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Blow-up for the stochastic nonlinear Schrödinger equations with quadratic potential and additive noise

Lixin Meng, Jingyu Li and Jian Tao*

*Correspondence:
taoj@nenu.edu.cn
School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, P.R. China

Abstract

We study the dynamics of a stochastic nonlinear Schrödinger equation with both a quadratic potential and an additive noise. We show that in both cases of repulsive potential and attractive one, any initial data with finite variance gives birth to a solution that blows up in arbitrarily small time. This is in contrast to the deterministic case when the potential is repulsive, where strong potentials could prevent the solutions from blowing up. Our result hence indicates that the additive noise rather than the potential dominates the dynamical behaviors of the solutions to the stochastic nonlinear Schrödinger equations.

MSC: Primary 60H15; 60H30; secondary 35Q55

Keywords: stochastic Schrödinger equation; Bose-Einstein condensation; quadratic potential; white noise; blow-up

1 Introduction

In this paper, we are interested in the blow-up problem for the following stochastic nonlinear Schrödinger equation with a quadratic potential

$$\begin{cases} iu_t + \Delta u + |u|^{2\sigma} u + \theta |x|^2 u - \dot{\eta} = 0, & t \geq 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x). \end{cases} \tag{1.1}$$

Here $\theta \in \mathbb{R}$, $\sigma > 0$, and $\dot{\eta}$ is a complex-valued noise that is white in time and correlated in space. When $\theta < 0$, (1.1) describes the evolution of the wave function of a Bose-Einstein condensation, where the potential $|x|^2$ models a magnetic field to confine the particles [1]. The additive noise $\dot{\eta}$ represents the fluctuation effect of physical process in random media [2].

When $\eta = 0$, (1.1) reduces to the deterministic nonlinear Schrödinger equation with a quadratic potential

$$\begin{cases} iu_t + \Delta u + |u|^{2\sigma} u + \theta |x|^2 u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.2}$$

which has been extensively studied in the last decade. Employing a Mehler type formula that is isometric in L^2, Carles [1] found the deep relation of solutions between the critical nonlinear Schrödinger equation with an attractive potential ($\theta < 0$) and the one without potential. He presented the similarities as well as differences of the blow-up dynamics of the two equations. Through decomposing its energy into two suitable parts, Carles [3] further studied the well-posedness and the blow-up problems for the supercritical equation with an attractive potential. Especially, he discovered that the singular solution of the nonlinear Schrödinger equation with an attractive potential blows up earlier than the one without potential. Subsequently, based on a new decomposition of energy, Carles [4] showed that in contrast to the attractive case, the repulsive potential ($\theta > 0$) has a tendency to delay or even prevent wave collapse. For more results on the nonlinear Schrödinger equation with quadratic potentials, we refer the reader to [5, 6].

If $\theta = 0$, (1.1) is the well-known stochastic nonlinear Schrödinger equation without potential. In [7] de Bouard and Debussche derived the global well-posedness of H^1 solution for a defocusing nonlinear Schrödinger equation. On the base of the local well-posedness theory established in [7], they subsequently investigated the effect of an additive noise on the finite time blow-up behavior of the solutions for a focusing nonlinear Schrödinger equation in [8]. It is usually an interesting question to study the effect of small noise on a deterministic model. As the additive noise converges to zero, for the nonlinear Schrödinger equation with a power-type nonlinearity, Gautier [9] established a large deviations principle, which demonstrates the rate of convergence to zero of the probability that paths are in sets which do not contain the deterministic solution. It is worth mentioning that if the noise acts as a potential, then a multiplicative noise arises, we refer the interested reader to [10–12] in this direction.

In this paper, we are interested in the mixed effect of noise and potential on the blow-up dynamics of the solution. As far as we know, the blow-up problem for the Schrödinger equations with a potential and a noise was first studied in [13], where Fang *et al.* showed that for the initial data with sufficiently negative energy, which is similar to the requirement of deterministic equation, the corresponding solutions blow up in finite time with positive probability. In the current paper, we continue to investigate the blow-up problem for nonlinear Schrödinger equations under both effects of an additive noise and a potential. We show that, regardless of the direction of the potential, any initial data with finite variance gives birth to a solution that blows up in arbitrarily small time, which improves the blow-up result in [13]. Recall that for the deterministic (1.2) with a repulsive potential ($\theta > 0$), Carles [4] proved that the potential could prevent the blow-up of the solution. Thus, our result implies that for the stochastic (1.1) the noise dominates the dynamics of the solution.

The rest of this paper is organized as follows. In Section 2, we present some preliminary lemmas and state our main results. Section 3 is devoted to the proof of the main results.

2 Preliminaries and main results

Before proceeding, let us briefly introduce some notation for convenience. $(\Omega, \mathscr{F}, \mathbb{P})$ denotes a probability space endowed with a filtration $(\mathscr{F}_t)_{t\geq 0}$. Let $(\beta_l)_{l\in\mathbb{N}}$ be a sequence of real-valued independent Brownian motions associated with this filtration. $L^p(\mathbb{R}^n)$ denotes the classical Lebesgue space of complex-valued functions on $\mathbb{R}^n$. As in the argument for the deterministic (1.2), we also need the Hilbert space

$$\Sigma := \{ f \in H^1(\mathbb{R}^n) : x \mapsto xf(x) \in L^2(\mathbb{R}^n) \}$$

equipped with the norm

$$\|f\|_{\Sigma}^2 := \|f\|_{H^1}^2 + \|xf\|_{L^2}^2.$$

Let $(e_l)_{l\in\mathbb{N}}$ be a Hilbertian basis of $L^2(\mathbb{R}^n)$ and ϕ denote a Hilbert-Schmidt operator from L^2 into Σ. $\mathscr{L}_2^{\Sigma}$ stands for the space of Hilbert-Schmidt operators ϕ from L^2 into Σ endowed with the norm

$$\|\phi\|_{\mathscr{L}_2^{\Sigma}} := \operatorname{tr}(\phi^*\phi) = \sum_{l\in\mathbb{N}} \|\phi e_l\|_{\Sigma}^2.$$

The noise we consider is $\dot{\eta} = \frac{\partial W}{\partial t}$, where the process W is given by

$$W(t,x,\omega) = \sum_{l=0}^{\infty} \beta_l(t,\omega)\phi e_l(x), \quad t \geq 0, x \in \mathbb{R}^n, \omega \in \Omega.$$

Then W is a complex-valued Wiener process on $L^2(\mathbb{R}^n)$ with covariance operator $\phi\phi^*$. Here ϕ^* denotes the adjoint operator of ϕ.

The integral $\int_{\mathbb{R}^n} f(x)\,dx$ will be abbreviated as $\int f(x)$ for convenience if no confusion is caused. If I is an interval of $\mathbb{R}$, B is a Banach space, $1 \leq q \leq \infty$, then $L^q(I;B)$ is the space of Lebesgue measurable function g from I into B such that the function $t \mapsto \|g(t)\|_B$ is in $L^q(I)$. $L^q((0,T);L^r(\mathbb{R}^n))$ will be abbreviated by $L_T^q L^r$. C denotes a generic positive constant which may change from one line to another.

$$c_\phi^\Sigma := \sum_{l\in\mathbb{N}} \int_{\mathbb{R}^n} |x|^2|\phi e_l|^2, \qquad c_\phi^0 := \sum_{l\in\mathbb{N}} \int_{\mathbb{R}^n} |\phi e_l|^2,$$
$$c_\phi^1 := \sum_{l\in\mathbb{N}} \int_{\mathbb{R}^n} |\nabla\phi e_l|^2, \qquad c_\phi^2 := \operatorname{Im} \sum_{l\in\mathbb{N}} \int_{\mathbb{R}^n} \overline{\phi e_l} x \nabla\phi e_l.$$

Throughout the paper, we assume that $\phi \in \mathscr{L}_2^{\Sigma}$. Thus, the four constants given above are finite and well defined.

We are now ready to state our main results.

Theorem 2.1 *Assume that* $\sigma \geq \frac{2}{n}$ *if* $n = 1,2$, *and that* $\frac{2}{n} \leq \sigma < \frac{2}{n-2}$ *if* $n \geq 3$. *Suppose that* $\phi \in \mathscr{L}_2^{\Sigma}$ *satisfying* $\ker\phi^* = \{0\}$. *Then for any* $u_0 \in \Sigma$ *with* $u_0 \neq 0$ *and any* $t > 0$ $(0 < t < \frac{\pi}{4\sqrt{-\theta}}$ *if* $\theta < 0)$, *we have*

$$\mathbb{P}\bigl(\tau^*(u_0) < t\bigr) > 0,$$

where $\tau^*(u_0)$ *is the existence time of the solution of* (1.1) *with initial data* u_0.

Remark 2.1 For the deterministic (1.2) with supercritical nonlinearities and repulsive potentials ($\theta > 0$), Carles [4] showed that for fixed initial data $u_0 \in \Sigma$, there exists $\theta_0 > 0$ such that for any $\theta > \theta_0$, the solution of (1.2) is global. Here we show that, in contrast to the result available for deterministic equations, under the effect of an additive white noise, any initial data develops a solution that blows up in arbitrarily small time. In this sense, Theorem 2.1 says that, for the stochastic Schrödinger equation (1.1) the white noise rather than the potential determines the dynamical behaviors of the solution.

Before the proof of Theorem 2.1, we need some preliminary lemmas. Set $U_\theta(t) := e^{it(\Delta+\theta|x|^2)}$. (q,r) denotes the admissible pair satisfying $\frac{2}{q} = n(\frac{1}{2} - \frac{1}{r})$ with $2 \le r < \frac{2n}{n-2}$ ($2 \le r < \infty$ if $n = 2$, $2 \le r \le \infty$ if $n = 1$). We recall the following Strichartz estimates for U_θ; see [3, 4] for details.

Lemma 2.1 (Strichartz estimates)

(i) *If $\theta > 0$, then for any admissible pair (q,r), (γ,ρ) and any interval I,*

$$\left\| U_\theta(t)\varphi \right\|_{L^q(I;L^r)} \le C_1 \|\varphi\|_{L^2}$$

for every $\varphi \in L^2(\mathbb{R}^n)$, and

$$\left\| \int_{I\cap\{s\le t\}} U_\theta(t-s)F(s)\,ds \right\|_{L^q(I;L^r)} \le C_2 \|F\|_{L^{\gamma'}(I;L^{\rho'})}$$

for every $F \in L^{\gamma'}(I;L^{\rho'})$, where $C_1 = C_1(q,r)$ and $C_2 = C_2(T,q,r,\gamma,\rho)$ are positive constants independent of I.

(ii) *If $\theta < 0$, then the both inequalities stated in* (i) *hold for any interval $I \subset [0, \frac{\pi}{4\sqrt{-\theta}}]$.*

To demonstrate the effect of quadratic potentials, we adopt the decomposition introduced by Carles [3, 4]. In the case of $\theta > 0$, set $\mu = \sqrt{\theta}$ and denote

$$J_+(t) := \mu x \sinh(2\mu t) + i\cosh(2\mu t)\nabla_x, \qquad K_+(t) := x\cosh(2\mu t) + \frac{i}{\mu}\sinh(2\mu t)\nabla_x,$$

then

$$i\nabla_x = \cosh(2\mu t)J_+(t) - \mu\sinh(2\mu t)K_+(t),$$
$$x = \cosh(2\mu t)K_+(t) - \frac{\sinh(2\mu t)}{\mu}J_+(t).$$

In the case of $\theta < 0$, set $\nu = \sqrt{-\theta}$ and denote

$$J_-(t) := \nu x\sin(2\nu t) - i\cos(2\nu t)\nabla_x, \qquad K_-(t) := x\cos(2\nu t) + \frac{i}{\nu}\sin(2\nu t)\nabla_x,$$

then

$$i\nabla_x = \nu\sin(2\nu t)K_-(t) - \cos(2\nu t)J_-(t), \qquad x = \cos(2\nu t)K_-(t) + \frac{\sin(2\nu t)}{\nu}J_-(t).$$

The operators $J_\pm$ and $K_\pm$ have the following properties; see [3, 4].

Lemma 2.2

$$J_\pm(t) = \pm U_\theta(t)i\nabla U_\theta(-t), \qquad K_\pm(t) = U_\theta(t)xU_\theta(-t). \tag{2.1}$$

If $F \in C^1(\mathbb{C},\mathbb{C})$ is of the form $F(z) = zG(|z|^2)$, then

$$\begin{aligned} J_\pm(t)F(u) &= \partial_u F(u)J_\pm(t)u - \partial_{\bar u}F(u)\overline{J_\pm(t)u}, \quad \text{if } \theta < 0, t \notin \frac{\pi}{2\nu}\mathbb{Z}, \\ K_\pm(t)F(u) &= \partial_u F(u)K_\pm(t)u - \partial_{\bar u}F(u)\overline{K_\pm(t)u}, \quad \text{if } \theta < 0, t \notin \frac{\pi}{4\nu} + \frac{\pi}{2\nu}\mathbb{Z}. \end{aligned} \tag{2.2}$$

On the base of Lemmas 2.1 and 2.2, Fang *et al.* (see Theorem 3.1 in [13]) proved the following local well-posedness result for (1.1).

Lemma 2.3 *Assume that* $0 \leq \sigma < \frac{2}{n-2}$ *if* $n \geq 3$ *or* $\sigma \geq 0$ *if* $n = 1, 2$, *and that* ϕ *is a Hilbert-Schmidt operator from* $L^2(\mathbb{R}^n)$ *into* Σ. *Then for any* $\mathscr{F}$ *measurable random variable* u_0 *with values in* Σ, *there exists a unique solution* $u(u_0, \cdot)$ *to* (1.1) *with continuous* Σ *values paths. The solution is defined on a random interval* $[0, \tau^*(u_0, \omega))$. *Here* $\tau^*(u_0, \omega)$ *is a stopping time such that*

$$\tau^*(u_0, \omega) = +\infty \quad \textit{or} \quad \lim_{t \to \tau^*(u_0,\omega)} \bigl\| u(t, \omega) \bigr\|_{\Sigma} = +\infty.$$

As in the deterministic case, to study the blow-up problem, we need to characterize the time evolution of the following three quantities: the energy

$$\begin{aligned} H(u) = {}& \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx - \frac{\theta}{2} \int_{\mathbb{R}^n} |x|^2 |u|^2 \, dx \\ & - \frac{1}{2\sigma + 2} \int_{\mathbb{R}^n} |u|^{2\sigma+2} \, dx \quad \text{for } u \in \Sigma, \end{aligned}$$

the momentum

$$G(u) = \operatorname{Im} \int_{\mathbb{R}^n} u x \cdot \nabla \bar{u} \, dx \quad \text{for } u \in \Sigma,$$

and the variance

$$V(u) = \int_{\mathbb{R}^n} |x|^2 |u|^2 \, dx \quad \text{for } u \in \Sigma.$$

By employing the Itô formula given in [14] and a regularization argument, we can derive the following identities; see [13] for details.

Lemma 2.4 *Under the assumptions of Theorem* 2.1, *for any stopping time* t *such that* $t < \tau^*(u_0)$ *a.s., we have*

$$\begin{aligned} H\bigl(u(t)\bigr) = {}& H(u_0) - \operatorname{Im} \int_0^t \int_{\mathbb{R}^n} \bigl(\Delta \bar{u} + \theta |x|^2 \bar{u} + |u|^{2\sigma} \bar{u} \bigr) \, dx \, dW \\ & - \frac{1}{2} \sum_{l \in \mathbb{N}} \int_0^t \int_{\mathbb{R}^n} |u|^{2\sigma} \bigl| \phi e_l(x) \bigr|^2 \, dx \, d\tau + \frac{c_\phi^1}{2} t - \frac{\theta c_\phi^\Sigma}{2} t \\ & - \sigma \sum_{l \in \mathbb{N}} \int_0^t \int_{\mathbb{R}^n} |u|^{2\sigma-2} \bigl(\operatorname{Re} \bigl(\bar{u} \phi e_l(x) \bigr) \bigr)^2 \, dx \, d\tau, \end{aligned} \tag{2.3}$$

$$\begin{aligned} G\bigl(u(t)\bigr) = {}& G(u_0) - 4 \int_0^t H\bigl(u(\tau)\bigr) \, d\tau - 4\theta \int_0^t \int_{\mathbb{R}^n} |x|^2 |u|^2 \, dx \, d\tau \\ & - \frac{2 - n\sigma}{\sigma + 1} \int_0^t \int_{\mathbb{R}^n} |u|^{2\sigma+2} \, dx \, d\tau \\ & + \operatorname{Re} \sum_{l \in \mathbb{N}} \int_0^t \int_{\mathbb{R}^n} (2 \bar{u} x \cdot \nabla \phi e_l + n \bar{u} \phi e_l) \, dx \, d\beta_l - t c_\phi^2 \end{aligned} \tag{2.4}$$

and

$$V\big(u(t)\big) = V(u_0) - 4\int_0^t G\big(u(\tau)\big)\,d\tau + 2\,\mathrm{Im}\int_0^t \int_{\mathbb{R}^n} |x|^2 \bar{u}\,dx\,dW + tc_\phi^\Sigma. \tag{2.5}$$

3 Proof of Theorem 2.1

In this section we will show Theorem 2.1, by exploring the compatibility of the quadratic potentials with the space Σ and by combining the energy decomposition technique [3, 4] with the framework of showing blow-up [8, 11]. To achieve this, we first revisit the blow-up result derived by Fang *et al.* ([13], Theorem 4.1) and present a stronger conclusion.

Lemma 3.1 (Blow-up for special initial data) *Assume that u_0, σ, and ϕ satisfy the assumptions of Theorem* 2.1.

(i) *In the case $\theta < 0$, suppose that for some $T > 0$,*

$$V(u_0) - 4G(u_0)T + 8H(u_0)T^2 + c_\phi^\Sigma T + 2c_\phi^2 T^2 + \frac{4}{3}\big(c_\phi^1 - \theta c_\phi^\Sigma\big)T^3 < 0, \tag{3.1}$$

then $\mathbb{P}(\tau^*(u_0) \le T) > 0$.

(ii) *In the case $\theta > 0$, suppose that for some $T > 0$,*

$$\begin{aligned}&2V(u_0) + \left[\frac{1}{2\mu}\big(c_\phi^\Sigma - 4G(u_0)\big) + \frac{1}{4\mu^3}\big(c_\phi^1 - \theta c_\phi^\Sigma\big)\right]\tanh(2\mu T)\\ &\quad + \frac{1}{2\mu^2}\big[c_\phi^2 + 4H(u_0)\big]\tanh^2(2\mu T) < 0,\end{aligned} \tag{3.2}$$

then $\mathbb{P}(\tau^*(u_0) \le T) > 0$.

Proof The proof is similar to that of Theorem 4.1 in [11]. For the convenience of the reader, we present the sketch of the proof here. We first consider the case of $\theta < 0$. Assume that the conclusion of Lemma 3.1 does not hold. That is, $\tau^*(u_0) > T$ almost surely. Define the stopping time

$$\tau_k := \inf\big\{s \in [0,T] : \big\|u(s)\big\|_\Sigma \ge k\big\} \quad \text{for } k \in \mathbb{N}.$$

Taking $t = \tau_k$ in Lemma 2.4, and substituting (2.3) and (2.4) into (2.5), by Fubini theorem, we obtain the following stochastic version of the variance identity:

$$\begin{aligned}&V\big(u(\tau_k)\big)\\ &\quad = V(u_0) - 4G(u_0)\tau_k + 8H(u_0)\tau_k^2 + c_\phi^\Sigma \tau_k + 2c_\phi^2\tau_k^2 + \frac{4}{3}\big(c_\phi^1 - \theta c_\phi^\Sigma\big)\tau_k^3\\ &\qquad + 16\theta\int_0^{\tau_k}(\tau_k - s)V\big(u(s)\big)\,ds + \frac{4(2-n\sigma)}{\sigma+1}\int_0^{\tau_k}(\tau_k - s)\big\|u(s)\big\|_{L^{2\sigma+2}}^{2\sigma+2}\,ds\\ &\qquad - 8\sigma\sum_l\int_0^{\tau_k}(\tau_k - s)^2\int_{\mathbb{R}^n}|u|^{2\sigma-2}\big(\mathrm{Re}(\bar{u}\phi e_l)\big)^2\,dx\,ds\\ &\qquad - 4\sum_l\int_0^{\tau_k}(\tau_k - s)^2\int_{\mathbb{R}^n}|u|^{2\sigma}|\phi e_l|^2\,dx\,ds + 2\,\mathrm{Im}\int_0^{\tau_k}\int_{\mathbb{R}^n}|x|^2\bar{u}\,dx\,dW\end{aligned}$$

$$
\begin{aligned}
&- 8\,\mathrm{Im}\int_0^{\tau_k}\int_{\mathbb{R}^n}(\tau_k-s)^2\bigl(\Delta\bar{u}+\theta|x|^2\bar{u}+|u|^{2\sigma}\bar{u}\bigr)\,dx\,dW \\
&- 4\,\mathrm{Re}\sum_{l\in\mathbb{N}}\int_0^{\tau_k}(\tau_k-s)\int_{\mathbb{R}^n}(2\bar{u}x\cdot\nabla\phi e_l+n\bar{u}\phi e_l)\,dx\,d\beta_l.
\end{aligned}
\tag{3.3}
$$

Noting that $\sigma \geq 2/n$ and $\theta < 0$, the seventh term to the tenth one in (3.3) are all nonpositive, we get, for any $k \in \mathbb{N}$,

$$
\begin{aligned}
V\bigl(u(\tau_k)\bigr) \leq{}& V(u_0) - 4G(u_0)\tau_k + 8H(u_0)\tau_k^2 + c_\phi^\Sigma\tau_k + 2c_\phi^2\tau_k^2 + \frac{4}{3}\bigl(c_\phi^1 - \theta c_\phi^\Sigma\bigr)\tau_k^3 \\
&+ \int_0^{\tau_k} g(\tau_k,s)\,dW(s) + \sum_{l\in\mathbb{N}}\int_0^{\tau_k} h(\tau_k,s)\,d\beta_l(s),
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
g(\tau_k,s) &= \mathrm{Im}\int_{\mathbb{R}^n}\bigl[2|x|^2\bar{u} - 8(\tau_k-s)^2\bigl(\Delta\bar{u}+\theta|x|^2\bar{u}+|u|^{2\sigma}\bar{u}\bigr)\bigr]\,dx, \\
h(\tau_k,s) &= -4(\tau_k-s)\,\mathrm{Re}\int_{\mathbb{R}^n}(2\bar{u}x\cdot\nabla\phi e_l + n\bar{u}\phi e_l)\,dx.
\end{aligned}
$$

Because $\tau_k \to T$ a.s. as $k \to \infty$, and $V(u(\tau_k)) \geq 0$, it follows from the assumption (3.1), that there exist constants $\delta > 0$ and $\hat{k} > 0$ such that for any $k > \hat{k}$,

$$
\int_0^{\tau_k} g(\tau_k,s)\,dW(s) + \sum_{l\in\mathbb{N}}\int_0^{\tau_k} h(\tau_k,s)\,d\beta_l(s) > \delta > 0 \quad \text{a.s.}
\tag{3.5}
$$

On the other hand, we will show that these two stochastic integrals are square integrable and thus their expectations are zero. By the Hölder inequality, we have

$$
\begin{aligned}
&\mathbb{E}\biggl(\sum_l\int_0^{\tau_k}\biggl|\int_{\mathbb{R}^n}|x|^2\bar{u}\phi e_l(x)\,dx\biggr|^2\,ds\biggr) \\
&\quad\leq \mathbb{E}\biggl(\sum_l\int_0^{\tau_k}\int_{\mathbb{R}^n}|x|^2|u|^2\,dx\int_{\mathbb{R}^n}|x|^2\bigl|\phi e_l(x)\bigr|^2\,dx\,ds\biggr) \\
&\quad= c_\phi^\Sigma\mathbb{E}\biggl(\int_0^{\tau_k}\bigl\|xu(s)\bigr\|_{L^2}^2\,ds\biggr) \\
&\quad\leq c_\phi^\Sigma T\sup_{s\in[0,T]}\mathbb{E}\bigl(\bigl\|xu(s)\bigr\|_{L^2}^2\bigr) \leq k^2Tc_\phi^\Sigma.
\end{aligned}
$$

By the Cauchy-Schwarz inequality and the Sobolev embedding theorem,

$$
\begin{aligned}
&\mathbb{E}\biggl(\sum_l\int_0^{\tau_k}(\tau_k-s)^4\biggl|\int_{\mathbb{R}^n}\bigl(\Delta\bar{u}+\theta|x|^2\bar{u}+|u|^{2\sigma}\bar{u}\bigr)\phi e_l\,dx\biggr|^2\,ds\biggr) \\
&\quad\leq \mathbb{E}\biggl(\sum_l\int_0^{\tau_k}(\tau_k-s)^4\biggl(\int_{\mathbb{R}^n}|\nabla u|^2\int_{\mathbb{R}^n}\bigl|\nabla(\phi e_l)\bigr|^2 + |\theta|\int_{\mathbb{R}^n}|x|^2|u|^2\int_{\mathbb{R}^n}|x|^2|\phi e_l|^2 \\
&\qquad+ \biggl(\int_{\mathbb{R}^n}|u|^{2\sigma+2}\biggr)^{\frac{4\sigma+2}{2\sigma+2}}\biggl(\int_{\mathbb{R}^n}|\phi e_l|^{2\sigma+2}\biggr)^{\frac{2}{2\sigma+2}}\biggr)\,ds\biggr) \\
&\quad\leq \frac{T^5}{5}\bigl(c_\phi^1k^2 + |\theta|c_\phi^\Sigma k^2 + c_\phi^0k^{4\sigma+2} + c_\phi^1k^{4\sigma+2}\bigr).
\end{aligned}
$$

Similarly,

$$\begin{aligned}
&\mathbb{E}\biggl(\sum_l \int_0^{\tau_k} (\tau_k - s)^2 \biggl| \int_{\mathbb{R}^n} \bigl[2\bar{u}x \cdot \nabla(\phi e_l) + n\bar{u}\phi e_l\bigr]\,dx \biggr|^2 \,ds\biggr) \\
&\quad \le \mathbb{E}\biggl(\sum_l \int_0^{\tau_k} (\tau_k - s)^2 \biggl(n^2 \int_{\mathbb{R}^n} |u|^2 \int_{\mathbb{R}^n} |\phi e_l|^2 + 4\int_{\mathbb{R}^n} |x|^2|u|^2 \int_{\mathbb{R}^n} \bigl|\nabla(\phi e_l)\bigr|^2\biggr)\,ds\biggr) \\
&\quad \le \mathbb{E}\biggl(\int_0^{\tau_k} (\tau_k - s)^2 \bigl(n^2\|u\|_{L^2}^2 c_\phi^0 + 4\|xu\|_{L^2}^2 c_\phi^1\bigr)\,ds\biggr) \\
&\quad \le \frac{T^3}{3} n^2 k^2 c_\phi^0 + \frac{4T^2}{3} k^2 c_\phi^1.
\end{aligned}$$

Therefore,

$$\mathbb{E}\biggl(\int_0^{\tau_k} g(\tau_k, s)\,dW(s)\biggr) + \mathbb{E}\biggl(\sum_{l\in\mathbb{N}} \int_0^{\tau_k} h(\tau_k, s)\,d\beta_l(s)\biggr) = 0, \tag{3.6}$$

which is in contradiction with (3.5).

We proceed to consider the case of $\theta > 0$. As in (3.4), we have

$$\begin{aligned}
V\bigl(u(\tau_k)\bigr) \le{}& V(u_0) - 4G(u_0)\tau_k + 8H(u_0)\tau_k^2 + c_\phi^\Sigma \tau_k + 2c_\phi^2\tau_k^2 + \frac{4}{3}\bigl(c_\phi^1 - \theta c_\phi^\Sigma\bigr)\tau_k^3 \\
&+ 16\theta \int_0^{\tau_k} (\tau_k - s) V\bigl(u(s)\bigr)\,ds + \int_0^{\tau_k} g(\tau_k, s)\,dW(s) \\
&+ \sum_{l\in\mathbb{N}} \int_0^{\tau_k} h(\tau_k, s)\,d\beta_l(s).
\end{aligned}$$

In view of (3.6) and noting $\tau_k \to T$ a.s. as $k \to \infty$, we have

$$\begin{aligned}
\mathbb{E}\bigl(V\bigl(u(T)\bigr)\bigr) \le{}& \mathbb{E}\bigl(V(u_0)\bigr) - 4\mathbb{E}\bigl(G(u_0)\bigr)T + 8\mathbb{E}\bigl(H(u_0)\bigr)T^2 + c_\phi^\Sigma T + 2c_\phi^2 T^2 \\
&+ \frac{4}{3}\bigl(c_\phi^1 - \theta c_\phi^\Sigma\bigr)T^3 + 16\theta \int_0^T (T - s)\mathbb{E}\bigl(V\bigl(u(s)\bigr)\bigr)\,ds.
\end{aligned}$$

For convenience, set $\mathfrak{A} := c_\phi^\Sigma - 4\mathbb{E}(G(u_0))$, $\mathfrak{B} := c_\phi^2 + 4\mathbb{E}(H(u_0))$, and $\mathfrak{C} := c_\phi^1 - \theta c_\phi^\Sigma$. Applying the theory of ordinary differential equations, we get

$$\begin{aligned}
&\mathbb{E}\bigl(V\bigl(u(T)\bigr)\bigr) \\
&\quad \le \cosh^2(2\mu T)\biggl[2\mathbb{E}\bigl(V(u_0)\bigr) + \biggl(\frac{\mathfrak{A}}{2\mu} + \frac{\mathfrak{C}}{4\mu^3}\biggr)\tanh(2\mu T) + \frac{\mathfrak{B}}{2\mu^2}\tanh^2(2\mu T)\biggr].
\end{aligned}$$

Thus if (3.2) holds, we then have $\mathbb{E}(V(u(T))) < 0$, which is a contradiction. □

We next show that blow-up occurs in arbitrarily small time for any initial data with finite variance. According to the framework established by [8, 11], there are three key ingredients to show blow-up: the controllability of the Schrödinger dynamics, the well-posedness of the perturbation equation and the continuous dependence of the solution on both the perturbation and the initial data. In this paper, we utilize the compatibility of the potential with the natural working space Σ to get the controllability of the Schrödinger equation in Σ; and then employ the energy decomposition technique together with the continuity argument to derive the well-posedness of the perturbation equation and the continuous dependence of the solution on the initial data in the natural space Σ. In particular, owing to the structure of the quadratic potential, we only need the assumption of finite variance regardless of higher order moments on the initial data which are required in [8, 11]. Analogous to the deterministic equation [4], we define for an admissible pair (q, r),

$$\begin{aligned} Y_r(0,T) := \bigl\{ & f \in \mathcal{C}\bigl([0,T];\Sigma\bigr) : A(t)f \in \mathcal{C}\bigl([0,T];L^2\bigr) \cap L^q\bigl((0,T);L^r\bigr) \\ & \text{for any } A(t) \in \bigl\{J(t), K(t), Id\bigr\}\bigr\}. \end{aligned}$$

Lemma 3.2 (Controllability) *For any $u_0 \in \Sigma$, $u_1 \in \Sigma$, $T_1 > 0$ ($\frac{\pi}{4\sqrt{-\theta}} > T_1 > 0$ if $\theta < 0$), there exists a function $z \in Y_{2\sigma+2}(0, T_1)$ such that $z(0) = 0$ and the solution of*

$$\begin{cases} iv_t + \Delta v + \theta|x|^2 v + |v+z|^{2\sigma}(v+z) = 0, & t \geq 0, x \in \mathbb{R}^n, \\ v(0,x) = u_0(x), \end{cases} \tag{3.7}$$

exists in $Y_{2\sigma+2}(0, T_1)$ and $z(T_1) + v(z, u_0, T_1) = u_1$.

Proof We only consider the case of $n \geq 3$ since the case of $n \leq 2$ is similar and easier. Consider a linear parabolic equation

$$\begin{cases} w_t + (-\Delta)^k w + |x|^{2k} w = 0, & t \geq 0, x \in \mathbb{R}^n, \\ w(0,x) = u_0(x), \end{cases} \tag{3.8}$$

where $k > [\frac{n}{2}] + 1$ is a positive integer. Denote by $S(t)$ the semigroup on Σ associated to (3.8). Define

$$u(t) := \frac{T_1 - t}{T_1} S(t) u_0 + \frac{t}{T_1} S(T_1 - t) u_1.$$

Obviously, $u(0) = u_0$ and $u(T_1) = u_1$. We shall investigate the regularity of u. First it is easy to see that $u \in \mathcal{C}([0, T_1]; \Sigma)$. Multiplying (3.8) by $\bar{w}$, integrating the resultant equation over $[0, T_1] \times \mathbb{R}^n$ gives

$$|x|^k u \in L^2\bigl((0,T_1);L^2\bigr) \quad \text{and} \quad u \in L^2\bigl((0,T_1);H^k\bigr). \tag{3.9}$$

Because $k > [\frac{n}{2}] + 1 \geq 2$, it follows from the Sobolev embedding theorem that

$$u \in L^2\bigl((0,T_1); W^{1,\frac{2n}{n-2}}\bigr).$$

We take η satisfying

$$\frac{1}{2\sigma+2} = \frac{(n-2)\eta}{2n} + \frac{1-\eta}{2}, \qquad \frac{1}{q} = \frac{\eta}{2} + \frac{1-\eta}{\infty}.$$

Clearly $\eta = \frac{n\sigma}{2(\sigma+1)} < 1$ since $\sigma < \frac{2}{n-2}$. Noting that $u \in L^\infty((0,T_1);H^1) \cap L^2((0,T_1); W^{1,\frac{2n}{n-2}})$, applying Hölder's inequality in space and then in time, we have

$$u \in L^q\bigl((0,T_1); W^{1,2\sigma+2}\bigr) \quad \text{with } q = \frac{4(\sigma+1)}{\sigma n}. \tag{3.10}$$

We next estimate xu. When $k > [\frac{n}{2}] + 1$, since $H^k \hookrightarrow L^\infty$ and $H^1 \hookrightarrow L^{\frac{2n}{n-2}}$, by Hölder's inequality, we obtain

$$\begin{aligned}\|xu\|_{L^{\frac{2n}{n-2}}} &\le \left(\int_{|x|\le 1} |xu|^{\frac{2n}{n-2}}\,dx + \int_{|x|\ge 1} |xu|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{2n}} \\ &\le \left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}\,dx + \int_{\mathbb{R}^n} |x|^{2k}|u|^2|u|^{\frac{4}{n-2}}\,dx\right)^{\frac{n-2}{2n}} \\ &\le C\|u\|_{H^1} + C\|u\|_{H^k}^{\frac{2}{n}}\big\||x|^k u\big\|_{L^2}^{\frac{n-2}{n}}.\end{aligned}$$

Thus,

$$\begin{aligned}&\|xu\|_{L^2((0,T_1);L^{\frac{2n}{n-2}})} \\ &\quad\le C\|u\|_{L^2((0,T_1);H^1)} + C\left(\int_0^{T_1} \big\|u(t)\big\|_{H^k}^{\frac{4}{n}}\big\||x|^k u(t)\big\|_{L^2}^{\frac{2(n-2)}{n}}\,dt\right)^{1/2} \\ &\quad\le C\|u\|_{L^2((0,T_1);H^1)} + C\|u\|_{L^2((0,T_1);H^k)}^{\frac{2}{n}}\big\||x|^k u\big\|_{L^2((0,T_1);L^2)}^{1-\frac{2}{n}},\end{aligned}$$

which in combination with (3.9) yields

$$xu \in L^2\big((0,T_1);L^{\frac{2n}{n-2}}\big). \tag{3.11}$$

Noting that $xu \in \mathcal{C}([0,T_1];L^2)$, as in (3.10), we get $xu \in L^q((0,T_1);L^{2\sigma+2})$. Thus

$$A(\cdot)u \in \mathcal{C}\big([0,T_1];L^2\big) \cap L^q\big((0,T_1);L^{2\sigma+2}\big). \tag{3.12}$$

We proceed to estimate $|u|^{2\sigma}u$. If $0 < \sigma \le 1$, by Hölder's inequality,

$$\|u\|_{L^{4\sigma+2}} \le \|u\|_{L^\infty}^{1-\lambda}\|u\|_{L^2}^{\lambda} \quad \text{with } \frac{1}{4\sigma+2} = \frac{\lambda}{2} + \frac{1-\lambda}{\infty},$$

from which it follows that

$$\begin{aligned}\big\||u|^{2\sigma}u\big\|_{L^1((0,T_1);L^2)} &\le C\|u\|_{\mathcal{C}([0,T_1];L^2)}^{\lambda(2\sigma+1)}\int_0^{T_1}\big\|u(t,\cdot)\big\|_{H^k}^{(1-\lambda)(2\sigma+1)}\,dt \\ &\le CT_1^{\beta}\|u\|_{\mathcal{C}([0,T_1];L^2)}^{\lambda(2\sigma+1)}\|u\|_{L^2((0,T_1);H^k)}^{1-\beta}\end{aligned}$$

for some $0 \le \beta < 1$, where we have used the embedding $H^k \hookrightarrow L^\infty$ and the fact that $0 < (1-\lambda)(2\sigma+1) \le 2$. If $1 < \sigma < \frac{2}{n-2}$, using the inequality

$$\|u\|_{L^{4\sigma+2}} \le \|u\|_{L^\infty}^{1-\hat{\lambda}}\|u\|_{L^{\frac{2n}{n-2}}}^{\hat{\lambda}} \quad \text{with } \frac{1}{4\sigma+2} = \frac{(n-2)\hat{\lambda}}{2n} + \frac{1-\hat{\lambda}}{\infty},$$

and the Sobolev embedding $H^1 \hookrightarrow L^{\frac{2n}{n-2}}$, we similarly have

$$\big\||u|^{2\sigma}u\big\|_{L^1((0,T_1);L^2)} \le C\|u\|_{\mathcal{C}([0,T_1];H^1)}^{\hat{\lambda}(2\sigma+1)}\|u\|_{L^2((0,T_1);H^k)}^{1-\hat{\beta}}.$$

Thus, using (3.9), we derive

$$|u|^{2\sigma}u \in L^1\big((0,T_1);L^2\big).$$

By Hölder's inequality,

$$\begin{aligned}\big\||u|^{2\sigma}\nabla u\big\|_{L^1((0,T_1);L^2)} &\le C\int_0^{T_1}\big\|u(t,\cdot)\big\|_{L^\infty}^{2\sigma}\big\|\nabla u(t,\cdot)\big\|_{L^2}\,dt\\ &\le C\|u\|_{\mathcal{C}([0,T_1];H^1)}\int_0^{T_1}\big\|u(t,\cdot)\big\|_{H^k}^{2\sigma}\,dt.\end{aligned}$$

If $0<\sigma\le 1$ (noting that when $n\ge 4$, $\sigma<\frac{2}{n-2}\le 1$), by Hölder's inequality in time, we get

$$\big\||u|^{2\sigma}\nabla u\big\|_{L^1((0,T_1);L^2)} \le CT^{1-\sigma}\|u\|_{\mathcal{C}([0,T_1];H^1)}\|u\|_{L^2((0,T_1);H^k)}^{\sigma}. \tag{3.13}$$

If $n=3$ and $1<\sigma<\frac{2}{n-2}=2$, it follows from Gagliardo-Nirenberg's inequality that

$$\|u\|_{L^{6\sigma}} \le C\|u\|_{H^2}^{\frac{1}{2}-\frac{1}{2\sigma}}\|u\|_{L^6}^{\frac{1}{2}+\frac{1}{2\sigma}}. \tag{3.14}$$

Thus observing that $H^2\hookrightarrow W^{1,6}$ when $n=3$, it holds that

$$\big\||u|^{2\sigma}\nabla u\big\|_{L^2} \le C\|\nabla u\|_{L^6}\|u\|_{L^{6\sigma}}^{2\sigma} \le C\|u\|_{H^2}^{\sigma}\|u\|_{H^1}^{\sigma+1}.$$

Since $\sigma<2$, $u\in\mathcal{C}([0,T_1];H^1)$, and $|\nabla(|u|^{2\sigma}u)|\le C|u|^{2\sigma}|\nabla u|$, we obtain

$$\nabla\big(|u|^{2\sigma}u\big)\in L^1\big((0,T_1);L^2\big).$$

We next show that $x|u|^{2\sigma}u\in L^1((0,T_1);L^2)$. As in (3.13), if $0<\sigma\le 1$, we have

$$\begin{aligned}\big\|x|u|^{2\sigma}u\big\|_{L^1((0,T_1);L^2)} &\le C\int_0^{T_1}\big\|u(t,\cdot)\big\|_{L^\infty}^{2\sigma}\big\|xu(t,\cdot)\big\|_{L^2}\,dt\\ &\le C\|u\|_{\mathcal{C}([0,T_1];\Sigma)}\int_0^{T_1}\big\|u(t,\cdot)\big\|_{H^k}^{2\sigma}\,dt\\ &\le C\|u\|_{\mathcal{C}([0,T_1];\Sigma)}\|u\|_{L^2((0,T_1);H^k)}^{\sigma}.\end{aligned}$$

If $n=3$ and $1<\sigma<2$, by (3.14),

$$\begin{aligned}\big\|x|u|^{2\sigma}u\big\|_{L^1((0,T_1);L^2)} &\le C\int_0^{T_1}\big\|xu(t,\cdot)\big\|_{L^6}\big\|u(t,\cdot)\big\|_{L^{6\sigma}}^{2\sigma}\,dt\\ &\le C\|xu\|_{L^2((0,T_1);L^6)}\|u\|_{\mathcal{C}([0,T_1];H^1)}^{\sigma+1}\|u\|_{L^2((0,T_1);H^2)}^{\sigma-1}.\end{aligned}$$

This inequality together with (3.11) implies $x|u|^{2\sigma}u\in L^1((0,T_1);L^2)$. Therefore, we obtain

$$|u|^{2\sigma}u\in L^1\big((0,T_1);\Sigma\big). \tag{3.15}$$

Now we define

$$v(t):=U_\theta(t)u_0+i\int_0^t U_\theta(t-s)|u|^{2\sigma}u(s)\,ds,$$

and $z(t) := u(t) - v(t)$. Clearly, $v(z, u_0, t) := v(t)$ is the solution of (3.7). By (2.1),

$$J(t)v(t) = U_\theta(t)i\nabla u_0 + i\int_0^t U_\theta(t-s)J(s)|u|^{2\sigma}u(s)\,ds,$$
$$K(t)v(t) = U_\theta(t)xu_0 + i\int_0^t U_\theta(t-s)K(s)|u|^{2\sigma}u(s)\,ds.$$

It thus follows from (3.15), Strichartz estimates and (2.2) that $v \in \mathcal{C}([0,T_1];\Sigma)$ and that

$$\|A(\cdot)v\|_{L^q((0,T_1);L^{2\sigma+2})} \le C\|u_0\|_\Sigma + C\||u|^{2\sigma}u\|_{L^1((0,T_1);\Sigma)},$$

which implies $A(\cdot)v \in L^q((0,T_1);L^{2\sigma+2})$. Finally, by (3.12), $z = (u - v)$ satisfies $A(\cdot)z \in \mathcal{C}([0,T_1];L^2) \cap L^q((0,T_1);L^{2\sigma+2})$. □

Lemma 3.3 (Local well-posedness of (3.7)) *Assume that $u_0 \in \Sigma$. Then for any z satisfying $z \in Y_{2\sigma+2}(0,T^*)$, there exist $T > 0$ with $0 < T \le T^*$ and a unique solution $v(z,u_0,\cdot)$ of (3.7) satisfying $v(z,u_0,\cdot) \in Y_{2\sigma+2}(0,T)$.*

Proof *Step* 1. *Existence.* Define the set

$$X_{T,M} := \bigl\{f \in Y_{2\sigma+2}(0,T) : \|A(\cdot)f\|_{L^\infty((0,T);L^2)} + \|A(\cdot)f\|_{L^q((0,T);L^{2\sigma+2})} \le M \text{ for any } A(t) \in \{J(t), K(t), Id\}\bigr\}$$

equipped with the distance

$$d(f,g) := \|f-g\|_{L^\infty((0,T);L^2)} + \|f-g\|_{L^q((0,T);L^{2\sigma+2})}.$$

It is easy to see that $(X_{T,M}, d)$ is a complete metric space. Consider the mapping $\mathbb{H}$ defined by

$$\mathbb{H}v(t) := U_\theta(t)u_0 + i\int_0^t U_\theta(t-s)|v+z|^{2\sigma}(v+z)(s)\,ds.$$

We shall show that there exist T and M satisfying (i) $\mathbb{H}$ maps $(X_{T,M}, d)$ into itself; (ii) $\mathbb{H}$ is a contraction in $(X_{T,M}, d)$. By (2.1),

$$J(t)\mathbb{H}v(t) = U_\theta(t)i\nabla u_0 + i\int_0^t U_\theta(t-s)J(s)|v+z|^{2\sigma}(v+z)(s)\,ds. \tag{3.16}$$

Thus, by the Strichartz estimates, (2.2), and Hölder's inequality, we have

$$\begin{aligned}
&\|J(\cdot)\mathbb{H}v\|_{L_T^\infty L^2} + \|J(\cdot)\mathbb{H}v\|_{L_T^q L^{2\sigma+2}} \\
&\quad\le C\|\nabla u_0\|_{L^2} + C\||v+z|^{2\sigma}J(\cdot)(v+z)\|_{L_T^{q'}L^{(2\sigma+2)'}} \\
&\quad\le C\|\nabla u_0\|_{L^2} + CT^{1-\delta(\sigma)}\|v+z\|_{L_T^\infty L^{2\sigma+2}}^{2\sigma}\|J(\cdot)(v+z)\|_{L_T^q L^{2\sigma+2}} \\
&\quad\le C\|\nabla u_0\|_{L^2} + CT^{1-\delta(\sigma)}\bigl(M^{2\sigma} + \|z\|_{L_T^\infty H^1}^{2\sigma}\bigr)\bigl(M + \|J(\cdot)z\|_{L_T^q L^{2\sigma+2}}\bigr),
\end{aligned}$$

where $\delta(\sigma) = \frac{n\sigma}{2(\sigma+1)} < 1$ since $\sigma < \frac{2}{n-2}$, and we have used the Sobolev embedding $H^1 \hookrightarrow L^{2\sigma+2}$. Applying similar arguments to operators $K(t)$ and Id, we get

$$\begin{aligned} &\bigl\|A(\cdot)\mathbb{H}v\bigr\|_{L^\infty_T L^2} + \bigl\|A(\cdot)\mathbb{H}v\bigr\|_{L^q_T L^{2\sigma+2}} \\ &\quad \le C\|u_0\|_\Sigma + CT^{1-\delta(\sigma)}\bigl(M^{2\sigma} + \|z\|^{2\sigma}_{L^\infty_T H^1}\bigr)\bigl(M + \bigl\|A(\cdot)z\bigr\|_{L^q_T L^{2\sigma+2}}\bigr). \end{aligned} \tag{3.17}$$

On the other hand, for any $w, v \in X_{T,M}$, we have

$$\mathbb{H}w(t) - \mathbb{H}v(t) = i\int_0^t U_\theta(t-s)\bigl(|w+z|^{2\sigma}(w+z) - |v+z|^{2\sigma}(v+z)\bigr)\,ds.$$

A simple calculation yields

$$\bigl||w+z|^{2\sigma}(w+z) - |v+z|^{2\sigma}(v+z)\bigr| \le C\bigl(|w+z|^{2\sigma} + |v+z|^{2\sigma}\bigr)|w-v|$$

for some constant $C > 0$. This inequality, in combination with the Strichartz estimates and Hölder's inequality, leads to

$$\begin{aligned} &\|\mathbb{H}w - \mathbb{H}v\|_{L^\infty_T L^2} + \|\mathbb{H}w - \mathbb{H}v\|_{L^q_T L^{2\sigma+2}} \\ &\quad \le CT^{1-\delta(\sigma)}\bigl(\|w+z\|^{2\sigma}_{L^\infty_T L^{2\sigma+2}} + \|v+z\|^{2\sigma}_{L^\infty_T L^{2\sigma+2}}\bigr)\|w-v\|_{L^q_T L^{2\sigma+2}} \\ &\quad \le CT^{1-\delta(\sigma)}\bigl(M^{2\sigma} + \|z\|^{2\sigma}_{L^\infty_T H^1}\bigr)d(w,v). \end{aligned} \tag{3.18}$$

Now choosing

$$M := \max\bigl\{\bigl\|J(\cdot)z\bigr\|_{L^q_T L^{2\sigma+2}}, \bigl\|K(\cdot)z\bigr\|_{L^q_T L^{2\sigma+2}}, \|z\|_{L^q_T L^{2\sigma+2}}\bigr\} + 2C\|u_0\|_\Sigma$$

and T small enough such that $CT^{1-\delta(\sigma)}(M^{2\sigma} + \|z\|^{2\sigma}_{L^\infty_T H^1}) \le \frac{1}{2}$, we get from (3.17) and (3.18) that $\|A(\cdot)\mathbb{H}v\|_{L^\infty_T L^2} + \|A(\cdot)\mathbb{H}v\|_{L^q_T L^{2\sigma+2}} \le M$ and $d(\mathbb{H}w, \mathbb{H}v) \le \frac{1}{2}d(w,v)$ which verify (i) and (ii). Applying the contraction mapping principle, we see that there exists a solution $v(z, u_0, \cdot)$ to (3.7) and $A(\cdot)v(z, u_0, \cdot) \in \mathcal{C}([0,T]; L^2) \cap L^q((0,T); L^{2\sigma+2})$. Moreover, we have the estimate

$$\bigl\|A(\cdot)v\bigr\|_{L^\infty_T L^2} + \bigl\|A(\cdot)v\bigr\|_{L^q_T L^{2\sigma+2}} \le C\bigl(\|u_0\|_\Sigma + \|z\|_{L^q_T W^{1,2\sigma+2}} + \|xz\|_{L^q_T L^{2\sigma+2}}\bigr). \tag{3.19}$$

Step 2. *Uniqueness*. Suppose that there are two mild solutions w and v satisfying $w, v \in \mathcal{C}([0,T]; \Sigma) \cap L^q((0,T); L^{2\sigma+2})$. Applying the Strichartz estimates and Hölder's inequality, as in (3.18), we get for any $t \in [0,T]$,

$$\begin{aligned} &\|w-v\|_{L^q((0,t);L^{2\sigma+2})} \\ &\quad \le Ct^{1-\delta(\sigma)}\bigl(\|w\|^{2\sigma}_{L^\infty_T H^1} + \|v\|^{2\sigma}_{L^\infty_T H^1} + \|z\|^{2\sigma}_{L^\infty_T H^1}\bigr)\|w-v\|_{L^q((0,t);L^{2\sigma+2})}. \end{aligned}$$

Thus if $\tau \in [0,T]$ is small enough such that

$$C\tau^{1-\delta(\sigma)}\bigl(\|w\|^{2\sigma}_{L^\infty_T H^1} + \|v\|^{2\sigma}_{L^\infty_T H^1} + \|z\|^{2\sigma}_{L^\infty_T H^1}\bigr) < 1,$$

we then have $\|w-v\|_{L^q((0,\tau);L^{2\sigma+2})}=0$, which gives $w=v$ on $[0,\tau]$. Set

$$\hat{\tau} := \sup\{\tau \in [0,T]; \|w-v\|_{L^q((0,\tau);L^{2\sigma+2})}=0\}.$$

Clearly, $\hat{\tau}>0$. Suppose $\hat{\tau}<T$, then we can take $\hat{\tau}$ as the initial time and deduce $\|w-v\|_{L^q((0,\hat{\tau}+\epsilon);L^{2\sigma+2})}=0$ for some $\epsilon>0$, which contradicts the definition of $\hat{\tau}$. Therefore, $\hat{\tau}=T$, and we complete the proof of the uniqueness part. □

Lemma 3.4 (Continuous dependence) *Let $\hat{u}_0 \in \Sigma$, $T>0$. Assume that $\hat{z} \in Y_{2\sigma+2}(0,T)$ and that the solution $v(\hat{z},\hat{u}_0,\cdot)$ of* (3.7) *exists on $[0,T]$. Then there exist neighborhoods $\mathcal{V}$ of $\hat{z}$ in $Y_{2\sigma+2}(0,T)$ and $\mathcal{W}$ of $\hat{u}_0$ in Σ, such that for any $(z,u_0)\in\mathcal{V}\times\mathcal{W}$, the solution $v(z,u_0,\cdot)$ of* (3.7) *exists and is unique in $\mathcal{C}([0,T];\Sigma)$. Moreover, the mapping $(z,u_0)\mapsto v(z,u_0,\cdot)$ is continuous from $\mathcal{V}\times\mathcal{W}$ into $\mathcal{C}([0,T];\Sigma)$.*

Proof Step 1. Let $r,R>0$ and take u_0, z satisfying

$$\|u_0\|_\Sigma \le r, \qquad \|z\|_{\mathcal{C}([0,T];H^1)} \le R \quad \text{and} \quad \|A(\cdot)z\|_{L^q((0,T);L^{2\sigma+2})} \le R$$

for any $A(t)\in\{J(t),K(t),Id\}$. Owing to Lemma 3.3, there exist $T_1=T_1(r,R)$ and a unique solution $v(z,u_0,\cdot)$ of (3.7) on $[0,T_1]$ such that $v(z,u_0,\cdot)\in Y_{2\sigma+2}(0,T_1)$.

We next show that the solution map $(z,u_0)\mapsto v(z,u_0,\cdot)$ is continuous on $[0,\tau]$ for some $0<\tau\le T_1$. To do this, we take $\{u_0^n\}_{n\ge1}\subset\Sigma$ and $\{z_n\}_{n\ge1}\subset Y_{2\sigma+2}(0,T)$ satisfying $u_0^n\to u_0$ in Σ and $A(\cdot)z_n\to A(\cdot)z$ in $\mathcal{C}([0,T];L^2)\cap L^q((0,T);L^{2\sigma+2})$ for any $A(t)\in\{J(t),K(t),Id\}$. Noting that

$$\begin{aligned}
&\|u_0^n\|_\Sigma \le 2\|u_0\|_\Sigma, \qquad \|z_n\|_{L_T^\infty H^1} \le 2\|z\|_{L_T^\infty H^1},\\
&\|A(\cdot)z_n\|_{L^q((0,T);L^{2\sigma+2})} \le 2\|A(\cdot)z\|_{L^q((0,T);L^{2\sigma+2})}
\end{aligned}$$

for n sufficiently large, according to Lemma 3.3, (3.7) has a unique solution v_n corresponding to (z_n,u_0^n), and there exists $\tau=\tau(r,R)$ such that v_n and v exist on $[0,\tau]\subset[0,T_1]$. Moreover, owing to (3.19), v_n satisfies

$$\begin{aligned}
&\|A(\cdot)v_n\|_{C([0,\tau];L^2)} + \|A(\cdot)v_n\|_{L^q((0,\tau);L^{2\sigma+2})}\\
&\quad \le C\left(\|u_0\|_\Sigma + \|z\|_{L^q((0,T);W^{1,2\sigma+2})} + \|xz\|_{L^q((0,T);L^{2\sigma+2})}\right).
\end{aligned} \tag{3.20}$$

We shall show that $v_n\to v$ in $\mathcal{C}([0,\tau];\Sigma)$ as $n\to\infty$. Since

$$\begin{aligned}
v_n - v = {}& i\int_0^t U_\theta(t-s)\left(|v_n+z_n|^{2\sigma}(v_n+z_n) - |v+z|^{2\sigma}(v+z)\right)ds\\
&+ U_\theta(t)\left(u_0^n-u_0\right)
\end{aligned} \tag{3.21}$$

as in (3.18), we have

$$\begin{aligned}
&\|v_n-v\|_{L_\tau^\infty L^2} + \|v_n-v\|_{L_\tau^q L^{2\sigma+2}}\\
&\quad \le C\tau^{1-\delta(\sigma)}\left(\|v_n+z_n\|_{L_\tau^\infty H^1}^{2\sigma} + \|v+z\|_{L_\tau^\infty H^1}^{2\sigma}\right)\|v_n-v+z_n-z\|_{L_\tau^q L^{2\sigma+2}}\\
&\qquad + C\|u_0^n-u_0\|_{L^2}.
\end{aligned} \tag{3.22}$$

Using (3.20) and choosing $\tau = \tau(r,R)$ small enough, it follows that

$$\|v_n - v\|_{L_\tau^\infty L^2} + \|v_n - v\|_{L_\tau^q L^{2\sigma+2}} \le C\|u_0^n - u_0\|_{L^2} + C\|z_n - z\|_{L_\tau^q L^{2\sigma+2}}. \tag{3.23}$$

Analogous to (3.17), applying $J(t)$ and $K(t)$ to (3.21) and setting $F(u) := |u|^{2\sigma}u$, we get

$$\begin{aligned}
&\|A(\cdot)(v_n - v)\|_{L_\tau^\infty L^2} + \|A(\cdot)(v_n - v)\|_{L_\tau^q L^{2\sigma+2}} \\
&\quad\le C\|A(\cdot)\big(|v_n + z_n|^{2\sigma}(v_n + z_n) - |v + z|^{2\sigma}(v + z)\big)\|_{L_\tau^{q'} L^{(2\sigma+2)'}} + C\|u_0^n - u_0\|_\Sigma \\
&\quad\le C\|\partial_u F(v_n + z_n)A(\cdot)(v_n + z_n) - \partial_u F(v + z)A(\cdot)(v + z)\|_{L_\tau^{q'} L^{(2\sigma+2)'}} \\
&\qquad + C\|\partial_{\bar u} F(v_n + z_n)\overline{A(\cdot)(v_n + z_n)} - \partial_{\bar u} F(v + z)\overline{A(\cdot)(v + z)}\|_{L_\tau^{q'} L^{(2\sigma+2)'}} \\
&\qquad + C\|u_0^n - u_0\|_\Sigma \\
&\quad:= I_1 + I_2 + C\|u_0^n - u_0\|_\Sigma.
\end{aligned} \tag{3.24}$$

I_1 can be estimated as follows. By Hölder's inequality,

$$\begin{aligned}
I_1 &\le C\|\partial_u F(v_n + z_n)A(\cdot)(v_n - v + z_n - z)\|_{L_\tau^{q'} L^{(2\sigma+2)'}} \\
&\quad + C\|\big[\partial_u F(v_n + z_n) - \partial_u F(v + z)\big]A(\cdot)(v + z)\|_{L_\tau^{q'} L^{(2\sigma+2)'}} \\
&\le C\tau^{1-\delta(\sigma)}\|v_n + z_n\|_{L_\tau^\infty H^1}^{2\sigma}\big(\|A(\cdot)(v_n - v)\|_{L_\tau^q L^{2\sigma+2}} + \|A(\cdot)(z_n - z)\|_{L_\tau^q L^{2\sigma+2}}\big) \\
&\quad + C\|\big[\partial_u F(v_n + z_n) - \partial_u F(v + z)\big]A(\cdot)(v + z)\|_{L_\tau^{q'} L^{(2\sigma+2)'}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &\le C\tau^{1-\delta(\sigma)}\|v_n + z_n\|_{L_\tau^\infty H^1}^{2\sigma}\big(\|A(\cdot)(v_n - v)\|_{L_\tau^q L^{2\sigma+2}} + \|A(\cdot)(z_n - z)\|_{L_\tau^q L^{2\sigma+2}}\big) \\
&\quad + C\|\big[\partial_{\bar u} F(v_n + z_n) - \partial_{\bar u} F(v + z)\big]A(\cdot)(v + z)\|_{L_\tau^{q'} L^{(2\sigma+2)'}}.
\end{aligned}$$

Substituting these two inequalities into (3.24), using (3.20) and choosing $\tau = \tau(r,R)$ small enough, we have

$$\begin{aligned}
&\|A(\cdot)(v_n - v)\|_{L_\tau^\infty L^2} + \|A(\cdot)(v_n - v)\|_{L_\tau^q L^{2\sigma+2}} \\
&\quad\le C\|u_0^n - u_0\|_\Sigma + C\|A(\cdot)(z_n - z)\|_{L_\tau^q L^{2\sigma+2}} \\
&\qquad + C\|\big[\partial_u F(v_n + z_n) - \partial_u F(v + z)\big]A(\cdot)(v + z)\|_{L_\tau^{q'} L^{(2\sigma+2)'}} \\
&\qquad + C\|\big[\partial_{\bar u} F(v_n + z_n) - \partial_{\bar u} F(v + z)\big]A(\cdot)(v + z)\|_{L_\tau^{q'} L^{(2\sigma+2)'}} \\
&\quad:= C\|u_0^n - u_0\|_\Sigma + C\|A(\cdot)(z_n - z)\|_{L_\tau^q L^{2\sigma+2}} + I_3^n + I_4^n.
\end{aligned} \tag{3.25}$$

To show that $v_n \to v$ in $\mathcal{C}([0,\tau];\Sigma)$ as $n \to \infty$, we only need to verify that $I_3^n + I_4^n \to 0$. Otherwise, without loss of generality, there exists $\varrho > 0$ and, up to a subsequence, $I_3^n \ge \varrho$. It then follows from (3.23) that there exists a subsequence, still denoted by $\{v_n\}_{n\ge1}$, such that $v_n \to v$ a.e. in $[0,\tau] \times \mathbb{R}^n$. Moreover, there exist $w, \varpi \in L_\tau^q L^{2\sigma+2}$ such that $v_n \le w$ and $z_n \le \varpi$ a.e. Noting that $|[\partial_u F(v_n + z_n) - \partial_u F(v + z)]A(\cdot)(v + z)| \le C(|w|^{2\sigma} + |\varpi|^{2\sigma})|A(\cdot)(v +$

$z)| \in L_\tau^{q'} L^{(2\sigma+2)'}$, by the dominated convergence theorem, we see that $I_3^n \to 0$, which is a contradiction.

Step 2. Now we obtain the solution of (3.7) denoted by $v_1(z, u_0, \cdot)$ on $[0, \tau]$ with $\tau = \tau(r, R)$. Set

$$\begin{aligned} r &= 1 + \left\| v(\hat{z}, \hat{u}_0, \cdot) \right\|_{\mathcal{C}([0,T];\Sigma)}, \\ R &= 1 + \|\hat{z}\|_{\mathcal{C}([0,T];H^1)} + \max\left\{ \left\| A(\cdot)\hat{z} \right\|_{L^q((0,T);L^{2\sigma+2})}, A(t) \in \left\{ J(t), K(t), Id \right\} \right\}. \end{aligned}$$

By the continuity of $v_1(z, u_0, \cdot)$ with respect to (z, u_0) on $[0, \tau]$, we see that, for $0 < \varepsilon_1 \ll 1$, there exists $0 < \delta_1 \ll 1$ such that

$$\begin{aligned} &\text{if } \|u_0 - \hat{u}_0\|_\Sigma + \|z - \hat{z}\|_{\mathcal{C}([0,\tau];H^1)} + \|z - \hat{z}\|_{L^q((0,\tau);W^{1,2\sigma+2})} \\ &\qquad + \left\| x(z - \hat{z}) \right\|_{L^q((0,\tau);L^{2\sigma+2})} \leq \delta_1, \\ &\text{then } \left\| v_1(z, u_0, \cdot) - v(\hat{z}, \hat{u}_0, \cdot) \right\|_{\mathcal{C}([0,\tau];\Sigma)} \leq \varepsilon_1. \end{aligned}$$

Set $u_1 = v_1(z, u_0, \tau)$, then

$$\|u_1\|_\Sigma \leq \left\| v(\hat{z}, \hat{u}_0, \tau) \right\|_\Sigma + \varepsilon_1 < r.$$

As in the argument of Step 1, the solution of (3.7) denoted by $v_2(z, u_1, \cdot)$ exists and is unique on $[\tau, 2\tau]$, and $v_2(z, u_1, \cdot)$ is continuous with respect to (z, u_1) on $[\tau, 2\tau]$.

By iteration, for every $1 \leq j \leq [\frac{T}{\tau}]$ and $0 < \varepsilon_j \ll 1$, there exists $0 < \delta_j \ll 1$ such that

$$\begin{aligned} &\text{if } \|u_{j-1} - \hat{u}_0\|_\Sigma + \|z - \hat{z}\|_{\mathcal{C}([(j-1)\tau,j\tau];H^1)} + \|z - \hat{z}\|_{L^q(((j-1)\tau,j\tau);W^{1,2\sigma+2})} \\ &\qquad + \left\| x(z - \hat{z}) \right\|_{L^r(((j-1)\tau,j\tau);L^{2\sigma+2})} \leq \delta_j, \\ &\text{then } \left\| v_j(z, u_{j-1}, \cdot) - v(\hat{z}, \hat{u}_0, \cdot) \right\|_{\mathcal{C}([(j-1)\tau,j\tau];\Sigma)} \leq \varepsilon_j. \end{aligned}$$

Set $u_j = v_j(z, u_{j-1}, j\tau)$, then $\|u_j\|_\Sigma \leq \|v(\hat{z}, \hat{u}_0, j\tau)\|_\Sigma + \varepsilon_j < r$. By Step 1, the solution of (3.7) denoted by $v_{j+1}(z, u_j, \cdot)$ exists and is unique on $[j\tau, (j+1)\tau]$. Moreover, every v_{j+1} is continuous with respect to (z, u_j) on $[j\tau, (j+1)\tau]$.

Step 3. Without loss of generality, we can take $\varepsilon_j \leq \varepsilon_{j+1} \ll 1$ and $\delta_j \leq \delta_{j+1} \ll 1$ in Step 2. Now we take $\mathcal{V}$ and $\mathcal{W}$ to be the balls in $Y_{2\sigma+2}(0, T)$ and Σ centered at $\hat{z}$ and $\hat{u}_0$ with radius ε_1 and δ_1. We define the solution of (3.7) on $[0, T]$ by

$$v(z, u_0, t) = v_{j+1}\big(z, v(z, u_0, j\tau), t\big) \quad \text{on } \big[j\tau, (j+1)\tau\big].$$

Then $v(z, u_0, \cdot)$ is the solution of (3.7) on $[0, T]$ and is continuous with respect to (z, u_0). Since (z, u_0) is arbitrary, we obtain the desired result. □

Proof of Theorem 2.1 Let $u_0 \in \Sigma$, and $T_1 > 0$ and $\bar{M} > 0$ be two constants. We choose a function $U \in \Sigma$ satisfying

$$0 < G(U) < \frac{\bar{M}}{2} \quad \text{and} \quad \frac{1}{2} \int_{\mathbb{R}^n} |\nabla U|^2 \, dx < \frac{1}{2\sigma + 2} \int_{\mathbb{R}^n} |U|^{2\sigma+2} \, dx. \tag{3.26}$$

Taking $U_{\alpha\beta}(x) = \alpha U(\beta x)$, with α and β being positive constants to be determined later, we get

$$\begin{aligned}
V(U_{\alpha\beta}) &= \alpha^2\beta^{-2-n}V(U),\\
G(U_{\alpha\beta}) &= \alpha^2\beta^{-n}G(U),\\
H(U_{\alpha\beta}) &= \alpha^2\beta^{2-n}\left[\frac{1}{2}\int_{\mathbb{R}^n}|\nabla U|^2\,dx - \frac{\theta\beta^{-4}}{2}\int_{\mathbb{R}^n}|x|^2|U|^2\,dx\right.\\
&\qquad \left. - \frac{\alpha^{2\sigma}\beta^{-2}}{2\sigma+2}\int_{\mathbb{R}^n}|U|^{2\sigma+2}\,dx\right].
\end{aligned}$$

If we take α and β satisfying $\alpha^2\beta^{-n} = \frac{\bar{M}}{2G(U)}$ and let β large enough, we then have

$$G(U_{\alpha\beta}) = \frac{\bar{M}}{2} \quad\text{and}\quad V(U_{\alpha\beta}) = \frac{\beta^{-2}\bar{M}V(U)}{2G(U)} \to 0 \quad \text{as } \beta\to\infty. \tag{3.27}$$

For $H(U_{\alpha\beta})$, since $\sigma \geq \frac{2}{n}$, by (3.26), we see that

$$\begin{aligned}
H(U_{\alpha\beta}) &= \frac{\beta^2\bar{M}}{2G(U)}\left[\frac{1}{2}\int_{\mathbb{R}^n}|\nabla U|^2\,dx - \frac{\theta\beta^{-4}}{2}\int_{\mathbb{R}^n}|x|^2|U|^2\,dx\right.\\
&\qquad \left. - \frac{\beta^{n\sigma-2}\bar{M}^\sigma}{[2G(U)]^\sigma(2\sigma+2)}\int_{\mathbb{R}^n}|U|^{2\sigma+2}\,dx\right]\\
&\leq \frac{\beta^2\bar{M}}{2G(U)}\left[\frac{1}{2}\int_{\mathbb{R}^n}|\nabla U|^2 - \frac{\theta\beta^{-4}}{2}\int_{\mathbb{R}^n}|x|^2|U|^2 - \frac{\beta^{n\sigma-2}}{2\sigma+2}\int_{\mathbb{R}^n}|U|^{2\sigma+2}\right]\\
&\to -\infty \quad \text{as } \beta\to\infty.
\end{aligned}$$

Thus, for any constant $\bar{H} > 0$, when β is large, setting $u_1 = U_{\alpha\beta}$, by (3.27) we obtain

$$V(u_1) \leq \frac{\bar{M}}{2}, \qquad G(u_1) = \frac{\bar{M}}{2} \quad\text{and}\quad H(u_1) \leq -2\bar{H}. \tag{3.28}$$

By Lemma 3.2, there exists $\hat{z}$ satisfying $A(\cdot)\hat{z} \in \mathcal{C}([0,T_1];L^2) \cap L^q((0,T_1);L^{2\sigma+2})$, $z(0) = 0$ and $\hat{z}(T_1) + v(\hat{z},u_0,T_1) = u_1$. By Lemma 3.4, there exists a ball B centered at $\hat{z}$ in $Y_{2\sigma+2}(0,T_1)$ such that for any $z \in B$, the solution $v(z,u_0,\cdot)$ of (3.7) exists and is continuous with respect to (z,u_0). Set $u = z + v(z,u_0,\cdot)$, then by (3.28),

$$V\bigl(u(T_1)\bigr) < \bar{M}, \qquad \bigl|G\bigl(u(T_1)\bigr)\bigr| < \bar{M}, \qquad H\bigl(u(T_1)\bigr) < -\bar{H}. \tag{3.29}$$

Noting that the solution of (1.1) is given by $u = z(t) + v(z,u_0,\cdot)$, where $z(t) = \int_0^t U_\theta(t-s)\,dW(s)$. By Lemma 3.1 in [13], z satisfies $A(\cdot)z \in \mathcal{C}([0,T_1];L^2) \cap L^q((0,T_1);L^{2\sigma+2})$ almost surely. Because $\ker\phi^* = \{0\}$, we see that z is non-degenerate and $\mathbb{P}(z\in B) > 0$. Now we set

$$\Omega_0 = \bigl\{\omega\in\Omega : \tau^*(u_0) \geq T_1 \text{ and } u(T_1) \text{ satisfies (3.29)}\bigr\},$$

then $\mathbb{P}(\Omega_0) > 0$. We choose $\bar{H}$ large enough such that

$$\bar{M} + 4\bar{M}T_1 - 8\bar{H}T_1^2 + c_\phi^\Sigma T_1 + 2c_\phi^2 T_1^2 + \frac{4}{3}\bigl(c_\phi^1 - \theta c_\phi^\Sigma\bigr)T_1^3 < 0 \quad \text{if } \theta < 0$$

and

$$2\bar{M}+\left(\frac{c_\phi^\Sigma+4\bar{M}}{2\mu}+\frac{c_\phi^1-\theta c_\phi^\Sigma}{4\mu^3}\right)\tanh(2\mu T_1)+\frac{c_\phi^2-4\bar{H}}{2\mu^2}\tanh^2(2\mu T_1)<0 \quad \text{if } \theta>0.$$

By Lemma 3.1 with u_0 replaced by $u(T_1)|_{\Omega_0}$, we have $\mathbb{P}(\tau^*(u_0)\le 2T_1)>0$. This completes the proof of Theorem 2.1. □

4 Conclusions

In this article, we study the mixed effect of additive noise and potential on the blow-up dynamics of solutions to a stochastic nonlinear Schrödinger equation, which could describe the evolution of the wave function of a Bose-Einstein condensation in random media. Our findings show that, regardless of the direction of the potential, any initial data with finite variance gives birth to a solution that blows up in arbitrarily small time with positive probability, which improves the result of [13].

Furthermore, our results show that, under the effect of an additive white noise, any initial data develops a solution that blows up in arbitrarily small time. This indicates that, for the stochastic Schrödinger equation, the white noise rather than the potential determines the dynamical behaviors of the solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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