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Existence of convex solutions for systems of Monge-Ampère equations

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Abstract
We study the existence, multiplicity, and nonexistence of convex solutions for systems of Monge-Ampère equations with multiparameters. The proof of the results is based on the method of upper and lower solutions and the fixed point index theory.

Keywords: convex solutions; Monge-Ampère equations; upper and lower solutions; fixed point index theory

1 Introduction

In this paper, we consider the existence, multiplicity, and nonexistence of convex solutions for the following boundary value problem:

$$\begin{cases} ((u_1'(r))^N)' = \lambda_1 N r^{N-1} f^1(-u_1, -u_2, \ldots, -u_n), \\ ((u_2'(r))^N)' = \lambda_2 N r^{N-1} f^2(-u_1, -u_2, \ldots, -u_n), \\ \ldots, \\ ((u_n'(r))^N)' = \lambda_n N r^{N-1} f^n(-u_1, -u_2, \ldots, -u_n), \\ u_i'(0) = u_i(1) = 0, \quad i = 1, 2, \ldots, n, 0 < r < 1, \end{cases} \tag{1.1}$$

where $N \geq 1$. Let $\mathbb{R}_+ =: [0, \infty)$. Throughout this paper, we assume that $f^i \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ ($i = 1, 2, \ldots, n$). Such a problem arises in the study of the existence of convex radial solutions for the following Dirichlet problem of the Monge-Ampère equations:

$$\begin{cases} \det(D^2 u_1) = \lambda f^1(-u_1, -u_2, \ldots, -u_n), \\ \det(D^2 u_2) = \lambda f^2(-u_1, -u_2, \ldots, -u_n), \\ \ldots, \\ \det(D^2 u_n) = \lambda f^n(-u_1, -u_2, \ldots, -u_n), \\ u_i = 0 \quad \text{on } \partial B, i = 1, 2, \ldots, n, \end{cases} \tag{1.2}$$

where $D^2 u_i = (\frac{\partial u_i}{\partial x_i \partial x_j})$ is the Hessian matrix of u_i and $B = \{x \in \mathbb{R}^N : |x| < 1\}$ is the unit ball in $\mathbb{R}^N$.

For the scalar equation, Kutev [1] obtained the existence of a unique nontrivial convex radially symmetric solution of

$$\begin{cases} \det(D^2 u) = \lambda f(-u) \quad \text{in } B, \\ u = 0 \quad \text{on } \partial B, \end{cases} \tag{1.3}$$

with $f(u) = u^p$ based on the Schauder fixed point theorem for positive, compact operators. Hu and Wang [2] established several criteria for the existence, multiplicity, and nonexistence of strictly convex solutions for (1.3) with or without an eigenvalue parameter based on the fixed point index, due to Krasnoselskii. For systems, the problem (1.2) has been studied by Wang [3]. They considered the existence, multiplicity, and nonexistence of nontrivial radial convex solutions with superlinearity or sublinearity assumptions based on Krasnoselskii's fixed point theorem in a cone. Therefore, it seems to be interesting to consider the convex radial solutions when the problem has multiparameters.

For the multiparameter problem, Dunninger and Wang [4, 5] considered the existence and multiplicity of positive radial solutions for the elliptic systems

$$\begin{cases} \Delta u + \lambda k_1(|x|)f(u,v) = 0, \\ \Delta v + \mu k_2(|x|)g(u,v) = 0 \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega = \{x \in \mathbb{R}^n : R_1 < |x| < R_2\}$, $R_1, R_2 > 0$, $n \geq 3$, $(\lambda,\mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$, $k_i \in C([R_1,R_2], \mathbb{R}_+)$, not vanishing identically on any subinterval of $[R_1,R_2]$ and $f, g \in C(\mathbb{R}^2_+, \mathbb{R}_+ \setminus \{0\})$. In particular, Dunninger and Wang [4] considered problem (1.4) for the case $f(0,0) > 0$, $g(0,0) > 0$ and the following two conditions are satisfied:

(A_1) f and g are nondecreasing on $\mathbb{R}^2_+$, *i.e.*,

$$f(u_1,v_1) \leq f(u_2,v_2) \quad \text{and} \quad g(u_1,v_1) \leq g(u_2,v_2)$$

whenever $(u_1,v_1) \leq (u_2,v_2)$, where the inequality on $\mathbb{R}^2_+$ can be understood componentwise;

(A_2) $f_\infty =: \lim_{(u,v)\to\infty} \frac{f(u,v)}{u+v} = \infty$, $g_\infty =: \lim_{(u,v)\to\infty} \frac{g(u,v)}{u+v} = \infty$.

They proved for the case $\lambda = \mu$ that there exists $\lambda^* > 0$ such that problem (1.4) has at least two, at least one, or no positive radial solutions according to $0 < \lambda < \lambda^*$, $\lambda = \lambda^*$, or $\lambda > \lambda^*$. Among other results, they considered the same problem for the case $f(0,0) = g(0,0) = 0$ in [5]. They proved under the assumptions $f_0 = g_0 = 0$ and $f_\infty = g_\infty = \infty$ that problem (1.4) has at least one positive radial solution for all $\lambda, \mu > 0$, where

$$f_0 = \lim_{(u,v)\to 0} \frac{f(u,v)}{u+v}, \qquad g_0 = \lim_{(u,v)\to 0} \frac{g(u,v)}{u+v}.$$

Lee [6] considered the multiplicity when the problem (1.4) has multiparameters for the first case and also when the problem has a perturbed boundary condition for the second case. Yang [7] proved the existence of positive solutions for Dirichlet boundary value problem of $2m$-order nonlinear differential systems with n different parameters based on the method of upper and lower solutions and the fixed point index theory. Inspired by these references, we will study the existence, multiplicity, and nonexistence of convex solutions for systems of Monge-Ampère equations with multiparameters.

The paper is organized as follows. In Section 2, we introduce the upper and lower solutions method for systems and the fixed point index theory. In Section 3, we state and prove the existence, multiplicity, and nonexistence results.

2 Preliminaries

A nontrivial convex solution of (1.1) is negative on $[0,1)$. With the simple transformation $v_i = -u_i$, (1.1) can be written as

$$\begin{cases} ((-v_1'(t))^N)' = \lambda_1 N t^{N-1} f^1(v_1, v_2, \ldots, v_n), \\ ((-v_2'(t))^N)' = \lambda_2 N t^{N-1} f^2(v_1, v_2, \ldots, v_n), \\ \ldots, \\ ((-v_n'(t))^N)' = \lambda_n N t^{N-1} f^n(v_1, v_2, \ldots, v_n), \\ v_i'(0) = v_i(1) = 0, \quad i = 1, 2, \ldots, n. \end{cases} \tag{2.1}$$

Therefore, throughout this paper we shall study the positive concave solution of (2.1).

Let $\varphi(t) = t^N$, $t \geq 0$. For $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, define the operators T_{λ_i} and T_λ as

$$\begin{aligned} T_{\lambda_i}\mathbf{v}(t) &= \int_t^1 \varphi^{-1}\left(\int_0^s \lambda_i N \tau^{N-1} f^i(\mathbf{v}(\tau))\, d\tau\right) ds, \quad i = 1, 2, \ldots, n, \\ T_\lambda \mathbf{v}(t) &= \left(T_{\lambda_1}\mathbf{v}(t), T_{\lambda_2}\mathbf{v}(t), \ldots, T_{\lambda_n}\mathbf{v}(t)\right). \end{aligned}$$

Problem (2.1) is equivalent to

$$T_\lambda \mathbf{v}(t) = \mathbf{v}(t), \quad t \in [0,1].$$

It implies that

$$v_i''(t) = -\frac{1}{N}\left(\int_0^t \lambda_i N \tau^{N-1} f^i(\mathbf{v}(\tau))\, d\tau\right)^{\frac{1}{N}-1} \left(\lambda_i N t^{N-1} f^i(\mathbf{v}(t))\right) \leq 0$$

for $t \in (0,1)$. Thus each component u_i must be convex.

Let X be the Banach space $\underbrace{C[0,1] \times \cdots \times C[0,1]}_{n}$ with the norm $\|\mathbf{v}\| = \sum_{i=1}^n \|v_i\|$ and $\|v_i\| = \max_{t\in[0,1]} |v_i(t)|$, $i = 1, 2, \ldots, n$. Let K be a cone in X defined as

$$K = \left\{\mathbf{v} = (v_1, v_2, \ldots, v_n) \in X : v_i(t) \geq 0, t \in [0,1] \text{ and } \min_{\frac{1}{4}\leq t\leq \frac{3}{4}} \sum_{i=1}^n v_i(t) \geq \frac{1}{4}\|\mathbf{v}\|\right\}.$$

It follows similarly from [4, 5], we can get the following lemma.

Lemma 2.1 *$T_\lambda(K) \subset K$ and T_λ is completely continuous on X.*

Consider the following boundary value problem:

$$\begin{cases} ((-x_1'(t))^N)' = F^1(t, x_1, x_2, \ldots, x_n), \\ ((-x_2'(t))^N)' = F^2(t, x_1, x_2, \ldots, x_n), \\ \ldots, \\ ((-x_n'(t))^N)' = F^n(t, x_1, x_2, \ldots, x_n), \\ x_i'(0) = x_i(1) = 0, \quad i = 1, 2, \ldots, n, \end{cases} \tag{2.2}$$

where $t \in (0,1)$, $F^i : D \to \mathbb{R}$ is continuous with $D \subset [0,1] \times \mathbb{R}^n$, $i = 1, 2, \ldots, n$.

Definition 2.1 Let $\alpha_i \in C^2([0,1],\mathbb{R})$, $i = 1,2,\dots,n$, we say $(\alpha_1,\alpha_2,\dots,\alpha_n)$ is a lower solution of (2.2) if $(t,\alpha_1(t),\alpha_2(t),\dots,\alpha_n(t)) \in D$ for all $t \in (0,1)$ and

$$\begin{cases} ((-\alpha_i'(t))^N)' \le F^i(t,\alpha_1,\alpha_2,\dots,\alpha_n), \\ \alpha_i'(0) = 0, \qquad \alpha_i(1) \le 0, \quad i = 1,2,\dots,n. \end{cases} \tag{2.3}$$

Definition 2.2 Let $\beta_i \in C^2([0,1],\mathbb{R})$, $i = 1,2,\dots,n$, we say $(\beta_1,\beta_2,\dots,\beta_n)$ is an upper solution of (2.2) if $(t,\beta_1(t),\beta_2(t),\dots,\beta_n(t)) \in D$ for all $t \in (0,1)$ and

$$\begin{cases} ((-\beta_i'(t))^N)' \ge F^i(t,\beta_1,\beta_2,\dots,\beta_n), \\ \beta_i'(0) = 0, \qquad \beta_i(1) \ge 0, \quad i = 1,2,\dots,n. \end{cases} \tag{2.4}$$

Let $D_\alpha^\beta = \{(t,x_1,x_2,\dots,x_n) \in [0,1] \times \mathbb{R}^n : \alpha_i(t) \le x_i \le \beta_i(t), i = 1,2,\dots,n\}$. We give a fundamental lemma of upper and lower solutions method.

Lemma 2.2 *Let* $(\alpha_1(t),\alpha_2(t),\dots,\alpha_n(t))$ *and* $(\beta_1(t),\beta_2(t),\dots,\beta_n(t))$ *be lower and upper solutions of* (2.2), *respectively, such that*

(h_1) $(\alpha_1(t),\alpha_2(t),\dots,\alpha_n(t)) \le (\beta_1(t),\beta_2(t),\dots,\beta_n(t))$, $\forall t \in (0,1)$;
(h_2) $D_\alpha^\beta \subset D$;
(h_3) $F^i(t,x_1,x_2,\dots,x_n)$ *is nondecreasing on* $\mathbb{R}^n$ *for fixed* $t \in [0,1]$, *that is,*

$$F^i(t,x_1,x_2,\dots,x_n) \le F^i(t,y_1,y_2,\dots,y_n), \quad i = 1,2,\dots,n,$$

whenever $(x_1,x_2,\dots,x_n) \le (y_1,y_2,\dots,y_n)$.

Then problem (2.2) *has at least one solution* $(x_1(t),x_2(t),\dots,x_n(t))$ *such that for all* $t \in (0,1)$,

$$\bigl(\alpha_1(t),\alpha_2(t),\dots,\alpha_n(t)\bigr) \le \bigl(x_1(t),x_2(t),\dots,x_n(t)\bigr) \le \bigl(\beta_1(t),\beta_2(t),\dots,\beta_n(t)\bigr).$$

Proof It is easy to verify that problem (2.2) is equivalent to the following system of integral equations:

$$x_i(t) = \int_t^1 \varphi^{-1}\left(\int_0^s F^i\bigl(\tau,x_1(\tau),x_2(\tau),\dots,x_n(\tau)\bigr)\,d\tau\right) ds, \quad i = 1,2,\dots,n,$$

where $t \in [0,1]$. Define the function series $\{x_i^{(k)}(t)\}_{k=0}^\infty$ by

$$\begin{cases} x_i^{(0)}(t) = \alpha_i(t), \\ x_i^{(k+1)}(t) = \int_t^1 \varphi^{-1}(\int_0^s F^i(\tau,x_1^{(k)}(\tau),x_2^{(k)}(\tau),\dots,x_n^{(k)}(\tau))\,d\tau)\,ds, \\ \quad k = 0,1,\dots, i = 1,2,\dots,n. \end{cases} \tag{2.5}$$

The inequalities in (2.3) are equivalent to

$$\alpha_i(t) \le \int_t^1 \varphi^{-1}\left(\int_0^s F^i\bigl(\tau,\alpha_1(\tau),\alpha_2(\tau),\dots,\alpha_n(\tau)\bigr)\,d\tau\right) ds, \\ t \in [0,1], i = 1,2,\dots,n. \tag{2.6}$$

It follows from the above inequalities that

$$x_i^{(1)}(t) \geq x_i^{(0)}(t), \quad t \in [0,1], i = 1,2,\ldots,n.$$

By induction, assume

$$x_i^{(k)}(t) \geq x_i^{(k-1)}(t), \quad t \in [0,1], i = 1,2,\ldots,n.$$

Then for $k \geq 1$ and by (h_3) and (2.5), we obtain

$$\begin{aligned} x_i^{(k+1)}(t) - x_i^{(k)}(t) &= \int_t^1 \left(\varphi^{-1}\left(\int_0^s F^i\big(\tau, x_1^{(k)}(\tau), x_2^{(k)}(\tau), \ldots, x_n^{(k)}(\tau)\big)\, d\tau \right) \right. \\ &\quad \left. - \varphi^{-1}\left(\int_0^s F^i\big(\tau, x_1^{(k-1)}(\tau), x_2^{(k-1)}(\tau), \ldots, x_n^{(k-1)}(\tau)\big)\, d\tau \right) \right) ds \\ &\geq 0, \quad t \in [0,1], i = 1,2,\ldots,n, \end{aligned}$$

which implies that

$$\alpha_i(t) = x_i^{(0)}(t) \leq x_i^{(1)}(t) \leq \cdots \leq x_i^{(k)}(t) \leq \cdots.$$

Since $(\beta_1(t), \beta_2(t), \ldots, \beta_n(t))$ is an upper solution of (2.2), we have

$$\begin{aligned} \beta_i(t) &\geq \int_t^1 \varphi^{-1}\left(\int_0^s F^i\big(\tau, \beta_1(\tau), \beta_2(\tau), \ldots, \beta_n(\tau)\big)\, d\tau \right) ds \\ &\geq \int_t^1 \varphi^{-1}\left(\int_0^s F^i\big(\tau, \alpha_1(\tau), \alpha_2(\tau), \ldots, \alpha_n(\tau)\big)\, d\tau \right) ds \\ &\geq \alpha_i(t) = x_i^{(0)}(t), \quad t \in [0,1], i = 1,2,\ldots,n. \end{aligned}$$

Assume $\beta_i(t) \geq x_i^{(k)}(t)$, $t \in [0,1]$, $i = 1,2,\ldots,n$, $k \in \mathbb{N}$, then by the definition of $x_i^{(k)}(t)$ and the above inequalities, we obtain

$$\begin{aligned} \beta_i(t) - x_i^{(k+1)}(t) &\geq \int_t^1 \left(\varphi^{-1}\left(\int_0^s F^i\big(\tau, \beta_1(\tau), \beta_2(\tau), \ldots, \beta_n(\tau)\big)\, d\tau \right) \right. \\ &\quad \left. - \varphi^{-1}\left(\int_0^s F^i\big(\tau, x_1^{(k)}(\tau), x_2^{(k)}(\tau), \ldots, x_n^{(k)}(\tau)\big)\, d\tau \right) \right) ds \\ &\geq 0, \quad t \in [0,1], i = 1,2,\ldots,n, \end{aligned}$$

which implies that $\{x_i^{(k)}(t)\}_{k=0}^{\infty}$ is bounded above by $\beta_i(t)$, hence the limit $x_i^*(t) = \lim_{k\to\infty} x_i^{(k)}(t)$ exists and satisfies

$$\alpha_i(t) \leq x_i^*(t) \leq \beta_i(t), \quad t \in [0,1], i = 1,2,\ldots,n.$$

Moreover, by taking the limits in both sides of (2.5), we obtain

$$x_i^*(t) = \int_t^1 \varphi^{-1}\left(\int_0^s F^i\big(\tau, x_1^*(\tau), x_2^*(\tau), \ldots, x_n^*(\tau)\big)\, d\tau \right) ds, \quad t \in [0,1], i = 1,2,\ldots,n,$$

which implies that $(x_1^*(t), x_2^*(t), \ldots, x_n^*(t))$ is a solution of (2.2). □

The following well-known results of the fixed point index are crucial in our arguments.

Lemma 2.3 [8] *Let X be a Banach space, K a cone in X and Ω bounded open in X. Let $0 \in \Omega$ and $T : K \cap \bar{\Omega} \to K$ be condensing. Suppose that $Tx \neq \lambda x$ for all $x \in K \cap \partial\Omega$ and all $\lambda \geq 1$. Then*

$$i(T, K \cap \Omega, K) = 1.$$

Lemma 2.4 [8] *Let X be a Banach space and K a cone in X. For $r > 0$, define $K_r = \{x \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is a compact map such that $Tx \neq x$ for $x \in \partial K_r$. If $\|x\| \leq \|Tx\|$ for all $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0.$$

3 Main results

Theorem 3.1 *Assume for all $i = 1, 2, \ldots, n$,*

(H_1) $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}_+^n \setminus \{(0, 0, \ldots, 0)\}$;
(H_2) $f^i \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ *is nondecreasing on* $\mathbb{R}_+^n$, *that is,*

$$f^i(u_1, u_2, \ldots, u_n) \leq f^i(v_1, v_2, \ldots, v_n), \quad \textit{if } (u_1, u_2, \ldots, u_n) \leq (v_1, v_2, \ldots, v_n)$$

and there exists at least one $j \in \{1, 2, \ldots, n\}$, such that $f^j(0, 0, \ldots, 0) > 0$;
(H_3) *there exist constants $m_i > 0$ such that*

$$f^i(\mathbf{v}) \geq m_i \varphi\left(\sum_{i=1}^n v_i\right);$$

(H_4)

$$\lim_{\|\mathbf{v}\| \to \infty} \frac{f^i(\mathbf{v})}{\varphi(\sum_{i=1}^n v_i)} = \infty.$$

Then there exists a bounded and continuous surface Γ separating $\mathbb{R}_+^n \setminus \{(0, 0, \ldots, 0)\}$ into two disjoint subsets Ω_1 and Ω_2 such that problem (1.1) *has at least two convex solutions for $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Omega_1$, at least one convex solution for $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Gamma$ and no solution for $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Omega_2$. Moreover, let $\Gamma_+ \cup \Gamma_0$ be the parametric representation of Γ, where*

$$\Gamma_+\colon \lambda_n = \lambda_n(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) > 0, \qquad \Gamma_0\colon \lambda_n = \lambda_n(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) = 0.$$

Then on Γ_+, the function $\lambda_n = \lambda_n(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ is continuous and nonincreasing on $\mathbb{R}_+^{n-1} \setminus \{(0, 0, \ldots, 0)\}$, that is, if $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \leq (\lambda'_1, \lambda'_2, \ldots, \lambda'_{n-1})$, then

$$\lambda_n(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \geq \lambda_n\left(\lambda'_1, \lambda'_2, \ldots, \lambda'_{n-1}\right)$$

and on Γ_0, the function $\lambda_{n-1} = \lambda_{n-1}(\lambda_1, \lambda_2, \ldots, \lambda_{n-2})$ is continuous and nonincreasing on $\mathbb{R}_+^{n-2} \setminus \{(0, 0, \ldots, 0)\}$.

We need some lemmas to prove Theorem 3.1. The following lemma is a prior estimate for solutions of problem (2.1).

Lemma 3.1 *Assume* (H_1)-(H_4) *hold. Let* Σ *be a compact subset of* $\mathbb{R}_+^n \setminus \{(0,0,\dots,0)\}$. *Then there exists a constant* $C_\Sigma > 0$ *such that for all* $(\lambda_1,\lambda_2,\dots,\lambda_n) \in \Sigma$ *and all possible positive solutions* $\mathbf{v} = (v_1, v_2, \dots, v_n)$ *of* (2.1) *at* $(\lambda_1,\lambda_2,\dots,\lambda_n)$, *one has*

$$\|\mathbf{v}\| \le C_\Sigma.$$

Proof Suppose by contradiction that there exists a sequence $\{(v_1^{(m)}, v_2^{(m)}, \dots, v_n^{(m)})\}_{m=1}^\infty$ of positive solutions of (2.1) at $(\lambda_1^{(m)}, \lambda_2^{(m)}, \dots, \lambda_n^{(m)})$ such that $(\lambda_1^{(m)}, \lambda_2^{(m)}, \dots, \lambda_n^{(m)}) \in \Sigma$ for all m and

$$\left\| \left(v_1^{(m)}, v_2^{(m)}, \dots, v_n^{(m)}\right) \right\| \to \infty.$$

Then $\mathbf{v}^{(m)} = (v_1^{(m)}, v_2^{(m)}, \dots, v_n^{(m)}) \in K$ and thus

$$\min_{\frac{1}{4} \le t \le \frac{3}{4}} \left(\sum_{i=1}^n v_i^{(m)}(t) \right) \ge \frac{1}{4} \left\| \mathbf{v}^{(m)} \right\|. \tag{3.1}$$

Since Σ is compact, the sequence $\{(\lambda_1^{(m)}, \lambda_2^{(m)}, \dots, \lambda_n^{(m)})\}_{m=1}^\infty$ has a convergent subsequence which we denote without loss of generality still by $\{(\lambda_1^{(m)}, \lambda_2^{(m)}, \dots, \lambda_n^{(m)})\}_{m=1}^\infty$ such that

$$\lim_{m\to\infty} \lambda_i^{(m)} = \lambda_i^*, \quad i = 1, 2, \dots, n$$

and at least one $\lambda_j^* > 0$, hence for m sufficiently large, we have $\lambda_j^{(m)} \ge \lambda_j^*/2 > 0$. Then from ($H_4$), we may choose $R_j > 0$ such that

$$f^j(\mathbf{v}) \ge L_1 \varphi\left(\sum_{i=1}^n v_i \right) \quad \text{for all } \sum_{i=1}^n v_i \ge R_j, \tag{3.2}$$

where L_1 satisfies

$$\frac{1}{4} \varphi^{-1}\left(\frac{\lambda_j^*}{2} L_1 \right) \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^s N\tau^{N-1}\, d\tau \right) ds > 1.$$

Combining (3.1) with (3.2), we get

$$\begin{aligned}
\left\| \mathbf{v}^{(m)} \right\| &= \left\| T_{\lambda^{(m)}} \mathbf{v}^{(m)} \right\| \\
&\ge \max_{t\in[0,1]} \left| T_{\lambda_j^{(m)}} \mathbf{v}^{(m)}(t) \right| \\
&= \int_0^1 \varphi^{-1}\left(\int_0^s \lambda_j^{(m)} N\tau^{N-1} f^j\left(\mathbf{v}^{(m)}(\tau)\right) d\tau \right) ds \\
&\ge \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^s \frac{\lambda_j^*}{2} N\tau^{N-1} f^j\left(\mathbf{v}^{(m)}(\tau)\right) d\tau \right) ds \\
&\ge \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^s \frac{\lambda_j^*}{2} N\tau^{N-1} L_1 \varphi\left(\sum_{i=1}^n v_i^{(m)}(\tau) \right) d\tau \right) ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^{s} \frac{\lambda_j^*}{2} N\tau^{N-1} L_1 \varphi\left(\frac{1}{4}\|\mathbf{v}^{(m)}\|\right) d\tau\right) ds \\
&= \frac{1}{4}\varphi^{-1}\left(\frac{\lambda_j^* L_1}{2}\right) \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^{s} N\tau^{N-1}\, d\tau\right) ds \cdot \|\mathbf{v}^{(m)}\| \\
&> \|\mathbf{v}^{(m)}\|
\end{aligned}
$$

for m sufficiently large. This is a contradiction. □

Lemma 3.2 *Assume* (H_1)-(H_2) *hold. If* (2.1) *has a positive solution at* $(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n)$. *Then* (2.1) *also has a positive solution at* $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *for all* $(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n)$.

Proof Let $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n)$ be a positive solution of (2.1) at $(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n)$ and let $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}_+^n \setminus \{(0, 0, \ldots, 0)\}$ with $(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n)$. Then $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n)$ is an upper solution and $(0, 0, \ldots, 0)$ is a lower solution of (2.1) at $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, respectively. It is easy to see that $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n) \not\equiv (0, 0, \ldots, 0)$ and $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n) \geq (0, 0, \ldots, 0)$. By ($H_2$), we obtain $(0, 0, \ldots, 0)$ is not a solution of (2.1) at $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, Lemma 2.2 implies that (2.1) has a positive solution at $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. □

Lemma 3.3 *Assume* (H_1)-(H_3) *hold. Then there exists* $(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) > (0, 0, \ldots, 0)$ *such that* (2.1) *has a positive solution for all* $(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$.

Proof Let $\beta_i(t) = \int_t^1 \varphi^{-1}(\int_0^s N\tau^{N-1}\, d\tau)\, ds = \frac{1}{2}(1 - t^2)$, $t \in [0, 1]$, $i = 1, 2, \ldots, n$, be the unique solution of

$$
\begin{cases}
((-v_i'(t))^N)' = Nt^{N-1}, \quad i = 1, 2, \ldots, n, \\
v_i'(0) = v_i(1) = 0.
\end{cases}
$$

Let $M_i = \max_{t\in[0,1]} f^i(\beta_1(t), \beta_2(t), \ldots, \beta_n(t))$, then by ($H_3$), $M_i > 0$, $i = 1, 2, \ldots, n$, and at $(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) = (\frac{1}{M_1}, \frac{1}{M_2}, \ldots, \frac{1}{M_n})$, we get

$$
\begin{aligned}
&\left(\left(-\beta_i'(t)\right)^N\right)' - \lambda_i^* Nt^{N-1} f^i\left(\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\right) \\
&\quad = Nt^{N-1} - \lambda_i^* Nt^{N-1} f^i\left(\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\right) \\
&\quad = -Nt^{N-1}\left[\lambda_i^* f^i\left(\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\right) - 1\right] \\
&\quad \geq 0, \quad t \in (0, 1), i = 1, 2, \ldots, n.
\end{aligned}
$$

This shows that $(\beta_1(t), \beta_2(t), \ldots, \beta_n(t))$ is an upper solution of (2.1) at $(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$. On the other hand, $(0, 0, \ldots, 0)$ is obviously a lower solution and $(0, 0, \ldots, 0) \leq (\beta_1(t), \beta_2(t), \ldots, \beta_n(t))$. Thus by Lemma 2.2, (2.1) has a positive solution at $(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$, and by Lemma 3.2 we complete the proof. □

Define

$$
\begin{aligned}
S = \Big\{&(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}_+^n \setminus \big\{(0, 0, \ldots, 0)\big\}: \text{(2.1) has a positive solution at} \\
&(\lambda_1, \lambda_2, \ldots, \lambda_n)\Big\}.
\end{aligned}
$$

Then by Lemma 3.3, $S \neq \emptyset$, and it is easy to see that $(S, \leq)$ is a partially ordered set.

Lemma 3.4 *Assume* (H_1)-(H_3) *hold. Then* $(S, \preceq)$ *is bounded above.*

Proof Let $(\lambda_1, \lambda_2, \dots, \lambda_n) \in S$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ be a positive solution of (2.1) at $(\lambda_1, \lambda_2, \dots, \lambda_n)$. Then by (H_3), we get

$$\begin{aligned}
\|v_i\| &= \int_0^1 \varphi^{-1}\left(\int_0^s \lambda_i N\tau^{N-1} f^i\big(\mathbf{v}(\tau)\big)\,d\tau\right) ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^s \lambda_i N\tau^{N-1} f^i\big(\mathbf{v}(\tau)\big)\,d\tau\right) ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^s \lambda_i N\tau^{N-1} m_i \varphi\left(\sum_{i=1}^n v_i(\tau)\right) d\tau\right) ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^s \lambda_i N\tau^{N-1} m_i \varphi\left(\frac{1}{4}\|\mathbf{v}\|\right) d\tau\right) ds \\
&= \frac{1}{4}\varphi^{-1}(\lambda_i m_i) \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^s N\tau^{N-1}\,d\tau\right) ds \cdot \|\mathbf{v}\|.
\end{aligned}$$

Thus

$$\lambda_i \leq \frac{1}{m_i}\varphi\left(\frac{4\|v_i\|}{\int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}(\int_{\frac{1}{4}}^s N\tau^{N-1}\,d\tau)\,ds\|\mathbf{v}\|}\right) \leq \frac{d}{m_i}, \quad i = 1, 2, \dots, n,$$

where

$$d = \varphi\left(\frac{4}{\int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}(\int_{\frac{1}{4}}^s N\tau^{N-1}\,d\tau)\,ds}\right).$$

Therefore S is bounded above by $(\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n) = (\frac{d}{m_1}, \frac{d}{m_2}, \dots, \frac{d}{m_n})$. □

Similar to Lemmas 2.6-2.8 in [7], we can prove the following lemmas.

Lemma 3.5 *Assume* (H_1)-(H_3) *hold. Then every chain in* S *has a unique supremum in* S.

Lemma 3.6 *Assume* (H_1)-(H_3) *hold. Then there exists* $\tilde{\lambda}_i \in [\lambda_i^*, \bar{\lambda}_i]$ *such that* (2.1) *has a positive solution at* $(0, \dots, 0, \lambda_i, 0, \dots, 0)$ *for all* $0 < \lambda_i \leq \tilde{\lambda}_i$ *and no solution at* $(0, \dots, 0, \lambda_i, 0, \dots, 0)$ *for all* $\lambda_i > \tilde{\lambda}_i$.

Lemma 3.7 *Assume* (H_1)-(H_3) *hold. Then there exists a continuous surface* Γ *separating* $\mathbb{R}_+^n \setminus \{(0, 0, \dots, 0)\}$ *into two disjoint subsets* Σ_1 *and* Σ_2 *such that* Σ_1 *is bounded and* Σ_2 *is unbounded,* (2.1) *has at least one solution for* $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Sigma_1 \cup \Gamma$ *and no solution for* $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Sigma_2$. *The function* $\lambda_n = \lambda_n(\lambda_1, \lambda_2, \dots, \lambda_{n-1})$ *is nonincreasing, that is, if*

$$(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) \leq \big(\lambda_1', \lambda_2', \dots, \lambda_{n-1}'\big) \leq (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_{n-1}),$$

then

$$\lambda_n(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) \geq \lambda_n\big(\lambda_1', \lambda_2', \dots, \lambda_{n-1}'\big).$$

Moreover, if $\lambda_n = 0$, *then the function* $\lambda_{n-1}(\lambda_1, \lambda_2, \dots, \lambda_{n-2})$ *is nonincreasing.*

Lemma 3.8 *Assume* (H_1)-(H_3) *hold and let* $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Omega_1$. *Then there exists* $\varepsilon_0 > 0$ *such that* $(v_1^* + \varepsilon, v_2^* + \varepsilon, \ldots, v_n^* + \varepsilon)$ *is an upper solution of* (2.1) *at* $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *for all* $\varepsilon \in (0, \varepsilon_0]$, *where* $(v_1^*, v_2^*, \ldots, v_n^*)$ *is the positive solution of* (2.1) *corresponding to some* $(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) \in \Gamma$ *satisfying*

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) < \left(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*\right).$$

Proof From (H_3), there exists a constant $M > 0$ such that

$$0 < M \le \min_{t \in [0,1]} f^i\left(v_1^*(t), v_2^*(t), \ldots, v_n^*(t)\right), \quad i = 1, 2, \ldots, n.$$

Then by the uniform continuity of f^i on a compact set, there exists $\varepsilon_0 > 0$ such that

$$\left|f^i\left(v_1^* + \varepsilon, v_2^* + \varepsilon, \ldots, v_n^* + \varepsilon\right) - f^i\left(v_1^*, v_2^*, \ldots, v_n^*\right)\right| < \frac{M(\lambda_i^* - \lambda_i)}{\lambda_i}$$

for all $t \in [0,1]$, $i = 1, 2, \ldots, n$ and $0 < \varepsilon \le \varepsilon_0$. Let $\tilde{v}_i^*(t) = v_i^*(t) + \varepsilon$, $i = 1, 2, \ldots, n$, then $\tilde{v}_i^{*\prime}(0) = 0$, $\tilde{v}_i^*(1) > 0$ and

$$\begin{aligned}
&\left(\left(-\tilde{v}_i^{*\prime}(t)\right)^N\right)' - \lambda_i N t^{N-1} f^i\left(\tilde{v}_1^*(t), \tilde{v}_2^*(t), \ldots, \tilde{v}_n^*(t)\right) \\
&\quad = \lambda_i^* N t^{N-1} f^i\left(v_1^*(t), v_2^*(t), \ldots, v_n^*(t)\right) - \lambda_i N t^{N-1} f^i\left(\tilde{v}_1^*(t), \tilde{v}_2^*(t), \ldots, \tilde{v}_n^*(t)\right) \\
&\quad = \lambda_i N t^{N-1}\left[f^i\left(v_1^*(t), v_2^*(t), \ldots, v_n^*(t)\right) - f^i\left(\tilde{v}_1^*(t), \tilde{v}_2^*(t), \ldots, \tilde{v}_n^*(t)\right)\right] \\
&\qquad + \left(\lambda_i^* - \lambda_i\right) N t^{N-1} f^i\left(v_1^*(t), v_2^*(t), \ldots, v_n^*(t)\right) \\
&\quad > -N t^{N-1} M\left(\lambda_i^* - \lambda_i\right) + \left(\lambda_i^* - \lambda_i\right) N t^{N-1} f^i\left(v_1^*(t), v_2^*(t), \ldots, v_n^*(t)\right) \\
&\quad = \left(\lambda_i^* - \lambda_i\right) N t^{N-1}\left[f^i\left(v_1^*(t), v_2^*(t), \ldots, v_n^*(t)\right) - M\right] \\
&\quad \ge 0, \quad i = 1, 2, \ldots, n
\end{aligned}$$

for all $t \in [0,1]$, $i = 1, 2, \ldots, n$. Hence $(\tilde{v}_1^*(t), \tilde{v}_2^*(t), \ldots, \tilde{v}_n^*(t))$ is an upper solution of (2.1) at $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ for all $\varepsilon \in (0, \varepsilon_0]$. □

Proof of Theorem 3.1 Because we have proved the above lemmas, we only need to prove the existence of the second positive solution of (2.1) for $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Omega_1$. Let $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Omega_1$. Denote

$$\tilde{v}_i^*(t) = v_i^*(t) + \varepsilon, \quad i = 1, 2, \ldots, n, t \in [0,1],$$

where ε is given in Lemma 3.8. Define the set

$$D = \left\{\mathbf{v} = (v_1, v_2, \ldots, v_n) \in X : -\varepsilon < v_i(t) < \tilde{v}_i^*(t), t \in [0,1], i = 1, 2, \ldots, n\right\}.$$

Then D is bounded open in X and $0 \in D$. The map $T_\lambda : K \cap \bar{D} \to K$ is condensing, since it is completely continuous. Let $\mathbf{v} \in K \cap \partial D$, then there exists $t_0 \in [0,1]$ such that $v_i(t_0) = \tilde{v}_i^*(t_0)$

for some $i \in \{1, 2, \ldots, n\}$. Let $\tilde{\mathbf{v}}^* = (\tilde{v}_1^*, \tilde{v}_2^*, \ldots, \tilde{v}_n^*)$, then by ($H_2$) and Lemma 3.8, we have

$$\begin{aligned} T_{\lambda_i}\mathbf{v}(t_0) &= \int_{t_0}^1 \varphi^{-1}\left(\int_0^s \lambda_i N\tau^{N-1} f^i\big(\mathbf{v}(\tau)\big)\,d\tau\right) ds \\ &\le \int_{t_0}^1 \varphi^{-1}\left(\int_0^s \lambda_i^* N\tau^{N-1} f^i\big(\tilde{\mathbf{v}}^*(\tau)\big)\,d\tau\right) ds \\ &< \tilde{v}_i^*(t_0) = v_i(t_0) \le \theta v_i(t_0) \end{aligned}$$

for all $\theta \ge 1$. Thus $\mathbf{v} \ne \theta\mathbf{v}$ for all $\mathbf{v} \in K \cap \partial D$ and all $\theta \ge 1$. Lemma 2.3 implies that

$$i(T_\lambda, K \cap D, K) = 1.$$

For some $\lambda_i > 0$, it follows from (H_4) that there exists $R_i > 0$ such that

$$f^i(\mathbf{v}) \ge L_2 \varphi\left(\sum_{i=1}^n v_i\right), \quad \forall \sum_{i=1}^n v_i \ge R_i, \tag{3.3}$$

where L_2 satisfies

$$\frac{1}{4}\varphi^{-1}(\lambda_i L_2) \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^s N\tau^{N-1}\,d\tau\right) ds > 1.$$

Let $R^* = \max\{C_\Sigma, 4R_i, \|\tilde{\mathbf{v}}^*\|\}$, where C_Σ is given in Lemma 3.1 with Σ a compact set in $\mathbb{R}_+^n \setminus \{(0, 0, \ldots, 0)\}$ containing $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Let

$$K_{R^*} = \{\mathbf{v} \in K : \|\mathbf{v}\| < R^*\},$$

then by Lemma 3.1,

$$T_\lambda \mathbf{v} \ne \mathbf{v}, \quad \forall \mathbf{v} \in \partial K_{R^*}.$$

Furthermore, if $\mathbf{v} \in \partial K_{R^*}$, then

$$\min_{t \in [\frac{1}{4}, \frac{3}{4}]} \left(\sum_{i=1}^n v_i(t)\right) \ge \frac{1}{4}\|\mathbf{v}\| \ge R_i.$$

Thus by (3.3),

$$f^i\big(\mathbf{v}(t)\big) \ge L_2 \varphi\left(\sum_{i=1}^n v_i(t)\right), \quad \forall t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Therefore

$$\begin{aligned} \big\|T_\lambda \mathbf{v}(t)\big\| &\ge \max_{t \in [0,1]} \big|T_{\lambda_i}\mathbf{v}(t)\big| \\ &\ge \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^s \lambda_i N\tau^{N-1} f^i\big(\mathbf{v}(\tau)\big)\,d\tau\right) ds \end{aligned}$$

$$
\begin{aligned}
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^{s} \lambda_i N\tau^{N-1} L_2 \varphi\left(\sum_{i=1}^{n} v_i(\tau)\right) d\tau\right) ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^{s} \lambda_i N\tau^{N-1} L_2 \varphi\left(\frac{1}{4}\|\mathbf{v}\|\right) d\tau\right) ds \\
&= \frac{1}{4}\varphi^{-1}(\lambda_i L_2) \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^{s} N\tau^{N-1}\, d\tau\right) ds \cdot \|\mathbf{v}\| \\
&> \|\mathbf{v}\|.
\end{aligned}
$$

It follows from Lemma 2.4 that

$$i(T_\lambda, K_{R^*}, K) = 0.$$

Consequently by the additivity of the fixed point index,

$$
\begin{aligned}
0 &= i(T_\lambda, K_{R^*}, K) \\
&= i(T_\lambda, K \cap D, K) + i(T_\lambda, K_{R^*} \setminus \overline{K \cap D}, K) \\
&= 1 + i(T_\lambda, K_{R^*} \setminus \overline{K \cap D}, K),
\end{aligned}
$$

which implies

$$i(T_\lambda, K_{R^*} \setminus \overline{K \cap D}, K) = -1.$$

Thus T_λ has at least one fixed point in $K \cap D$ and another in $K_{R^*} \setminus \overline{K \cap D}$. This implies that (2.1) has at least two positive solutions at $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Omega_1$. Thus (1.1) has at least two negative solutions at $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Omega_1$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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References

1. Kutev, ND: Nontrivial solutions for the equations of Monge-Ampère type. J. Math. Anal. Appl. **132**, 424-433 (1988)
2. Hu, S, Wang, H: Convex solutions of boundary value problems arising from Monge-Ampère equations. Discrete Contin. Dyn. Syst. **16**, 705-720 (2006)
3. Wang, H: Convex solutions of systems of Monge-Ampère equations. arXiv:1007.3013v2
4. Dunninger, DR, Wang, H: Multiplicity of positive radial solutions for an elliptic systems on an annulus domain. Nonlinear Anal. **42**, 803-811 (2000)
5. Dunninger, DR, Wang, H: Existence and multiplicity of positive solutions for elliptic systems. Nonlinear Anal. **29**, 1051-1060 (1997)
6. Lee, Y: Multiplicity of positive radial solutions for multiparameter semilinear elliptic systems on an annulus. J. Differ. Equ. **174**, 420-441 (2001)
7. Yang, X: Existence of positive solutions for 2*m*-order nonlinear differential systems. Nonlinear Anal. **61**, 77-95 (2005)
8. Guo, D, Lakshmikantham, V: Nonlinear Problems in Abstract Cones. Academic Press, Orlando (1988)