Jin and Liu *Boundary Value Problems* (2015) 2015:144
DOI 10.1186/s13661-015-0408-3

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

On the periodic boundary value problem for Duffing type fractional differential equation with p-Laplacian operator

Hua Jin* and Wenbin Liu

*Correspondence:
jinhua197927@163.com
Department of Mathematics, China University of Mining and Technology, Xuzhou, 221116, PR China

Abstract
By using the continuation theorem of coincidence degree theory, we study the existence of solutions of Duffing type fractional differential equations with a p-Laplacian operator. Under certain nonlinear growth conditions of the nonlinearity, we obtain a new result on the existence of solutions.

MSC: 34A08; 34B15

Keywords: continuation theory; fractional differential equation; p-Laplacian operator; periodic boundary conditions

1 Introduction

Fractional calculus is a generalization of ordinary differentiation and integration on an arbitrary order that may be noninteger. Fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. We can see numerous applications in viscoelasticity, neurons, electrochemistry, control, etc. (see [1–6]). Recently, with the intensive development of the theory of fractional calculus itself and its applications, there have many important results of fractional differential equations on initial value problems, and boundary value problems at nonresonance and resonance (see [7–12]).

In the study of the turbulent flow in a porous medium, Leibenson (see [13]) introduced the p-Laplacian equation as follows:

$$\left(\phi_p\left(x'(t)\right)\right)' = f\left(t, x(t), x'(t)\right), \tag{1.1}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q > 1$ is a constant such that $1/p + 1/q = 1$. In the past few decades, many important results relative to (1.1) with certain boundary value conditions have been obtained. See the papers [14–20] and the references therein. However, as far as we know, work on the existence of solutions for periodic boundary value problems (PBVPs for short) of fractional differential equations was discussed less.

The aim of this paper is to concentrate on the following periodic boundary value problem for Duffing type fractional differential equations with p-Laplacian operator:

$$\begin{cases} D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}x(t))) + g(t,x(t)) = e(t), & t \in [0,T], \\ x(0) = x(T), & D_{0^+}^{\alpha}x(0) = D_{0^+}^{\alpha}x(T), \end{cases} \tag{1.2}$$

where $0 < \alpha, \beta \leq 1$, $D_{0^+}^{\alpha}, D_{0^+}^{\beta}$ are Caputo fractional derivatives, $T > 0$ is a given constant, and $g : [0,T] \times \mathbb{R} \to \mathbb{R}$, $e : [0,T] \to \mathbb{R}$ are continuous. Throughout this paper, we assume that

$$\int_0^T (T-s)^{\beta-1} e(s)\,ds = 0.$$

The choice of periodic boundary conditions is motivated by the difficulty in the study of the PBVP

$$\begin{cases} D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}x(t))) = h(t), & t \in [0,T], \\ x(0) = x(T), & D_{0^+}^{\alpha}x(0) = D_{0^+}^{\alpha}x(T). \end{cases} \tag{1.3}$$

As we know, PBVP (1.3) is not solvable for each $h \in C([0,T],\mathbb{R})$, and, when solvable, has no unique solution because $x(t) + c$, $\forall c \in \mathbb{R}$ is a solution together with $x(t)$. A trivial necessary condition for the solvability of PBVP (1.3) is that $\int_0^T (T-s)^{\beta-1} h(s)\,ds = 0$.

Notice that $D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}))$ is a nonlinear operator, so the coincidence degree theory for linear differential operators is invalid in the direct application to it.

The rest of this paper is organized as follows. In Section 2, we describe the fractional differential operator and some lemmas. In Section 3, some sufficient conditions for the existence of solutions for PBVP (1.2) are established, and a new result on the existence of solutions is obtained. Finally, in Section 4, an example is given to illustrate the main result.

2 Preliminaries

Some definitions of the fractional derivative have emerged over the years (see [21, 22]), and in this paper we restrict our attention to the use of the Caputo fractional derivative. In this section, we introduce some basic definitions and lemmas which will be used in what follows. For details, we refer the reader to [21–25].

Definition 2.1 The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $u : (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,ds,$$

provided that the right side integral is pointwise defined on $(0,+\infty)$, where $\Gamma(\cdot) > 0$ is the Gamma function.

Definition 2.2 The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u : (0,+\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha}u(t) = I_{0^+}^{n-\alpha}\frac{d^n u(t)}{dt^n} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\,ds,$$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0, +\infty)$.

Lemma 2.1 (see [23]) *Let $\alpha > 0$. Assume that u, $D_{0^+}^{\alpha} u \in L([0,T], \mathbb{R})$. Then the following equality holds*:

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, here n is the smallest integer greater than or equal to α.

Lemma 2.2 (see [24]) *For any $u, v \geq 0$,*

$$\phi_p(u+v) \leq \begin{cases} \phi_p(u) + \phi_p(v), & \text{if } p < 2; \\ 2^{p-2}(\phi_p(u) + \phi_p(v)), & \text{if } p \geq 2. \end{cases}$$

Now we briefly recall some notations and an abstract existence result, which can be found in [25].

Let X, Y be real Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator with index zero, and $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L,$$
$$X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible. We denote the inverse by K_P.

If Ω is an open bounded subset of X such that $\operatorname{dom} L \cap \bar{\Omega} \neq \emptyset$, then the map $N : X \to Y$ will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact.

Lemma 2.3 (see [25]) *Let X and Y be two Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator with index zero, $\Omega \subset X$ be an open bounded set, and $N : \bar{\Omega} \to Y$ be L-compact on $\bar{\Omega}$. Suppose that all of the following conditions hold*:

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap \operatorname{dom} L$, $\lambda \in (0,1)$;

(2) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, *where $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism map.*

Then the equation $Lx = Nx$ has at least one solution on $\bar{\Omega} \cap \operatorname{dom} L$.

3 Existence result

For making use of the continuation theorem to study the existence of solutions for PBVP (1.2), we consider a system as follows:

$$\begin{cases} D_{0^+}^{\alpha} x_1(t) = \phi_q(x_2(t)), \\ D_{0^+}^{\beta} x_2(t) = e(t) - g(t, x_1(t)), \\ x_1(0) = x_1(T), \qquad x_2(0) = x_2(T). \end{cases} \tag{3.1}$$

Clearly, if $x(\cdot) = (x_1(\cdot), x_2(\cdot))^{\mathrm{T}}$ is a solution of PBVP (3.1), then $x_1(\cdot)$ must be a solution of PBVP (1.2). So, to prove PBVP (1.2) has solutions, we only need to show that PBVP (3.1) has solutions.

In this paper, we take $X = \{x = (x_1, x_2)^{\mathrm{T}} \mid x_1, x_2 \in C([0,T], \mathbb{R})\}$ with the norm $\|x\| = \max\{\|x_1\|_0, \|x_2\|_0\}$, where $\|x_i\|_0 = \max_{t\in[0,T]} |x_i(t)|$ ($i \in \{1,2\}$). By means of the linear functional analysis theory, we can prove X is a Banach space.

Define the operator $L : \operatorname{dom} L \subset X \to X$ by

$$Lx = \begin{pmatrix} D_{0^+}^{\alpha} x_1 \\ D_{0^+}^{\beta} x_2 \end{pmatrix}, \tag{3.2}$$

where

$$\begin{aligned}\operatorname{dom} L = \bigl\{ & x \in X \mid D_{0^+}^{\alpha} x_1, D_{0^+}^{\beta} x_2 \in C\bigl([0,T], \mathbb{R}\bigr), \\ & x_1(0) = x_1(T), x_2(0) = x_2(T)\bigr\}.\end{aligned}$$

Let $N : X \to X$ be defined by

$$Nx(t) = \begin{pmatrix} \phi_q(x_2(t)) \\ e(t) - g(t, x_1(t)) \end{pmatrix}, \quad \forall t \in [0,T]. \tag{3.3}$$

It is easy to see that PBVP (3.1) can be converted to the operator equation

$$Lx = Nx, \quad x \in \operatorname{dom} L.$$

Now let us introduce some lemmas.

Lemma 3.1 *Let L be defined by* (3.2), *then*

$$\operatorname{Ker} L = \bigl\{ x \in X \mid x(t) = c, \forall t \in [0,T], c \in \mathbb{R}^2 \bigr\}, \tag{3.4}$$

$$\operatorname{Im} L = \left\{ y \in X \;\middle|\; \begin{pmatrix} \int_0^T (T-s)^{\alpha-1} y_1(s)\,ds \\ \int_0^T (T-s)^{\beta-1} y_2(s)\,ds \end{pmatrix} = 0 \right\}. \tag{3.5}$$

Proof Obviously, from Lemma 2.1, we can see that (3.4) holds.

If $y \in \operatorname{Im} L$, then there exists $x \in \operatorname{dom} L$ such that $y = Lx$. That is, $y_1(t) = D_{0^+}^{\alpha} x_1(t)$, $y_2(t) = D_{0^+}^{\beta} x_2(t)$. By using Lemma 2.1, we have

$$\begin{aligned} x_1(t) &= c_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_1(s)\,ds, \quad c_1 \in \mathbb{R}, \\ x_2(t) &= c_2 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y_2(s)\,ds, \quad c_2 \in \mathbb{R}. \end{aligned}$$

From conditions $x_1(0) = x_1(T)$, $x_2(0) = x_2(T)$, we obtain

$$\begin{pmatrix} \int_0^T (T-s)^{\alpha-1} y_1(s)\,ds \\ \int_0^T (T-s)^{\beta-1} y_2(s)\,ds \end{pmatrix} = 0. \tag{3.6}$$

On the other hand, suppose $y \in X$ and satisfies (3.6). Let $x_1(t) = I_{0^+}^{\alpha} y_1(t)$, $x_2(t) = I_{0^+}^{\beta} y_2(t)$. Obviously $x_1(0) = x_1(T)$, $x_2(0) = x_2(T)$. Hence $x = (x_1, x_2)^{\mathrm{T}} \in \operatorname{dom} L$ and $Lx = y$. So $y \in \operatorname{Im} L$. The proof is complete. □

Lemma 3.2 *Let L be defined by* (3.2), *then L is a Fredholm operator of index zero. The projectors $P : X \to X$ and $Q : X \to X$ can be defined as*

$$Px(t) = x(0), \quad \forall t \in [0, T],$$

$$Qy(t) = \begin{pmatrix} \frac{\alpha}{T^{\alpha}} \int_0^T (T-s)^{\alpha-1} y_1(s)\,ds \\ \frac{\beta}{T^{\beta}} \int_0^T (T-s)^{\beta-1} y_2(s)\,ds \end{pmatrix} := \begin{pmatrix} (Qy)_1 \\ (Qy)_2 \end{pmatrix}, \quad \forall t \in [0, T].$$

Furthermore, the operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be written as

$$K_P y = \begin{pmatrix} I_{0^+}^{\alpha} y_1 \\ I_{0^+}^{\beta} y_2 \end{pmatrix},$$

which is $(L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$.

Proof For any $y \in X$, we have

$$Q^2 y = Q \begin{pmatrix} (Qy)_1 \\ (Qy)_2 \end{pmatrix} = \begin{pmatrix} (Qy)_1 \frac{\alpha}{T^{\alpha}} \int_0^T (T-s)^{\alpha-1}\,ds \\ (Qy)_2 \frac{\beta}{T^{\beta}} \int_0^T (T-s)^{\beta-1}\,ds \end{pmatrix} = Qy.$$

Let $y^* = y - Qy = \binom{y_1^*}{y_2^*}$, then we get

$$\begin{aligned} \int_0^T (T-s)^{\alpha-1} y_1^*(s)\,ds &= \int_0^T (T-s)^{\alpha-1} y_1(s)\,ds - \int_0^T (T-s)^{\alpha-1} (Qy)_1\,ds \\ &= \frac{T^{\alpha}}{\alpha} \bigl((Qy)_1 - \bigl(Q^2 y\bigr)_1 \bigr) = 0. \end{aligned}$$

Similarly, we have $\int_0^T (T-s)^{\beta-1} y_2^*(s)\,ds = 0$. So $y^* \in \operatorname{Im} L$. Hence $X = \operatorname{Im} L + \operatorname{Im} Q$. Since $\operatorname{Im} L \cap \operatorname{Im} Q = \{0\}$, we have $X = \operatorname{Im} L \oplus \operatorname{Im} Q$. Thus

$$\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L = 2.$$

This means that L is a Fredholm operator of index zero.

From the definition of K_P, for $y \in \operatorname{Im} L$, we have

$$LK_P y = \begin{pmatrix} D_{0^+}^{\alpha} I_{0^+}^{\alpha} y_1 \\ D_{0^+}^{\beta} I_{0^+}^{\beta} y_2 \end{pmatrix} = y.$$

On the other hand, for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, we have $x_1(0) = x_2(0) = 0$. By Lemma 2.1, we get

$$K_P L x = \begin{pmatrix} x_1 - x_1(0) \\ x_2 - x_2(0) \end{pmatrix} = x.$$

So, we know that $K_P = (L_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$. The proof is complete. □

Lemma 3.3 *Let N be defined by* (3.3). *Assume $\Omega \subset X$ is an open bounded subset such that* $\operatorname{dom} L \cap \bar{\Omega} \neq \varnothing$, *then N is L-compact on $\bar{\Omega}$.*

Proof By the continuity of ϕ_q, e, g, we find that $QN(\bar{\Omega})$ and $K_P(I-Q)N(\bar{\Omega})$ are bounded. Moreover, there exists a constant $M > 0$ such that $\|(I-Q)Nx\| \leq M$, $\forall x \in \bar{\Omega}$, $t \in [0,T]$. Thus, in view of the Arzelà-Ascoli theorem, we only need prove that $K_P(I-Q)N(\bar{\Omega}) \subset X$ is equicontinuous.

For $0 \leq t_1 < t_2 \leq T$, $x \in \bar{\Omega}$, we have

$$
\begin{aligned}
&K_P(I-Q)Nx(t_2) - K_P(I-Q)Nx(t_1) \\
&\quad = \begin{pmatrix} I_{0^+}^{\alpha}((I-Q)Nx)_1(t_2) - I_{0^+}^{\alpha}((I-Q)Nx)_1(t_1) \\ I_{0^+}^{\beta}((I-Q)Nx)_2(t_2) - I_{0^+}^{\beta}((I-Q)Nx)_2(t_1) \end{pmatrix}.
\end{aligned}
$$

From $\|(I-Q)Nx\| \leq M$, $\forall x \in \bar{\Omega}$, $t \in [0,T]$, we can see that

$$
\begin{aligned}
&\left| \frac{1}{\Gamma(\alpha)} \left(\int_0^{t_2} (t_2 - s)^{\alpha-1} \big((I-Q)Nx\big)_1(s)\,ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} \big((I-Q)Nx\big)_1(s)\,ds \right) \right| \\
&\quad \leq \frac{M}{\Gamma(\alpha)} \left(\int_0^{t_1} \big((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\big)\,ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\,ds \right) \\
&\quad = \frac{M}{\Gamma(\alpha+1)} \big(t_1^{\alpha} - t_2^{\alpha} + 2(t_2 - t_1)^{\alpha}\big) \\
&\quad \leq \frac{M}{\Gamma(\alpha+1)} \big(t_2^{\alpha} - t_1^{\alpha} + 2(t_2 - t_1)^{\alpha}\big).
\end{aligned}
$$

Since t^{α} is uniformly continuous on $[0,T]$, we find that $(K_P(I-Q)N(\bar{\Omega}))_1 \subset C([0,T],\mathbb{R})$ is equicontinuous. A similar proof can show that $(K_P(I-Q)N(\bar{\Omega}))_2 \subset C([0,T],\mathbb{R})$ is equicontinuous. So we find that $K_P(I-Q)N : \bar{\Omega} \to X$ is compact. The proof is complete. □

Now we give the main result as regards the existence of solutions for PBVP (1.2).

Theorem 3.1 *Assume that*:

(H$_1$) *there exists a constant $d_1 > 0$ such that*

$$
(-1)^i x g(t,x) > 0 \quad \big(i \in \{1,2\}\big), \forall t \in [0,T], |x| > d_1;
$$

(H$_2$) *there exist a constant $d_2 > 0$ and nonnegative functions $a, b \in C([0,T],\mathbb{R})$ such that*

$$
\big|g(t,x)\big| \leq a(t)|x|^{p-1} + b(t), \quad \forall t \in [0,T], |x| > d_2.
$$

Then PBVP (1.2) *has at least one solution, provided that*

$$
\begin{aligned}
\gamma_1 &:= \frac{2^p T^{\beta+\alpha p-\alpha} \|a\|_0}{\Gamma(\beta+1)(\Gamma(\alpha+1))^{p-1}} < 1, \quad \text{if } p < 2; \\
\gamma_2 &:= \frac{2^{2p-2} T^{\beta+\alpha p-\alpha} \|a\|_0}{\Gamma(\beta+1)(\Gamma(\alpha+1))^{p-1}} < 1, \quad \text{if } p \geq 2.
\end{aligned}
\tag{3.7}
$$

Proof Set

$$\Omega' = \{x \in \operatorname{dom} L \mid Lx = \lambda Nx, \lambda \in (0,1)\}.$$

For $x \in \Omega'$, we get $Nx \in \operatorname{Im} L$. So by (3.5), we have

$$\int_0^T (T-s)^{\alpha-1}\phi_q(x_2(s))\,ds = 0,$$
$$\int_0^T (T-s)^{\beta-1}\bigl(e(s) - g(s, x_1(s))\bigr)\,ds = 0.$$

From the integral mean value theorem and $\int_0^T (T-s)^{\beta-1}e(s)\,ds = 0$, there exist constants $\zeta, \eta \in (0, T)$ such that $x_2(\zeta) = 0$, $g(\eta, x_1(\eta)) = 0$. Together with the condition (H_1), we have $|x_1(\eta)| \le d_1$. By Lemma 2.1, we have

$$x_1(t) = x_1(\eta) - I_{0^+}^{\alpha} D_{0^+}^{\alpha} x_1(\eta) + I_{0^+}^{\alpha} D_{0^+}^{\alpha} x_1(t),$$

which, together with

$$\begin{aligned} \bigl|I_{0^+}^{\alpha} D_{0^+}^{\alpha} x_1(t)\bigr| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^t (t-s)^{\alpha-1} D_{0^+}^{\alpha} x_1(s)\,ds\right| \\ &\le \frac{1}{\Gamma(\alpha)}\bigl\|D_{0^+}^{\alpha} x_1\bigr\|_0 \cdot \frac{1}{\alpha} t^{\alpha} \\ &\le \frac{T^{\alpha}}{\Gamma(\alpha+1)}\bigl\|D_{0^+}^{\alpha} x_1\bigr\|_0, \quad \forall t \in [0, T], \end{aligned}$$

and $|x_1(\eta)| \le d_1$, yields

$$\|x_1\|_0 \le d_1 + \frac{2T^{\alpha}}{\Gamma(\alpha+1)}\bigl\|D_{0^+}^{\alpha} x_1\bigr\|_0. \tag{3.8}$$

On the other hand, if $x \in \Omega'$, we have

$$\begin{cases} D_{0^+}^{\alpha} x_1(t) = \lambda \phi_q(x_2(t)), \\ D_{0^+}^{\beta} x_2(t) = \lambda(e(t) - g(t, x_1(t))). \end{cases} \tag{3.9}$$

From the first equation of (3.9), we get $x_2(t) = \phi_p(\lambda^{-1} D_{0^+}^{\alpha} x_1(t))$. By substituting it into the second equation of (3.9), we get

$$D_{0^+}^{\beta}\bigl(\phi_p\bigl(D_{0^+}^{\alpha} x_1(t)\bigr)\bigr) = \lambda^p e(t) - \lambda^p g(t, x_1) := \lambda^p N_g x_1(t).$$

Thus, by Lemma 2.1, we obtain

$$\phi_p\bigl(D_{0^+}^{\alpha} x_1(t)\bigr) = c_0 + \lambda^p I_{0^+}^{\beta} N_g x_1(t). \tag{3.10}$$

Then we have

$$x_1(t) = c_1 + I_{0^+}^{\alpha}\phi_q\bigl(c_0 + \lambda^p I_{0^+}^{\beta} N_g x_1\bigr)(t).$$

By the boundary condition $x_1(0) = x_1(T)$, we get

$$\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\phi_q\big(c_0 + \lambda^p I_{0^+}^{\beta} N_g x_1(s)\big)\,ds = 0.$$

Obviously, there exists a constant $\xi \in (0,T)$ such that $\phi_q(c_0 + \lambda^p I_{0^+}^{\beta} N_g x_1(\xi)) = 0$, which implies that $c_0 = -\lambda^p I_{0^+}^{\beta} N_g x_1(\xi)$. By substituting it into (3.10), we have

$$\phi_p\big(D_{0^+}^{\alpha} x_1(t)\big) = -\lambda^p I_{0^+}^{\beta} N_g x_1(\xi) + \lambda^p I_{0^+}^{\beta} N_g x_1(t). \tag{3.11}$$

From the hypothesis (H_2), we get

$$\begin{aligned}
\big|I_{0^+}^{\beta} N_g x_1(t)\big| &= \frac{1}{\Gamma(\beta)}\left|\int_0^t (t-s)^{\beta-1} e(s)\,ds - \int_0^t (t-s)^{\beta-1} g\big(s,x_1(s)\big)\,ds\right| \\
&\le \frac{T^{\beta}}{\Gamma(\beta+1)}\|e\|_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\big|g\big(s,x_1(s)\big)\big|\,ds \\
&\le \frac{T^{\beta}}{\Gamma(\beta+1)}\big(\|e\|_0 + \|a\|_0\|x_1\|_0^{p-1} + \|b\|_0 + G_{d_2}\big), \quad \forall t \in [0,T],
\end{aligned}$$

where $G_{d_2} = \max\{|g(t,x)| \mid t \in [0,T], |x| \le d_2\}$. Together with (3.8), (3.11), and

$$\big|\phi_p\big(D_{0^+}^{\alpha} x_1(t)\big)\big| = \big|D_{0^+}^{\alpha} x_1(t)\big|^{p-1},$$

we have

$$\begin{aligned}
\big\|D_{0^+}^{\alpha} x_1\big\|_0^{p-1} &\le \frac{2T^{\beta}}{\Gamma(\beta+1)}\big(\|e\|_0 + \|a\|_0\|x_1\|_0^{p-1} + \|b\|_0 + G_{d_2}\big) \\
&\le \frac{2T^{\beta}}{\Gamma(\beta+1)}\left[\|e\|_0 + \|b\|_0 + G_{d_2} + \|a\|_0\left(d_1 + \frac{2T^{\alpha}}{\Gamma(\alpha+1)}\big\|D_{0^+}^{\alpha} x_1\big\|_0\right)^{p-1}\right].
\end{aligned}$$

If $p < 2$, by using Lemma 2.2, we get

$$\begin{aligned}
\big\|D_{0^+}^{\alpha} x_1\big\|_0^{p-1} &\le \frac{2T^{\beta}}{\Gamma(\beta+1)}\big(\|e\|_0 + \|b\|_0 + G_{d_2}\big) \\
&\quad + \frac{2T^{\beta}\|a\|_0}{\Gamma(\beta+1)}\left(d_1^{p-1} + \frac{(2T^{\alpha})^{p-1}}{(\Gamma(\alpha+1))^{p-1}}\big\|D_{0^+}^{\alpha} x_1\big\|_0^{p-1}\right) \\
&= A_1 + \frac{2^p T^{\beta+\alpha p-\alpha}\|a\|_0}{\Gamma(\beta+1)(\Gamma(\alpha+1))^{p-1}}\big\|D_{0^+}^{\alpha} x_1\big\|_0^{p-1},
\end{aligned}$$

where $A_1 = \frac{2T^{\beta}}{\Gamma(\beta+1)}(\|e\|_0 + \|b\|_0 + G_{d_2}) + \frac{2T^{\beta}\|a\|_0}{\Gamma(\beta+1)} d_1^{p-1}$. Then, from (3.7), we have

$$\big\|D_{0^+}^{\alpha} x_1\big\|_0 \le \left(\frac{A_1}{1-\gamma_1}\right)^{q-1} := M_1.$$

Thus, from (3.8), we get

$$\|x_1\|_0 \le d_1 + \frac{2T^{\alpha}}{\Gamma(\alpha+1)} M_1. \tag{3.12}$$

If $p \geq 2$, similar to the above argument, let $A_2 = \frac{2T^\beta}{\Gamma(\beta+1)}(\|e\|_0 + \|b\|_0 + G_{d_2}) + \frac{2^{p-1}T^\beta\|a\|_0}{\Gamma(\beta+1)}d_1^{p-1}$, we obtain

$$\|x_1\|_0 \leq d_1 + \frac{2T^\alpha}{\Gamma(\alpha+1)}M_2, \tag{3.13}$$

where $M_2 = (\frac{A_2}{1-\gamma_2})^{q-1}$. So, combining (3.12) with (3.13), we get

$$\|x_1\|_0 \leq \max\left\{d_1 + \frac{2T^\alpha}{\Gamma(\alpha+1)}M_1, d_1 + \frac{2T^\alpha}{\Gamma(\alpha+1)}M_2\right\} := M. \tag{3.14}$$

From the second equation of (3.9) and Lemma 2.1, we have

$$x_2(t) = c + \lambda I_{0^+}^\beta N_g x_1(t),$$

which together with $x_2(\zeta) = 0$ yields

$$x_2(t) = -\lambda I_{0^+}^\beta N_g x_1(\zeta) + \lambda I_{0^+}^\beta N_g x_1(t).$$

Then we have

$$\|x_2\|_0 \leq \frac{2T^\beta}{\Gamma(\beta+1)}\left(\|e\|_0 + G_M\right) := \overline{M},$$

where $G_M = \max\{|g(t,x)| \mid t \in [0,T], |x| \leq M\}$. Together with (3.14), we obtain

$$\|x\| = \max\left\{\|x_1\|_0, \|x_2\|_0\right\} \leq \max\{M, \overline{M}\} := M_0.$$

Let $\Omega = \{x \in X \mid \|x\| < M_0 + 1\}$. From the above argument, we know that the equation

$$Lx = \lambda Nx, \quad \forall \lambda \in (0,1)$$

has no solution on $\partial\Omega \cap \operatorname{dom} L$. So the condition (1) of Lemma 2.3 is satisfied. Next the other two conditions of Lemma 2.3 are to be verified.

For $x \in \operatorname{Ker} L$, we have $x_1(t) = c_1$, $x_2(t) = c_2$, $\forall t \in [0,T]$, $c_1, c_2 \in \mathbb{R}$. If $QNx = 0$, we obtain

$$\int_0^T (T-s)^{\alpha-1}\phi_q(c_2)\,ds = 0,$$
$$\int_0^T (T-s)^{\beta-1}g(s,c_1)\,ds = 0.$$

From the first equality, we get $c_2 = 0$. From the second equality and (H_1), we have $|c_1| \leq d_1$. So $\|x\| = \max\{|c_1|, |c_2|\} \leq d_1 < M_0 + 1$. Then the condition (2) of Lemma 2.3 is satisfied.

Define the operators $J : \operatorname{Im} Q \to \operatorname{Ker} L$ by

$$J(x_1, x_2)^{\mathrm{T}} = \left((-1)^{i+1}x_2, x_1\right)^{\mathrm{T}},$$

and $F : [0,1] \times \bar{\Omega} \to X$ by

$$F(\mu, x) = \mu x + (1-\mu)JQNx = \begin{pmatrix} \mu x_1 + (-1)^i(1-\mu)\frac{\beta}{T^\beta}\int_0^T (T-s)^{\beta-1}g(s,x_1(s))\,ds \\ \mu x_2 + (1-\mu)\frac{\alpha}{T^\alpha}\int_0^T (T-s)^{\alpha-1}\phi_q(x_2(s))\,ds \end{pmatrix},$$

where $i \in \{1,2\}$. Let $x \in \operatorname{Ker} L$ satisfying $F(\mu,x) = 0$, we get $x_1(t) = c_1$, $x_2(t) = c_2$, $\forall t \in [0,T]$, $c_1, c_2 \in \mathbb{R}$, and

$$\mu c_1 + (-1)^i(1-\mu)\frac{\beta}{T^\beta}\int_0^T (T-s)^{\beta-1} g(s,c_1)\,ds = 0, \tag{3.15}$$

$$\mu c_2 + (1-\mu)\phi_q(c_2) = 0. \tag{3.16}$$

From (3.16), we get $c_2 = 0$ because c_2 and $\phi_q(c_2)$ have the same sign. From (3.15), if $\mu = 0$, we get $|c_1| \le d_1$ because of (H_1). If $\mu \in (0,1]$, we also get $|c_1| \le d_1$. In fact, if $|c_1| > d_1$, in view of (H_1), one has

$$\mu c_1^2 + (1-\mu)\frac{\beta}{T^\beta}\int_0^T (T-s)^{\beta-1}(-1)^i c_1 g(s,c_1)\,ds > 0,$$

which contradicts (3.15). From the argument above, we obtain $\|x\| < M_0 + 1$. Thus

$$F(\mu,x) \neq 0, \quad \forall (\mu,x) \in [0,1] \times (\partial\Omega \cap \operatorname{Ker} L).$$

Hence, by the homotopy property of the degree, we have

$$\begin{aligned}\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) &= \deg\bigl(F(0,\cdot), \Omega \cap \operatorname{Ker} L, 0\bigr)\\ &= \deg\bigl(F(1,\cdot), \Omega \cap \operatorname{Ker} L, 0\bigr) = \deg(I, \Omega \cap \operatorname{Ker} L, 0) \neq 0.\end{aligned}$$

So the condition (3) of Lemma 2.3 is satisfied.

Consequently, by using Lemma 2.3, the operator equation $Lx = Nx$ has at least one solution $x(\cdot) = (x_1(\cdot), x_2(\cdot))^{\mathrm{T}}$ on $\bar{\Omega} \cap \operatorname{dom} L$. Namely, PBVP (1.2) has at least one solution $x_1(\cdot)$. The proof is complete. □

4 An example

In this section, we will give an example to illustrate our main result.

Example 4.1 Consider the following PBVP for a fractional p-Laplacian equation:

$$\begin{cases} D_{0^+}^{\frac{3}{4}}(\phi_4(D_{0^+}^{\frac{1}{2}}x(t))) - \frac{1}{120}x^3(t) + \frac{1}{2} = (1-t)^{\frac{1}{4}}\sin 2\pi t, & t \in [0,1],\\ x(0) = x(1), \qquad D_{0^+}^{\frac{1}{2}}x(0) = D_{0^+}^{\frac{1}{2}}x(1). \end{cases} \tag{4.1}$$

Corresponding to PBVP (1.2), we get $p = 4$, $\alpha = 1/2$, $\beta = 3/4$, $T = 1$, $e(t) = (1-t)^{\frac{1}{4}}\sin 2\pi t$, and

$$g(t,x) = -\frac{1}{120}x^3 + \frac{1}{2}.$$

Choose $a(t) = \frac{1}{120}$, $b(t) = 1$. By a simple calculation, we obtain

$$xg(t,x) = -\frac{x}{120}\bigl(x^3 - 60\bigr) < 0, \quad \forall t \in [0,1], |x| > 4,$$

$$\gamma_2 = \frac{2^3/120}{\Gamma(\frac{3}{4}+1)}\left[\frac{2}{\Gamma(\frac{1}{2}+1)}\right]^3 < 1.$$

Obviously, PBVP (4.1) satisfies all assumptions of Theorem 3.1. Hence, PBVP (4.1) has at least one solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the anonymous referee for his/her valuable comments, which have improved the correctness and presentation of the manuscript. This work is supported by the National Natural Science Foundation of China (11271364).

Received: 10 October 2014 Accepted: 2 March 2015 Published online: 21 August 2015

References
1. Diethelm, K, Freed, AD: On the solution of nonlinear fractional order differential equations used in the modeling of viscoelasticity. In: Keil, F, Mackens, W, Voss, H, Werther, J (eds.) Scientific Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, pp. 217-224. Springer, Heidelberg (1999)
2. Lundstrom, BN, Higgs, MH, Spain, WJ, Fairhall, AL: Fractional differentiation by neocortical pyramidal neurons. Nat. Neurosci. **11**, 1335-1342 (2008)
3. Glockle, WG, Nonnenmacher, TF: A fractional calculus approach of self-similar protein dynamics. Biophys. J. **68**, 46-53 (1995)
4. Hilfer, R: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
5. Mainardi, F: Fractional calculus: some basic problems in continuum and statistical mechanics. In: Carpinteri, A, Mainardi, F (eds.) Fractals and Fractional Calculus in Continuum Mechanics, pp. 291-348. Springer, Wien (1997)
6. Kirchner, JW, Feng, X, Neal, C: Fractal stream chemistry and its implications for contaminant transport in catchments. Nature **403**, 524-526 (2000)
7. Babakhani, A, Gejji, VD: Existence of positive solutions of nonlinear fractional differential equations. J. Math. Anal. Appl. **278**, 434-442 (2003)
8. Delbosco, D, Rodino, L: Existence and uniqueness for a nonlinear fractional differential equation. J. Math. Anal. Appl. **204**, 609-625 (1996)
9. Kilbas, AA, Trujillo, JJ: Differential equations of fractional order: methods, results and problems-I. Appl. Anal. **78**, 153-192 (2001)
10. Kilbas, AA, Trujillo, JJ: Differential equations of fractional order: methods, results and problems-II. Appl. Anal. **81**, 435-493 (2002)
11. Agarwal, RP, O'Regan, D, Stanek, S: Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. J. Math. Anal. Appl. **371**, 57-68 (2010)
12. Bai, Z: On solutions of some fractional *m*-point boundary value problems at resonance. Electron. J. Qual. Theory Differ. Equ. **2010**, Article ID 37 (2010)
13. Leibenson, LS: General problem of the movement of a compressible fluid in a porous medium. Izv. Akad. Nauk Kirg. SSSR **9**, 7-10 (1983) (in Russian)
14. Chen, T, Liu, W, Yang, C: Antiperiodic solutions for Liénard-type differential equation with *p*-Laplacian operator. Bound. Value Probl. **2010**, Article ID 194824 (2010)
15. Jiang, D, Gao, W: Upper and lower solution method and a singular boundary value problem for the one-dimensional *p*-Laplacian. J. Math. Anal. Appl. **252**, 631-648 (2000)
16. Lian, L, Ge, W: The existence of solutions of m-point *p*-Laplacian boundary value problems at resonance. Acta Math. Appl. Sin. **28**, 288-295 (2005)
17. Liu, B, Yu, J: On the existence of solutions for the periodic boundary value problems with *p*-Laplacian operator. J. Syst. Sci. Math. Sci. **23**, 76-85 (2003)
18. Pang, H, Ge, W, Tian, M: Solvability of nonlocal boundary value problems for ordinary differential equation of higher order with a *p*-Laplacian. Comput. Math. Appl. **56**, 127-142 (2008)
19. Su, H, Wang, B, Wei, Z, Zhang, X: Positive solutions of four-point boundary value problems for higher-order *p*-Laplacian operator. J. Math. Anal. Appl. **330**, 836-851 (2007)
20. Zhang, J, Liu, W, Ni, J, Chen, T: Multiple periodic solutions of *p*-Laplacian equation with one-side Nagumo condition. J. Korean Math. Soc. **45**, 1549-1559 (2008)
21. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
22. Samko, SG, Kilbas, AA, Marichev, OI: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Switzerland (1993)
23. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
24. Lian, H: Boundary value problems for nonlinear ordinary differential equations on infinite intervals, Doctoral thesis (2007)
25. Gaines, R, Mawhin, J: Coincidence Degree and Nonlinear Differential Equations. Springer, Berlin (1977)