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Successive iteration and positive solutions for a third-order boundary value problem involving integral conditions

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Abstract

This paper investigates the existence of concave positive solutions and establishes corresponding iterative schemes for a third-order boundary value problem with Riemann-Stieltjes integral boundary conditions. The main tool is a monotone iterative technique. Meanwhile, an example is worked out to demonstrate the main results.

Keywords: Riemann-Stieltjes integral boundary conditions; iterative; monotone positive solution; completely continuous

1 Introduction

Third-order differential equation arises from many branches of applied mathematics and physics. For example, in the deflection of a curved beam having a constant or varying cross-section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [1].

In the past ten years or so, many authors also studied some third-order differential equation boundary value problems by different types of techniques. For example, in [2], Sun applied the Krasnosel'skii's fixed point theorem to obtain the existence of positive solutions for third-order three-point boundary value problem. In [3], authors studied the uniqueness and existence results for a third-order multi-point boundary value problem by the method of upper and lower solutions.

In [4], Zhou and Ma obtained the existence of positive solutions and established a corresponding iterative scheme for the following third-order boundary value problem:

$$\begin{cases} (\Phi_p(u''))'(t) = q(t)f(t,u(t)), \quad 0 \le t \le 1, \\ u(0) = \sum_{i=1}^m \alpha_i u(\zeta_i), \qquad u'(\eta) = 0, \qquad u''(1) = \sum_{i=1}^n \beta_i u''(\theta_i), \end{cases} \tag{1.1}$$

the main tool was the monotone iterative technique.

Boundary value problems with Riemann-Stieltjes integral boundary condition have been considered recently as both multi-point and integral type boundary conditions are treated in a single framework. For more comments on the Riemann-Stieltjes integral boundary condition and its importance, we refer the reader to the papers by Webb and Infante [5–7] and other related works, such as [8, 9].

In the existing literature, very few papers have dealt with third-order differential equations with Riemann-Stieltjes integral boundary conditions. We found that Graef and Webb [10] studied the following problem:

$$\begin{cases} u'''(t) = g(t)f(t,u(t)), \quad t \in (0,1), \\ u(0) = \alpha[u], \qquad u'(p) = 0, \qquad u''(1) + \beta[u] = \lambda[u''], \end{cases} \tag{1.2}$$

where $p > \frac{1}{2}$, and $\alpha[u]$, $\beta[u]$ and $\lambda[v]$ are linear functionals on $C[0,1]$ given by a Riemann-Stieltjes integral. The existence of multiple positive solutions is obtained by the application of fixed point index theory. Since α, β and λ can include both sums and integrals, this boundary condition is more general setup than in (1.1).

In [11], Zhang and Sun investigated the existence of monotone positive solution for the following third-order nonlocal boundary value problem:

$$\begin{cases} u'''(t) + f(t,u(t),u'(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \\ au'(0) - bu''(0) = \alpha[u], \\ cu'(1) + du''(1) = \beta[u], \end{cases} \tag{1.3}$$

where $\alpha[u] = \int_0^1 u(t)\,dA(t)$ and $\beta[u] = \int_0^1 u(t)\,dB(t)$ are linear functionals on $C[0,1]$ given by Riemann-Stieltjes integrals. The main tool is monotone iterative techniques.

Inspired by [5, 10, 11], in this paper, we apply monotone iterative techniques to obtain the existence and iteration of monotone positive solutions for the following boundary value problem:

$$\begin{cases} u'''(t) = g(t)f(t,u(t),u'(t),u''(t)), \quad t \in (0,1), \\ u(0) = \alpha[u], \qquad u'(p) = 0, \qquad u''(1) + \beta[u] = \lambda[u''], \end{cases} \tag{1.4}$$

where $p > \frac{1}{2}$, $\alpha[u] = \int_0^1 u(s)\,dA(s)$, $\beta[u] = \int_0^1 u(s)\,dB(s)$, $\lambda[v] = \int_0^1 v(t)\,d\Lambda(t)$, and $\alpha[u]$, $\beta[u]$, $\lambda[v]$ are linear functionals on $C[0,1]$ given by the Riemann-Stieltjes integral, $A(t)$, $B(t)$ and $\Lambda(t)$ are suitable functions of bounded variation.

Compared with (1.1)-(1.3), the difficulty of this paper is that nonlinear term f depends on all the lower derivatives of u, which leads to complexities to prove the properties of the operator T, especially the monotonicity of the operator T. In addition, it is worth stating that the first term of our iterative scheme is a simple function or a constant function. Therefore, the iterative scheme is feasible. Under the appropriate assumptions on nonlinear term, this paper is to establish a new and general result on the existence of positive solution to boundary value problem (1.4). An example is also included to illustrate the main results.

2 Preliminaries

In this section, we give the definitions and some preliminaries.

Definition 2.1 Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided the following hypotheses are satisfied:

(i) if $u \in P$, $\lambda \geq 0$, then $\lambda u \in P$;

(ii) if $u \in P$ and $-u \in P$, then $u = 0$.

Definition 2.2 Let the Banach space $E = C^2[0,1]$, $u \in E$ is said to be concave on $[0,1]$ if

$$u\big(\lambda t_1 + (1-\lambda)t_2\big) \geq \lambda u(t_1) + (1-\lambda)u(t_2)$$

for any $t_1, t_2 \in [0,1]$ and $\lambda \in [0,1]$.

We consider the Banach space $E = C^2[0,1]$ equipped with the norm

$$\|u\| = \max\Big\{\max_{0\leq t\leq 1}\big|u(t)\big|, \max_{0\leq t\leq 1}\big|u'(t)\big|, \max_{0\leq t\leq 1}\big|u''(t)\big|\Big\}.$$

Denote

$$E_+ = C_+^2[0,1] = \big\{u \in E | u(t) \geq 0, t \in [0,1]\big\},$$

and define the cone $P \subset E$ by

$$P = \big\{u \in E | u(t) \geq 0, u \text{ is concave and } u'(p) = 0\big\}.$$

Lemma 2.1 *For the following boundary value problems,*

$$\gamma'''(t) = 0, \qquad \gamma(0) = 1, \qquad \gamma'(p) = 0, \qquad \gamma''(1) = 0 \tag{2.1}$$

and

$$\delta'''(t) = 0, \qquad \delta(0) = 0, \qquad \delta'(p) = 0, \qquad \delta''(1) = 1, \tag{2.2}$$

we have $\gamma(t) \equiv 1$, $\delta(t) = pt - t^2/2$, *for* $t \in [0,1]$.

Lemma 2.2 [10] *Suppose* $\lambda[1] \neq 1$. *For any* $y \in C[0,1]$, *the unique solution of the boundary value problem*

$$\begin{cases} u'''(t) = y(t), \\ u(0) = 0, \qquad u'(p) = 0, \qquad u''(1) = \lambda[u''] \end{cases} \tag{2.3}$$

is given by

$$u(t) = \big(tp - t^2/2\big)\int_0^1 \left(1 + \frac{\Lambda(s)}{1-\lambda[1]}\right)y(s)\,ds - t\int_0^p (p-s)y(s)\,ds + \int_0^t \frac{(t-s)^2}{2}y(s)\,ds. \tag{2.4}$$

Lemma 2.3 *Suppose that* $1/2 \leq p \leq 1$, $\Lambda(s) \geq 0$ *and* $\lambda[1] < 1$, *let* $G(t,s)$ *be the Green function*

$$G(t,s) := \big(tp - t^2/2\big)\left(1 + \frac{\Lambda(s)}{1-\lambda[1]}\right) - t(p-s)H(p-s) + \frac{(t-s)^2}{2}H(t-s) \tag{2.5}$$

for $0 \leq t \leq 1$, $0 \leq s \leq 1$, *where* $H(x-y) = \begin{cases} 0, & x < y, \\ 1, & x \geq y, \end{cases}$ *we have*

$$0 \leq G(t,s) \leq \Phi(s) := \begin{cases} \frac{p^2}{2} + \frac{p^2}{2}\left(\frac{\Lambda(s)}{1-\lambda[1]}\right) & \text{if } s \geq p, \\ \frac{s^2}{2} + \frac{p^2}{2}\left(\frac{\Lambda(s)}{1-\lambda[1]}\right) & \text{if } s < p. \end{cases}$$

Proof The upper bounds are obtained by finding $\max_{t\in[0,1]} G(t,s)$ for each fixed s. Let $Q(s) := 1 + \frac{\Lambda(s)}{1-\lambda[1]}$. Since $\Lambda(s) \geq 0$, we can get $Q(s) \geq 1$. From (2.5), we have the following formula:

$$(\partial/\partial t)G(t,s) = (p-t)Q(s) - (p-s)H(p-s) + (t-s)H(t-s).$$

We first consider fixed $s \geq p$. The derivative, which is positive for $t < p$, negative for $p < t < s$ and $p \leq s \leq t$. When $t = p$, the derivative becomes zero. Therefore, under our hypothesis, the maximum of $G(t,s)$ for this fixed s occurs when $t = p$. This gives the upper half of the expression for Φ.

When $s < p$, using the fact that $\Lambda(s) > 0$, we have, for $t > s$,

$$\begin{aligned} G(t,s) &= \bigl(tp - t^2/2\bigr)\left(1 + \frac{\Lambda(s)}{1-\lambda[1]}\right) - t(p-s) + \frac{(t-s)^2}{2} \\ &= \bigl(tp - t^2/2\bigr)\frac{\Lambda(s)}{1-\lambda[1]} + \frac{s^2}{2} \leq \frac{p^2}{2}\frac{\Lambda(s)}{1-\lambda[1]} + \frac{s^2}{2}. \end{aligned}$$

When $s < p$ and $t \leq s$, we have

$$\begin{aligned} G(t,s) &= \bigl(tp - t^2/2\bigr)\left(1 + \frac{\Lambda(s)}{1-\lambda[1]}\right) - t(p-s) \\ &= \bigl(ts - t^2/2\bigr) + \bigl(tp - t^2/2\bigr)\frac{\Lambda(s)}{1-\lambda[1]} \leq \frac{s^2}{2} + \frac{p^2}{2}\frac{\Lambda(s)}{1-\lambda[1]}. \end{aligned}$$

Since $Q(s) > 1$ and $p - \frac{t}{2} \geq 0$, for $0 \leq t,s \leq 1$, we can easily obtain

$$0 \leq G(t,s) = \begin{cases} t(p-\frac{t}{2})Q(s) + \frac{(t-2)^2}{2}, & p \leq s < t, \\ t(p-\frac{t}{2})Q(s), & p \leq s, t < s, \\ t(p-\frac{t}{2})(\frac{\Lambda(s)}{1-\lambda[1]}) + \frac{s^2}{2}, & s < p, s < t, \\ t(p-\frac{t}{2})(\frac{\Lambda(s)}{1-\lambda[1]}) + t(s-\frac{t}{2}), & t < s < p. \end{cases}$$ □

We always suppose that the following assumptions hold:

(H_1) $g \in L^1[0,1]$, $g \geq 0$, and $\int_0^1 g(s)\,ds > 0$;
(H_2) A, B are of bounded variation and $g_A(s), g_B(s) \geq 0$ for a.e. s, where

$$g_A(s) := \int_0^1 G(t,s)\,dA(t) \quad \text{and} \quad g_B(s) := \int_0^1 G(t,s)\,dB(t);$$

(H_3) $\gamma \in C[0,1]$, $\gamma(t) \geq 0$, $0 \leq \alpha[\gamma] < 1$, $\beta[\gamma] \geq 0$;
(H_4) $\delta \in C[0,1]$, $\delta(t) \geq 0$, $0 \leq \beta[\delta] < 1$, $\alpha[\delta] \geq 0$;
(H_5) $D := (1-\alpha[\gamma])(1-\beta[\delta]) - \alpha[\delta]\beta[\gamma] > 0$.

Lemma 2.4 [10] *For any $y \in C[0,1]$, suppose that u is a solution of the following boundary value problem*:

$$\begin{cases} u'''(t) = y(t), \quad t \in (0,1), \\ u(0) = \alpha[u], \qquad u'(p) = 0, \qquad u''(1) + \beta[u] = \lambda[u''], \end{cases} \tag{2.6}$$

then we have

$$\begin{aligned} u(t) &= \frac{\gamma(t)}{D}\left[\left(1-\beta[\delta]\right)\int_0^1 g_A(s)y(s)\,ds + \alpha[\delta]\int_0^1 g_B(s)y(s)\,ds\right] \\ &\quad + \frac{\delta(t)}{D}\left[\beta[\gamma]\int_0^1 g_A(s)y(s)\,ds + \left(1-\alpha[\gamma]\right)\int_0^1 g_B(s)y(s)\,ds\right] \\ &\quad + \int_0^1 G(t,s)y(s)\,ds. \end{aligned} \tag{2.7}$$

For any $u \in E_+$, $T: P \to E$ is defined

$$(Tu)(t) = \gamma(t)\alpha[gf_u] + \delta(t)\beta[gf_u] + \int_0^1 G(t,s)g(s)f_u(s)\,ds, \tag{2.8}$$

where $f_u(s) = f(s, u(s), u'(s), u''(s))$, and

$$\begin{aligned} \alpha[gf_u] &= \frac{1}{D}\left[\left(1-\beta[\delta]\right)\int_0^1 g_A(s)g(s)f_u(s)\,ds + \alpha[\delta]\int_0^1 g_B(s)g(s)f_u(s)\,ds\right], \\ \beta[gf_u] &= \frac{1}{D}\left[\beta[\gamma]\int_0^1 g_A(s)g(s)f_u(s)\,ds + \left(1-\alpha[\gamma]\right)\int_0^1 g_B(s)g(s)f_u(s)\,ds\right]. \end{aligned} \tag{2.9}$$

Combining the expression of $\alpha[gf_u]$, $\beta[gf_u]$ with (H_1)-(H_5), we can obtain that

$$\alpha[gf_u], \beta[gf_u] \geq 0.$$

From the definition of T, it is obvious that

$$(Tu)''(t) = \begin{cases} -\beta[gf_u] + \int_0^1(-Q(s))g(s)f_u(s)\,ds, & t < s, \\ -\beta[gf_u] + \int_0^1(1-Q(s))g(s)f_u(s)\,ds, & t > s, \end{cases}$$

$$(Tu)'(t) = \begin{cases} (p-t)\beta[gf_u] + \int_0^1(p-t)Q(s)g(s)f_u(s)\,ds, & t < p \leq s \text{ or } p \leq t \leq s, \\ (p-t)\beta[gf_u] + \int_0^1[(p-t)Q(s) + (s-p)]g(s)f_u(s)\,ds, & t \leq s < p, \\ (p-t)\beta[gf_u] + \int_0^1[(p-t)Q(s) + (t-s)]g(s)f_u(s)\,ds, & p \leq s \leq t, \\ (p-t)\beta[gf_u] + \int_0^1[(p-t)Q(s) + (t-p)]g(s)f_u(s)\,ds, & s \leq t < p \text{ or } s < p \leq t. \end{cases}$$

Lemma 2.5 *If* (H_1)-(H_5) *are satisfied*, $T: P \to P$ *is completely continuous.*

Proof Since $(Tu)''(t) \leq 0$, Tu is concave. From $(Tu)'(t) \geq 0$ on $[0,p]$ and $(Tu)'(t) \leq 0$ on $[p,1]$, we obtain that $(Tu)(t)$ is nondecreasing on $[0,p]$ and nonincreasing on $[p,1]$. Moreover,

$$(Tu)(0) = \gamma(0)\alpha[gf_u] + \delta(0)\beta(gf_u) + \int_0^1 G(0,s)g(s)f_u(s)\,ds \geq 0,$$

then

$$(Tu)(1) = \gamma(1)\alpha[gf_u] + \delta(1)\beta[gf_u] + \int_0^1 G(1,s)g(s)f_u(s)\,ds \geq 0.$$

So, by the concavity of T, then $(Tu)(t) \geq 0$, $0 \leq t \leq 1$. Hence, $T: P \to P$.

In what follows, we will prove that $T : P \to P$ is completely continuous. The continuity of T is obvious. Now, we prove that T is compact. Let $\Omega \subset P$ be a bounded set. It is easy to prove that $T(\Omega)$ is bounded and equicontinuous. Then the Arzelà-Ascoli theorem guarantees that $T(\Omega)$ is relatively compact, which means T is compact. □

3 Main results

For notational convenience, we denote

$$\begin{aligned} &L_A = \int_0^1 g_A(s)g(s)\,ds, \qquad L_B = \int_0^1 g_B(s)g(s)\,ds, \qquad L_C = \int_0^1 Q(s)g(s)\,ds, \\ &L_D = \int_0^1 g(s)\,ds, \qquad L_E = \int_0^1 sg(s)\,ds, \qquad L_F = \int_0^1 \bigl[Q(s)-1\bigr]g(s)\,ds, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &L_G = \int_0^1 \Phi(s)g(s)\,ds, \qquad L_H = \frac{(1-\beta[\delta])L_A + \alpha[\delta]L_B}{D}, \\ &L_I = \frac{\beta[\gamma]L_A + (1-\alpha[\gamma])L_B}{D}. \end{aligned} \tag{3.2}$$

Remark From (H_1)-(H_5), we can easily get that the signs of (3.1) and (3.2) are nonnegative. If $g_A(s)$ is a piecewise function related to t for $t \in [0,1]$, L_A is an expression of t. In this case, we denote $L_A = \max_{0\le t\le 1}\{\int_0^1 g_A(s)g(s)\,ds\}$. The same condition is satisfied with $g_B(s)$ and L_B.

We will prove the following existence result.

Theorem 3.1 *Assume that* (H_1)-(H_5) *hold, if there are two positive numbers* a_1, a *satisfying* $a = \max\{L_H + L_G + \frac{p^2}{2}(L_I + L_C + L_D), L_I + L_C + L_D\}a_1$, *and*

(S1) $f(t,u_1,v_1,w_1) \le f(t,u_2,v_2,w_2)$ *for* $0 \le t \le 1$, $0 \le u_1 \le u_2 \le a$, $0 \le |v_1| \le |v_2| \le a$, $-a \le w_2 \le w_1 \le 0$;

(S2) $\max_{0\le t\le 1} f(t,a,a,-a) \le a_1$;

(S3) $f(t,0,0,0) \not\equiv 0$ *for* $0 \le t \le 1$.

Then the boundary value problem (1.4) *has positive, nondecreasing on* $[0,p]$ *nonincreasing on* $[p,1]$ *and concave solutions* ω^*, ν^* *such that*

$$\begin{aligned} &0 < \omega^* \le a, \qquad 0 \le \bigl|(\omega^*)'\bigr| \le a, \qquad -a \le (\omega^*)'' \le 0 \quad \textit{and} \\ &\lim_{n\to\infty} \omega_n = \lim_{n\to\infty} T^n\omega_0 = \omega^*, \qquad \lim_{n\to\infty} (\omega_n)' = \lim_{n\to\infty} \bigl(T^n\omega_0\bigr)' = \bigl(\omega^*\bigr)', \\ &\lim_{n\to\infty} (\omega_n)'' = \lim_{n\to\infty} \bigl(T^n\omega_0\bigr)'' = \bigl(\omega^*\bigr)'', \end{aligned}$$

where $\omega_0(t) = a_1(L_H + L_G) + a_1 t(p - \frac{t}{2})(L_I + L_C + L_D)$, *and*

$$\begin{aligned} &0 < \nu^* \le a, \qquad 0 \le \bigl|(\nu^*)'\bigr| \le a, \qquad -a \le (\nu^*)'' \le 0 \quad \textit{and} \\ &\lim_{n\to\infty} \nu_n = \lim_{n\to\infty} T^n\nu_0 = \nu^*, \qquad \lim_{n\to\infty} (\nu_n)' = \lim_{n\to\infty} \bigl(T^n\nu_0\bigr)' = \bigl(\nu^*\bigr)', \\ &\lim_{n\to\infty} (\nu_n)'' = \lim_{n\to\infty} \bigl(T^n\nu_0\bigr)'' = \bigl(\nu^*\bigr)'', \end{aligned}$$

where $\nu_0(t) = 0$.

Proof We denote

$$\bar{P}_a = \bigl\{u \in P \mid \|u\| \le a\bigr\}.$$

Then, in what follows, we first prove that $T:\bar{P}_a \to \bar{P}_a$. If $u \in \bar{P}_a$, then $\|u\| \le a$, we have

$$\begin{aligned}
&0 \le u(t) \le u(p) = \max_{0\le t\le 1}\bigl|u(t)\bigr| \le \|u\| \le a, \\
&0 \le \max_{0\le t\le 1}\bigl|u'(t)\bigr| \le \|u\| \le a, \\
&-a \le -\|u\| \le -\max_{0\le t\le 1}\bigl|u''(t)\bigr| \le u''(t) \le 0.
\end{aligned}$$

So by (S1), (S2) we have

$$0 \le f\bigl(t,u(t),u'(t),u''(t)\bigr) \le \max_{0\le t\le 1} f(t,a,a,-a) \le a_1 \quad \text{for } 0 \le t \le 1.$$

In fact,

$$\begin{aligned}
\|Tu\| &= \max\Bigl\{\max_{0\le t\le 1}\bigl|(Tu)(t)\bigr|, \max_{0\le t\le 1}\bigl|(Tu)'(t)\bigr|, \max_{0\le t\le 1}\bigl|(Tu)''(t)\bigr|\Bigr\} \\
&= \max\bigl\{(Tu)(p), (Tu)'(0), -(Tu)'(1), -(Tu)''(0)\bigr\}.
\end{aligned} \tag{3.3}$$

By (2.9), (3.1) and (S2), we have

$$\begin{aligned}
\alpha[gf_u] &= \frac{1}{D}\biggl[\bigl(1-\beta[\delta]\bigr)\int_0^1 g_A(s)g(s)f_u(s)\,ds + \alpha[\delta]\int_0^1 g_B(s)g(s)f_u(s)\,ds\biggr] \\
&\le \frac{a_1}{D}\bigl[\bigl(1-\beta[\delta]\bigr)L_A + \alpha[\delta]L_B\bigr] = a_1 L_H, \qquad (3.4)\\
\beta[gf_u] &= \frac{1}{D}\biggl[\beta[\gamma]\int_0^1 g_A(s)g(s)f_u(s)\,ds + \bigl(1-\alpha[\gamma]\bigr)\int_0^1 g_B(s)g(s)f_u(s)\,ds\biggr] \\
&\le \frac{a_1}{D}\bigl[\beta[\gamma]L_A + \bigl(1-\alpha[\gamma]\bigr)L_B\bigr] = a_1 L_I. \qquad (3.5)
\end{aligned}$$

From (3.4), (3.5) and Lemma 2.3, we have

$$\begin{aligned}
(Tu)(p) &= \max_{t\in[0,1]}(Tu)(t) = \alpha[gf_u] + \frac{p^2}{2}\beta[gf_u] + \int_0^1 G(p,s)g(s)f_u(s)\,ds \\
&\le a_1L_H + \frac{p^2}{2}a_1L_I + a_1\int_0^1 \Phi(s)g(s)\,ds \\
&\le a_1L_H + \frac{p^2}{2}a_1L_I + a_1L_G \le a, \\
-(Tu)'(1) &= \begin{cases} -[(p-1)\beta[gf_u] + (p-1)\int_0^1(Q(s)-1)g(s)f_u(s)\,ds], & p > s, \\ -[(p-1)\beta[gf_u] + \int_0^1[(p-1)Q(s)+1-s]g(s)f_u(s)\,ds], & p \le s \end{cases} \\
&\le (1-p)a_1L_I + (1-p)a_1L_C \le a_1(L_I+L_C) \le a, \\
(Tu)'(0) &= \begin{cases} p\beta[gf_u] + \int_0^1[pQ(s)+s-p]g(s)f_u(s)\,ds, & p > s, \\ p\beta[gf_u] + \int_0^1 pQ(s)g(s)f_u(s)\,ds, & p \le s \end{cases} \\
&\le pa_1(L_I+L_C) \le a, \\
-(Tu)''(0) &= \beta[gf_u] + \int_0^1 Q(s)g(s)f_u(s)\,ds \le a_1(L_I+L_C) \le a.
\end{aligned}$$

Thus, we obtain that

$$\|Tu\| = \max\bigl\{(Tu)(p), (Tu)'(0), -(Tu)'(1), -(Tu)''(0)\bigr\} \le a.$$

Hence, we assert that $T : \bar{P}_a \to \bar{P}_a$.

Let $\omega_0 = a_1 L_H + a_1 L_G + a_1 t(p - \frac{t}{2})(L_I + L_C + L_D)$, for $0 \le t \le 1$, then $\omega_0(t) \in \bar{P}_a$. Let $\omega_1 = T\omega_0$, $\omega_2 = T^2\omega_0$, then $\omega_1 \in \bar{P}_a$ and $\omega_2 \in \bar{P}_a$. We denote $\omega_{n+1} = T\omega_n = T^n\omega_0$, $n = 0, 1, 2, \ldots$. Since $T : \bar{P}_a \to \bar{P}_a$, we have $\omega_n \in T\bar{P}_a \subseteq \bar{P}_a$, $n = 0, 1, 2, \ldots$. Since T is completely continuous, we assert that $\{\omega_n\}_{n=1}^{\infty}$ is a sequentially compact set. We have

$$\begin{aligned}
\omega_1(t) &= T\omega_0(t) \\
&= \alpha[gf_{\omega_0}] + t\left(p - \frac{t}{2}\right)\beta[gf_{\omega_0}] + \int_0^1 G(t,s)g(s)f_{\omega_0}(s)\,ds \\
&\le a_1 L_H + t\left(p - \frac{t}{2}\right)a_1 L_I + \int_0^1 \Phi(s)g(s)f_{\omega_0}(s)\,ds \\
&\le a_1 L_H + t\left(p - \frac{t}{2}\right)a_1 L_I + a_1 L_G \\
&\le a_1 L_H + a_1 L_G + a_1 t\left(p - \frac{t}{2}\right)(L_I + L_C + L_D) \\
&= \omega_0(t), \quad 0 \le t \le 1, \\
\bigl|\omega_1'(t)\bigr| &= \bigl|(T\omega_0)'(t)\bigr| \\
&\le \begin{cases} |p-t|\beta[gf_{\omega_0}] + \int_0^1 |p-t|Q(s)g(s)f_{\omega_0}(s)\,ds, & t < p \le s \text{ or } p \le t < s, \\ |p-t|\beta[gf_{\omega_0}] + \int_0^1 |(p-t)Q(s) - (p-s)|g(s)f_{\omega_0}(s)\,ds, & t < s < p, \\ |p-t|\beta[gf_{\omega_0}] + \int_0^1 |(p-t)Q(s) + (t-s)|g(s)f_{\omega_0}(s)\,ds, & p \le s \le t, \\ |p-t|\beta[gf_{\omega_0}] + \int_0^1 |(p-t)Q(s) - (p-t)|g(s)f_{\omega_0}(s)\,ds, & s \le t < p \text{ or } s < p \le t \end{cases} \\
&\le \begin{cases} |p-t|[a_1 L_I + a_1 L_C], & t < s, \\ |p-t|[a_1 L_I + a_1 L_C + a_1 L_D], & t \ge s \end{cases} \\
&\le a_1|p-t|(L_I + L_C + L_D) = \bigl|\omega_0'(t)\bigr|, \quad 0 \le t \le 1, \\
\omega_1''(t) &= (T\omega_0)''(t) \\
&= \begin{cases} -\beta[gf_{\omega_0}] + \int_0^1 (-Q(s))g(s)f_{\omega_0}(s)\,ds, & t < s, \\ -\beta[gf_{\omega_0}] + \int_0^1 (1 - Q(s))g(s)f_{\omega_0}(s)\,ds, & t \ge s \end{cases} \\
&\ge \begin{cases} -[a_1 L_I + a_1 L_C], & t < s, \\ -[a_1 L_I + a_1 L_F], & t \ge s \end{cases} \\
&\ge -a_1(L_I + L_C + L_D) = \omega_0''(t), \quad 0 \le t \le 1.
\end{aligned}$$

So

$$\begin{aligned}
&\omega_2(t) = T\omega_1(t) \le T\omega_0(t) = \omega_1(t), \quad 0 \le t \le 1, \\
&\bigl|\omega_2'(t)\bigr| = \bigl|(T\omega_1)'(t)\bigr| \le \bigl|(T\omega_0)'(t)\bigr| = \bigl|\omega_1'(t)\bigr|, \quad 0 \le t \le 1, \\
&\omega_2''(t) = (T\omega_1)''(t) \ge (T\omega_0)''(t) = \omega_1''(t), \quad 0 \le t \le 1.
\end{aligned}$$

Hence, by induction, we have

$$\omega_{n+1} \leq \omega_n, \qquad \left|\omega'_{n+1}(t)\right| \leq \left|\omega'_n(t)\right|, \qquad \omega''_{n+1}(t) \geq \omega''_n(t), \quad 0 \leq t \leq 1, n = 0,1,2,\ldots.$$

Thus, we assert that $\omega_n \to \omega^*$, we get $T\omega^* = \omega^*$ since T is continuous and $\omega_{n+1} = T\omega_n$.

Let $\nu_0 = 0$, $0 \leq t \leq 1$, then $\nu_0(t) \in \bar{P}_a$. Let $\nu_1 = T\nu_0$, $\nu_2 = T^2\nu_0$, then $\nu_1 \in \bar{P}_a$ and $\nu_2 \in \bar{P}_a$. We denote $\nu_{n+1} = T\nu_n = T^n\nu_0$, $n = 0,1,2,\ldots$. Since $T : \bar{P}_a \to \bar{P}_a$, we have $\nu_n \in T\bar{P}_a \subseteq \bar{P}_a$, $n = 0,1,2,\ldots$. Since T is completely continuous, we assert that $\{\nu_n\}_{n=1}^{\infty}$ is a sequentially compact set. We have

$$\begin{aligned}
&\nu_1(t) = T\nu_0(t) = T\nu_0(t) \geq 0, \quad 0 \leq t \leq 1, \\
&\left|\nu'_1(t)\right| = \left|(T\nu_0)'(t)\right| = \left|(T\nu_0)'(t)\right| \geq 0, \quad 0 \leq t \leq 1, \\
&\nu''_1(t) = (T\nu_0)''(t) = (T\nu_0)''(t) \leq 0, \quad 0 \leq t \leq 1.
\end{aligned}$$

So

$$\begin{aligned}
&\nu_2(t) = T\nu_1(t) \geq T\nu_0(t) = \nu_1(t), \quad 0 \leq t \leq 1, \\
&\left|\nu'_2(t)\right| = \left|(T\nu_1)'(t)\right| \geq \left|(T\nu_0)'(t)\right| = \left|\nu'_1(t)\right|, \quad 0 \leq t \leq 1, \\
&\nu''_2(t) = (T\nu_1)''(t) \leq (T\nu_0)''(t) = \nu''_1(t), \quad 0 \leq t \leq 1.
\end{aligned}$$

Hence, by induction, we have

$$\nu_{n+1} \geq \nu_n, \qquad \left|\nu'_{n+1}(t)\right| \geq \left|\nu'_n(t)\right|, \qquad \nu''_{n+1}(t) \leq \nu''_n(t), \quad 0 \leq t \leq 1, n = 0,1,2,\ldots.$$

Thus, we assert that $\nu_n \to \nu^*$, we get $T\nu^* = \nu^*$ since T is continuous and $\nu_{n+1} = T\nu_n$.

It is well known that the fixed point of the operator T is the solution of boundary value problem (1.4). Therefore, ω^* and ν^* are positive nondecreasing on $[0,p]$, nonincreasing on $[p,1]$ and concave solutions of problem (1.4). □

Example Let $p = \frac{2}{3}$ and $g(s) = 1$, we consider the following boundary value problem:

$$\begin{cases} u'''(t) = \frac{tu}{70} + \frac{1}{540}u'^2 + e^{-u''}, \quad t \in (0,1), \\ u(0) = \alpha[u], \qquad u'(\frac{2}{3}) = 0, \qquad u''(1) + \beta[u] = \lambda[u''], \end{cases} \tag{3.6}$$

where $\alpha[u] = \int_0^1 (1-s)u(s)\,ds$ and $\beta[u] = \int_0^1 su(s)\,ds$ are nonlocal boundary conditions of integral type. For these boundary conditions, we have $\gamma(t) = 1$ and $\delta(t) = \frac{2}{3}t - \frac{t^2}{2}$ corresponding to Lemma 2.1. A simple calculation shows that

$$\begin{aligned}
&\alpha[\gamma] = \frac{1}{2}, \qquad \beta[\gamma] = \frac{1}{2}, \qquad \alpha[\delta] = \frac{5}{72}, \qquad \beta[\delta] = \frac{7}{72}, \\
&D = \left(1 - \alpha[\gamma]\right)\left(1 - \beta[\delta]\right) - \alpha[\delta]\beta[\gamma] = \frac{5}{12}, \\
&L_A = \frac{355}{1{,}296}, \qquad L_B = \frac{409}{1{,}296}, \qquad L_C = \frac{7}{2}, \qquad L_D = 1, \qquad L_E = \frac{1}{2}, \\
&L_F = \frac{5}{2}, \qquad L_G = \frac{7}{9}, \qquad L_H = \frac{157}{243}, \qquad L_I = \frac{191}{270},
\end{aligned}$$

and $a = \max\{L_H + L_G + \frac{p^2}{2}(L_I + L_C + L_D), L_I + L_C + L_D\}a_1 \approx 5.2a_1$. Then all the hypotheses of Theorem 3.1 are fulfilled with $a = 52$ and $a_1 = 10$. It follows from Theorem 3.1 that the boundary value problem (3.6) has two monotone positive solutions ω^* and ν^* such that

$$\begin{aligned}
&0 < \omega^* \leq 52, \qquad 0 \leq \left|\left(\omega^*\right)'\right| \leq 52, \qquad -52 \leq \left(\omega^*\right)'' \leq 0 \quad \text{and} \\
&\lim_{n\to\infty} \omega_n = \lim_{n\to\infty} T^n\omega_0 = \omega^*, \qquad \lim_{n\to\infty} (\omega_n)' = \lim_{n\to\infty} \left(T^n\omega_0\right)' = \left(\omega^*\right)', \\
&\lim_{n\to\infty} (\omega_n)'' = \lim_{n\to\infty} \left(T^n\omega_0\right)'' = \left(\omega^*\right)'',
\end{aligned}$$

where $\omega_0(t) = -\frac{433}{27}t^2 + \frac{1{,}732}{81}t + \frac{3{,}460}{243}$, and

$$\begin{aligned}
&0 < \nu^* \leq 66, \qquad 0 \leq \left|\left(\nu^*\right)'\right| \leq 66, \qquad -66 \leq \left(\nu^*\right)'' \leq 0 \quad \text{and} \\
&\lim_{n\to\infty} \nu_n = \lim_{n\to\infty} T^n\nu_0 = \nu^*, \qquad \lim_{n\to\infty} (\nu_n)' = \lim_{n\to\infty} \left(T^n\nu_0\right)' = \left(\nu^*\right)', \\
&\lim_{n\to\infty} (\nu_n)'' = \lim_{n\to\infty} \left(T^n\nu_0\right)'' = \left(\nu^*\right)'',
\end{aligned}$$

where $\nu_0(t) = 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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