le t\le T}\big(\|m_t\|_{H^1} + \|n_t\|_{H^1} + \|P_t\|_{H^1}\big) + \int_0^T\big(\|m_{tt}\|_{L^2}^2 + \|n_{tt}\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2\big)\,dt \le C. \tag{2.87}$$

Proof From $(1.1)_1$, (2.1), (2.60) and (2.65), we get

$$\|m_t\|_{L^2} \le C\|u\|_{L^\infty}\|\nabla m\|_{L^2} + C\|\nabla u\|_{L^2} \le C. \tag{2.88}$$

Similarly, we have

$$\|n_t\|_{L^2} \le C\|u\|_{L^\infty}\|\nabla n\|_{L^2} + C\|\nabla u\|_{L^2} \le C. \tag{2.89}$$

Applying the ∇ operator to $(1.1)_1$ yields

$$\partial_j m_t + \partial_j u^i\partial_i m + u^i\partial_i\partial_j m + \partial_j m \operatorname{div} u + m\partial_j \operatorname{div} u = 0. \tag{2.90}$$

By (2.1), (2.60), (2.65), and (2.84), we can obtain

$$\begin{aligned} \|\nabla m_t\|_{L^2} &\le C\|\nabla u\|_{L^3}\|\nabla m\|_{L^6} + C\|u\|_{L^\infty}\|\nabla^2 m\|_{L^2} + \|\nabla^2 u\|_{L^2} \\ &\le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 m\|_{L^2} + C\|u\|_{L^2}^{\frac{1}{4}}\|\nabla^2 u\|_{L^2}^{\frac{3}{4}} + C\|\nabla^2 u\|_{L^2} \le C. \end{aligned} \tag{2.91}$$

Similarly, we have

$$\|\nabla n_t\|_{L^2} \leq C. \tag{2.92}$$

Next, differentiating $(1.1)_1$ with respect to t yields

$$m_{tt} + u_t \cdot \nabla m + u \cdot \nabla m_t + m_t \operatorname{div} u + m \operatorname{div} u_t = 0. \tag{2.93}$$

Thus, we get from Lemma (2.1), (2.65), (2.81), (2.84), (2.92), and (2.93)

$$\begin{aligned}\int_0^T \|m_{tt}\|_{L^2}^2\,dt &\leq C\int_0^T \left(\|u_t\|_{L^6}^2\|\nabla m\|_{L^3}^2 + \|\nabla m_t\|_{L^2}^2 + \|m_t\|_{L^6}^2\|\nabla u\|_{L^3}^2 + \|\nabla u_t\|_{L^2}^2\right)dt \\ &\leq C. \end{aligned}\tag{2.94}$$

Similarly, we have

$$\int_0^T \|\nabla n_{tt}\|_{L^2}^2\,dt \leq C. \tag{2.95}$$

The estimates for P_t and P_{tt} can be dealt with in a similar way. □

Corollary 2.3 [20, 22] *If* (m, n, u) *is a classical solution of* (1.1)-(1.3) *in* $[0, T]$, *we have the following estimates*:

$$\sup_{0\leq t\leq T}\int |\nabla u_t|^2\,dx + \int_0^T\int m u_{tt}^2\,dx\,dt \leq C. \tag{2.96}$$

Proposition 2.7 *If* (m, n, u) *is a classical solution of* (1.1)-(1.3) *in* $[0, T]$, *we have the following estimates*:

$$\sup_{0\leq t\leq T}\left(\|m-\tilde{m}\|_{H^3} + \|n-\tilde{n}\|_{H^3} + \left\|P(m,n)-P(\tilde{m},\tilde{n})\right\|_{H^3}\right) \leq C, \tag{2.97}$$

$$\sup_{0\leq t\leq T}\|\nabla u\|_{H^2} + \int_0^T \left(\|\nabla u\|_{H^3}^2 + \|\nabla u_t\|_{H^1}^2\right)dt \leq C. \tag{2.98}$$

Proof By using the Hölder inequality, Young's inequality, Lemma 2.1, (2.60), (2.65), (2.84), and (2.87), we have

$$\begin{aligned}\left\|\nabla(m\dot{u})\right\|_{L^2} &= \left\|\nabla(mu_t + mu\cdot\nabla u)\right\|_{L^2} \leq C\left\||\nabla m||u_t|\right\|_{L^2} + C\|m\nabla u_t\|_{L^2} \\ &\quad + C\left\||\nabla m||u||\nabla u|\right\|_{L^2} + C\left\|m|\nabla u|^2\right\|_{L^2} + C\left\|m|u||\nabla^2 u|\right\|_{L^2} \\ &\leq C\|\nabla m\|_{L^3}\|u_t\|_{L^6} + C\|\nabla u_t\|_{L^2} + C\|u\|_{L^\infty}\|\nabla m\|_{L^3}\|\nabla u\|_{L^6} \\ &\quad + C\|\nabla u\|_{L^3}\|\nabla u\|_{L^6} + C\|u\|_{L^\infty}\left\|\nabla^2 u\right\|_{L^2} \\ &\leq C, \end{aligned}\tag{2.99}$$

which together with (2.64) imply

$$\sup_{0\leq t\leq T}\|m\dot{u}\|_{H^1} \leq C. \tag{2.100}$$

The standard H^1-estimate for the elliptic equations gives

$$\begin{aligned}\|\nabla^2 u\|_{H^1} &\le C\|\mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u\|_{H^1}\\ &\le C\|m\dot{u} + \nabla P\|_{H^1}\\ &\le C\|m\dot{u}\|_{H^1} + C\|\nabla P\|_{H^1} \le C,\end{aligned} \tag{2.101}$$

where we have used $(1.1)_3$, Lemma 2.1, (2.60), (2.65), (2.84), and (2.87) give

$$\sup_{0<t\le T}\|\nabla u\|_{H^2} \le C. \tag{2.102}$$

Next, by using the standard L^2-estimate for the elliptic system, we can obtain

$$\begin{aligned}\|\nabla^2 u_t\|_{L^2} &\le C\|\mu\Delta u_t + (\mu+\lambda)\nabla \operatorname{div} u_t\|_{L^2}\\ &= C\|m u_{tt} + m_t u_t + m_t u\cdot\nabla u + m u_t\cdot\nabla u + m u\cdot\nabla u_t + \nabla P_t\|_{L^2}\\ &\le C\|m u_{tt}\|_{L^2} + C\|m_t\|_{L^3}\|u_t\|_{L^6} + C\|u\|_{L^\infty}\|m_t\|_{L^3}\|\nabla u\|_{L^6}\\ &\quad + \|u_t\|_{L^6}\|\nabla u\|_{L^3} + C\|u\|_{L^\infty}\|\nabla u_t\|_{L^2} + C\|\nabla P_t\|_{L^2}\\ &\le C\|m u_{tt}\|_{L^2} + C,\end{aligned} \tag{2.103}$$

which together with (2.96) imply

$$\int_0^T \|\nabla u_t\|_{H^1}^2\,dt \le C. \tag{2.104}$$

In order to estimate $\|\nabla^2 u\|_{H^2}$, we use the standard H^2-estimate of the elliptic equations again to get

$$\begin{aligned}\|\nabla^2 u\|_{H^2} &\le C\|\mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u\|_{H^2}\\ &\le C\|m\dot{u}\|_{H^2} + C\|\nabla P\|_{H^2}\\ &\le C + C\|\nabla u_t\|_{H^1} + C\big(\|\nabla^3 m\|_{L^2} + \|\nabla^3 n\|_{L^2}\big),\end{aligned} \tag{2.105}$$

here we also have used $(1.1)_3$, Lemma 2.1, Lemma 2.4, (2.101), and the following simple facts:

$$\begin{aligned}\|\nabla^2(m u_t)\|_{L^2} &\le C\big\||\nabla^2 m||u_t|\big\|_{L^2} + C\big\||\nabla m||\nabla u_t|\big\|_{L^2} + C\|\nabla^2 u_t\|_{L^2}\\ &\le C\|\nabla^2 m\|_{L^2}\|\nabla u_t\|_{L^2} + C\|\nabla m\|_{L^3}\|\nabla u_t\|_{L^6} + C\|\nabla^2 u\|_{L^2}\\ &\le C + C\|\nabla u_t\|_{L^2}\end{aligned} \tag{2.106}$$

and

$$\begin{aligned}\|\nabla^2(m u\cdot\nabla u)\|_{L^2} &\le C\big\||\nabla^2(m u)||\nabla u|\big\|_{L^2} + C\big\||\nabla(m u)||\nabla^2 u|\big\|_{L^2} + C\|\nabla^3 u\|_{L^2}\\ &\le C + C\|\nabla^2(m u)\|_{L^2}\|\nabla u\|_{H^2} + C\|\nabla(m u)\|_{L^3}\|\nabla^2 u\|_{L^6}\end{aligned}$$

$$\begin{aligned} &\le C + C\|\nabla^2 m\|_{L^2}\|u\|_{L^\infty} + C\|\nabla m\|_{L^6}\|\nabla u\|_{L^3} + C\|\nabla^2 u\|_{L^2} \\ &\le C. \end{aligned} \tag{2.107}$$

Applying the ∇^3 operator to $(1.1)_1$, multiplying with $\nabla^3 m$, and integrating the resulting equation over $\mathbb{R}^3 \times [0,T]$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int|\nabla^3 m|^2\,dx &= -\int_0^T\int\nabla^3\big(\operatorname{div}(mu)\big)\cdot\nabla^3 m\,dx\,dt \\ &\le C\big(\big\||\nabla^3 u||\nabla m|\big\|_{L^2} + \big\||\nabla^2 u||\nabla^2 m|\big\|_{L^2} + \big\||\nabla u||\nabla^3 m|\big\|_{L^2} \\ &\quad + \|\nabla^4 u\|_{L^2}\big)\|\nabla^3 m\|_{L^2} \\ &\le C\big(\|\nabla^3 u\|_{L^2}\|\nabla m\|_{H^2} + \|\nabla^2 u\|_{L^3}\|\nabla^2 m\|_{L^6} \\ &\quad + \|\nabla^3 m\|_{L^2}\|\nabla u\|_{L^\infty}\big)\|\nabla^3 m\|_{L^2} \\ &\quad + C\big(1 + \|\nabla^2 u_t\|_{L^2} + \|\nabla^3 m\|_{L^2}\big)\|\nabla^3 m\|_{L^2} \\ &\le C + C\|\nabla^2 u_t\|_{L^2}^2 + C\|\nabla^3 m\|_{L^2}^2. \end{aligned} \tag{2.108}$$

Similarly, we have

$$\frac{1}{2}\frac{d}{dt}\int|\nabla^3 n|^2\,dx \le C + C\|\nabla^2 u_t\|_{L^2}^2 + C\|\nabla^3 n\|_{L^2}^2, \tag{2.109}$$

where we have used (2.105). From (2.108)-(2.109), we have

$$\frac{d}{dt}\big(\|\nabla^3 m\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2\big) \le C + C\|\nabla^2 u_t\|_{H^2}^2 + C\big(\|\nabla^3 m\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2\big). \tag{2.110}$$

Then by using Gronwall's inequality and (2.104), we can get

$$\sup_{0\le t\le T}\big(\|\nabla^3 m\|_{L^2} + \|\nabla^3 n\|_{L^2}\big) \le C. \tag{2.111}$$

Collecting the estimates (2.105)-(2.101), we obtain

$$\int_0^T\|\nabla u\|_{H^3}^2\,dt \le C. \tag{2.112}$$

This completes the proof of Proposition 2.7. □

Proposition 2.8 ([22]) *If* (m,n,u) *is a classical solution of* (1.1)-(1.3) *in* $[0,T]$, *we have the following estimates*:

$$\sup_{0\le t\le T}\sigma\big(\|\nabla^2 u_t\|_{L^2} + \|\nabla^4 u\|_{L^2}\big) + \int_0^T\sigma^2\|\nabla u_{tt}\|_{L^2}^2\,dt \le C. \tag{2.113}$$

3 Proof of Theorem 1.1

We now give the main result of this paper with the estimates in Section 2. From the local existence results, there exists a T^* such that (1.1)-(1.4) have a unique classical solution (m,n,u) on $(0,T^*]$.

From Lemma 2.6 and Propositions 2.2 and 2.3, we know that there exists a $\tilde{T} \in (0, T^*]$ such that (2.5)-(2.6) hold for $T = \tilde{T}$. Set

$$\bar{T} = \sup\bigl\{T \mid (2.5) \text{ and } (2.6) \text{ hold}\bigr\}, \tag{3.1}$$

then $\bar{T} \geq \tilde{T} > 0$. From Propositions 2.7 and 2.8 and the standard embedding

$$L^\infty\bigl(\tau, T; H^1\bigr) \cap H^1\bigl(\tau, T; H^{-1}\bigr) \hookrightarrow C\bigl([\tau, T] : L^q\bigr), \quad \text{for any } q \in [2, 6), \tag{3.2}$$

we have $\nabla u_t, \nabla^3 u \in C([\tau, T]; L^2 \cap L^4)$, $\nabla u, \nabla^2 u \in C([\tau, T]; L^2 \cap C(\bar{\Omega}))$. From (2.81), (2.96), and (2.113) we can obtain $\int_\tau^T \|(m|u_t|^2)_t\|_{L^1}\, dt \leq C$, which implies $m^{\frac{1}{2}} u_t \in C([\tau, T]; L^2)$. This together with (3.2) gives $m^{\frac{1}{2}} \dot{u}, \nabla \dot{u} \in C([\tau, T]; L^2)$.

Now, we claim that $\bar{T} = \infty$. Otherwise, $\bar{T} < \infty$. Then by Lemma 2.3 together with the estimates (2.14), (2.60) and the estimates in Propositions 2.4-2.8, and Corollaries 2.2 and 2.3, we see that $m(x, \bar{T})$, $n(x, \bar{T})$, $u(x, \bar{T})$ satisfy (1.10)-(1.12) with $g(x) = \dot{u}(x, \bar{T})$. Then, from the local existence results, there exists $\bar{T}' > \bar{T}$ such that (2.5) and (2.6) hold for $T = \bar{T}'$, which contradicts (3.1). Hence, we can obtain $\bar{T} = \infty$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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