Yu et al. *Boundary Value Problems* (2015) 2015:147
DOI 10.1186/s13661-015-0409-2

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Positive solutions for nonlinear double impulsive differential equations with p-Laplacian on infinite intervals

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Abstract
In this paper, we investigate nonlinear second-order double impulsive differential equations integral boundary value problem with p-Laplacian on an infinite interval with the infinite number of impulsive times. Based on the cone theory and monotone iterative technique, we establish the existence of minimal nonnegative solution and iteration of positive solutions for such a boundary value problem. The main results are new and extend the existing results. At last, some examples are worked out to demonstrate the use of the main results.

Keywords: monotone iterative technique; double impulsive differential equations; integral boundary value problem; infinite intervals

1 Introduction

Boundary value problems on infinite intervals appear often in applied mathematics and physics, for example, in the study of the unsteady flow of a gas through semi-infinite porous medium, in analyzing the heat transfer in radial flow between circular disks, in the study of plasma physics, and in an analysis of the mass transfer on a rotating disk in non-Newtonian fluid, see [1, 2] and the references therein. For extensive applications, this kind of BVPs attract lots of scholars to devote themselves to developing them. Scholars do some work and apply many techniques to deal with such problems, see [3–10] and the references therein. While boundary value problems with integral boundary conditions for ordinary differential equations on an infinite interval also arise in different fields such as heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics. In the past few years, many people have started to be active in studying the existence of the solutions to nonlinear integral boundary value problems (IBVPs) on infinite intervals. Some conclusions appeared in the meantime, see [11–18] and the references therein.

Considering the theory of impulsive differential equations, it has been emerging as an important area of investigation in recent years and has been extensively applied in chemical technology, population dynamics, and so on. It is much wider because all the structure of its emergence has deep physical background and realistic mathematical model and coincides with many phenomena in nature. For an introduction of the basic theory of impulsive differential equations in R^n, see [19–21] and the references therein.

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We notice that there has been increasing interest in studying nonlinear differential equation and impulsive integro differential equation on an infinite interval with an infinite number of impulsive times to identify a few, see [22–26] and the references therein. There are relatively few papers available for integral boundary value problems for impulsive differential equations on an infinite interval with an infinite number of impulsive times up to now, see [27–31] and the references therein.

Recently, in [32], Zhang *et al.* investigated the existence of minimal nonnegative solution for the following second-order impulsive differential equation IBVP:

$$\begin{cases} -x''(t) = f(t, x(t), x'(t)), \quad t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \ldots, \\ \Delta x'|_{t=t_k} = \overline{I}_k(x(t_k)), \quad k = 1, 2, \ldots, \\ x(0) = \int_0^{+\infty} g(t)x(t)\,dt, \qquad x'(\infty) = 0, \end{cases}$$

where $f \in C(J \times J \times J, J)$, $I_k, \overline{I}_k \in C(R, R)$, $J = [0, +\infty)$, $0 = t_0 < t_1 < \cdots < t_k < \cdots$, $t_k \to \infty$ for $k = 1, 2, \ldots$, and $g(t) \in L[J, J]$ with $0 < \int_0^{+\infty} g(t)\,dt < 1$. $\Delta x|_{t_k}$ denotes the jump of $x(t)$ at $t = t_k$, that is,

$$\Delta x|_{t=t_k} = x\bigl(t_k^+\bigr) - x\bigl(t_k^-\bigr),$$

where $x(t_k^+)$ and $x(t_k^-)$ represent the right-hand limit and left-hand limit of $x(t)$ at $t = t_k$, respectively, $\Delta x'|_{t_k}$ has a similar meaning to $x'(t)$.

More recently, in [33], Zhang studied the existence and iteration of positive solutions for nonlinear second-order impulsive IBVP with p-Laplacian on infinite intervals

$$\begin{cases} (\phi_p(x'(t)))' + q(t)f(t, x(t), x'(t)), \quad t \in J'_+, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \ldots, \\ x(0) = \int_0^{+\infty} g(t)x(t)\,dt, \qquad x'(\infty) = x_\infty, \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $J = [0, +\infty)$, $J_+ = (0, +\infty)$, $J'_+ = J_+ \backslash \{t_1, t_2, \ldots, t_k, \ldots\}$, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_k \to \infty$ for $k = 1, 2, \ldots$, and $g(t) \in L[J, J]$ with $\int_0^{+\infty} g(t)\,dt < 1$, $\int_0^{+\infty} tg(t)\,dt < \infty$, and $0 \leq x'(\infty) = \lim_{t \to +\infty} x'(t)$.

However, to the authors' knowledge, the corresponding theory for double impulsive integral boundary value problems with p-Laplacian operator and infinite impulsive times on infinite intervals has not been considered till now. Motivated by the above mentioned works, in this paper, we study the existence of solutions for nonlinear double impulsive IBVPs of second-order differential equations with p-Laplacian on an infinite interval

$$\begin{cases} (\phi_p(x'(t)))' + a(t)f(t, x(t), x'(t)) = 0, \quad t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \ldots, \\ \Delta \phi_p(x')|_{t=t_k} = \overline{I}_k(x(t_k)), \quad k = 1, 2, \ldots, \\ x(0) = \int_\eta^{+\infty} g(t)x(t)\,dt, \qquad x'(\infty) = 0, \end{cases} \tag{1.1}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\eta \geq 0$ is a constant, $f \in C(J^3, J)$, $J = [0, +\infty)$, $0 = t_0 < t_1 < \cdots < t_k < \cdots$, $t_k \to \infty$ for $k = 1, 2, \ldots$ and note $J_0 = [0, t_1]$ and $J_i = (t_i, t_{i+1}]$ $(i = 1, 2, \ldots)$, $g(t) \in L^1[J, J]$ with $\int_\eta^{+\infty} g(t)\,dt < 1$, $\int_\eta^{+\infty} tg(t)\,dt < +\infty$, $I_k, \overline{I}_k \in C(R, R)$, and

$$\Delta x|_{t=t_k} = x\bigl(t_k^+\bigr) - x\bigl(t_k^-\bigr), \qquad \Delta \phi_p\bigl(x'\bigr)|_{t=t_k} = \phi_p\bigl(x'\bigl(t_k^+\bigr)\bigr) - \phi_p\bigl(x'\bigl(t_k^-\bigr)\bigr).$$

It is clear that

$$\phi_p(s+t) \le \begin{cases} 2^{p-1}(\phi_p(s)+\phi_p(t)), & p \ge 2, s, t > 0, \\ \phi_p(s)+\phi_p(t), & 1 < p < 2, s, t > 0, \end{cases} \tag{1.2}$$

$$\phi_p^{-1}(s+t) \le \begin{cases} 2^{\frac{1}{p-1}}(\phi_p^{-1}(s)+\phi_p^{-1}(t)), & p \ge 2, s, t > 0, \\ \phi_p^{-1}(s)+\phi_p^{-1}(t), & 1 < p < 2, s, t > 0. \end{cases} \tag{1.3}$$

Throughout this paper, we adopt the following assumptions:

(H_1) Suppose that $f \in C[J \times J \times J, J]$, and there exist $p, q, r \in C(J, J)$ such that

$$f(t,u,v) \le p(t)\phi_p(u) + q(t)\phi_p(v) + r(t), \quad \forall t \in J \text{ and } \forall u, v \in J,$$

and note

$$p^* = \int_0^{+\infty} a(t)p(t)(1+t)^{p-1}\,dt < +\infty, \qquad q^* = \int_0^{+\infty} a(t)q(t)\,dt < +\infty,$$
$$r^* = \int_0^{+\infty} a(t)r(t)\,dt < +\infty.$$

(H_2) $I_k, \bar{I}_k \in C(J,J)$ and there exist nonnegative constants $a_k \ge 0$, $b_k \ge 0$, $c_k \ge 0$, $d_k \ge 0$ such that

$$0 \le I_k(u) \le a_k u + b_k, \quad \forall u \in J \ (k = 1,2,3,\ldots),$$
$$0 \le \bar{I}_k(u) \le c_k\phi_p(u) + d_k, \quad \forall u \in J \ (k = 1,2,3,\ldots)$$

and note

$$a^* = \sum_{k=1}^{\infty}(t_k+1)a_k < +\infty, \qquad b^* = \sum_{k=1}^{\infty} b_k < +\infty,$$
$$c^* = \sum_{k=1}^{\infty}(t_k+1)^{p-1}c_k < +\infty, \qquad d^* = \sum_{k=1}^{\infty} d_k < +\infty,$$

with $a^* < (1/3)(1 - \int_\eta^\infty g(t)\,dt)$ and $c^* < \phi_p(1/(3n))$, where n is a constant and it firstly appears in (3.12).

(H_3) $f(t,u_1,v_1) \le f(t,u_2,v_2)$, $I_k(u_1) \le I_k(u_2)$, $\bar{I}_k(u_1) \le \bar{I}_k(u_2)$ for $t \in J$, $u_1 \le u_2$, $v_1 \le v_2$ ($k = 1,2,3,\ldots$).

(H_4) Suppose that $f \in C[J \times J \times J, J]$, $f(t,0,0) \not\equiv 0$ on any subinterval of J, and u, v are bounded, $f(t,(1+t)u,v)$ is bounded on J.

(H_5) $a(t)$ is a nonnegative measurable function defined in $J\backslash\{0\}$, and $a(t)$ does not identically vanish on any subinterval of $J\backslash\{0\}$, and

$$0 < \int_0^{+\infty} a(t)\,dt < +\infty, \qquad 0 < \int_0^{+\infty} \phi_p^{-1}\left(\int_t^{+\infty} a(s)\,ds\right)dt < +\infty.$$

2 Preliminary results

In this section, we firstly present some definitions and lemmas, which will be needed in the proof of the main results.

Definition 2.1 Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that

(1) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0$, $b \geq 0$,
(2) $u, -u \in P$ implies that $u = 0$.

Definition 2.2 A map $\alpha : P \to [0, +\infty)$ is said to be concave on P if

$$\alpha\big(tu + (1-t)v\big) \geq \alpha(u) + (1-t)\alpha(v) \quad \text{for all } u, v \in P \text{ and } t \in [0,1].$$

Definition 2.3 (see [8]) Let $V = \{x \in X : \|x\| < l\}$ $(l > 0)$, $V_1 := \{\frac{x(t)}{1+t}, x \in V\} \cup \{x'(t), x \in V\}$ is called equiconvergent at infinity if and only if for all $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that for all $x \in V_1$, the following holds:

$$\left|\frac{x(t_1)}{1+t_1} - \frac{x(t_2)}{1+t_2}\right| < \varepsilon, \qquad \left|x'(t_1) - x'(t_2)\right| < \varepsilon, \quad \text{for all } t_1, t_2 \geq T.$$

Lemma 2.4 (see [34, 35]) *If* $\{\frac{x(t)}{1+t}, x \in V\}$ *and* $\{x'(t), x \in V\}$ *are both equicontinuous on any compact intervals of* $[0, +\infty)$ *and equiconvergent at infinity, then* V *is relatively compact on* X.

Definition 2.5 Let $J' = J \setminus \{t_1, t_2, \dots t_k, \dots\}$, $x \in E \cap C^2[J', R]$ is called a nonnegative solution of IBVP (1.1) if $x(t) \geq 0$, $x'(t) \geq 0$ and $x(t)$ satisfies IBVP (1.1). Moreover, $\overline{x}(t)$ is called a minimal nonnegative solution if x is an arbitrary nonnegative solution of (1.1), then $x(t) \geq \overline{x}(t)$, $x'(t) \geq \overline{x}'(t)$ for all $t \in J'$.

Let

$$\begin{aligned} PC[J,R] = \big\{&x : x \text{ is a map from } J \text{ into } R \text{ such that } x(t) \text{ is continuous at } t \neq t_k, \\ &\text{left continuous at } t = t_k \text{ and } x\big(t_k^+\big) \text{ exist for } k = 1, 2, \dots\big\}, \\ PC^1[J,R] = \big\{&x \in PC[J,R] : x'(t) \text{ exists and is continuous at } t \neq t_k, \\ &\text{left continuous at } t = t_k \text{ and } x'\big(t_k^+\big) \text{ exist for } k = 1, 2, \dots\big\}, \\ E = \Big\{&x \in PC^1[J,R] : \sup_{t \in J}\big(\left|x(t)\right|/(1+t)\big) < \infty, \sup_{t \in J}\left|x'(t)\right| < \infty\Big\} \end{aligned}$$

with the norm $\|x\| = \max\{\|x\|_1, \|x'\|_\infty\}$, where $\|x\|_1 = \sup_{t \in J} \frac{|x(t)|}{1+t}$, $\|x'\|_\infty = \sup_{t \in J} |x'(t)|$. At the same time, define a cone $P \subset E$ by

$$P = \big\{x \in E : x(t) \geq 0, x'(t) \geq 0\big\}.$$

Lemma 2.6 *Suppose that* (H_1), (H_2) *hold. Then, for all* $x \in P$, $\int_0^{+\infty} a(t) f(t, x(t), x'(t))\,dt$, $\sum_{k=1}^{\infty} I_k(x(t_x))$ *and* $\sum_{k=1}^{\infty} \overline{I}_k(x(t_x))$ *are convergent.*

Proof By (H_1) and (H_2), we have

$$f(t,x(t),x'(t)) \le p(t)(1+t)^{p-1}\phi_p\left(\frac{x(t)}{1+t}\right) + q(t)\phi_p(x'(t)) + r(t),$$
$$I_k(x(t_k)) \le a_k(1+t_k)\frac{x(t_k)}{1+t_k} + b_k,$$
$$\bar{I}_k(x(t_k)) \le c_k(1+t_k)^{p-1}\phi_p\left(\frac{x(t_k)}{1+t_k}\right) + d_k.$$

Thus

$$\int_0^{+\infty} a(t)f(t,x(t),x'(t)) \le p^*\phi_p(\|x(t)\|_1) + q^*\phi_p(\|x(t)\|_\infty) + r^* < \infty,$$
$$\sum_{k=1}^{\infty} I_k(x(t_k)) \le a^*\|x(t)\|_1 + b^* < \infty,$$
$$\sum_{k=1}^{\infty} \bar{I}_k(x(t_k)) \le c^*\phi_p(\|x(t)\|_1) + d^* < \infty.$$

The proof is complete. □

Lemma 2.7 *Let $y(t) \in L^1[0,+\infty)$ and $\int_\eta^{+\infty} g(t)\,dt < 1$, then the IBVP*

$$\begin{cases} (\phi_p(x'(t)))' = -y(t), & t \in J, t \ne t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & k = 1,2,\ldots, \\ \Delta\phi_p(x')|_{t=t_k} = \bar{I}_k(x(t_k)), & k = 1,2,\ldots, \\ x(0) = \int_\eta^{+\infty} g(t)x(t)\,dt, & x'(\infty) = 0, \end{cases} \tag{2.1}$$

has a unique solution

$$\begin{aligned} x(t) = {} & \frac{1}{1-\int_\eta^{+\infty} g(t)\,dt}\Bigg[\sum_{t_i<t}\int_{t_{i-1}}^{t_i}\phi_p^{-1}\left(\int_s^{+\infty} y(\tau)\,d\tau - \sum_{t_k\ge s}\bar{I}_k(x(t_k))\right)ds \\ & + \int_\eta^{+\infty} g(t)\int_{t_i}^{t}\phi_p^{-1}\left(\int_s^{+\infty} y(\tau)\,d\tau - \sum_{t_k\ge s}\bar{I}_k(x(t_k))\right)ds\,dt + \sum_{t_i<t} I_i(x(t_i))\Bigg] \\ & + \int_{t_i}^{t}\phi_p^{-1}\left(\int_s^{+\infty} y(\tau)\,d\tau - \sum_{t_k\ge s}\bar{I}_k(x(t_k))\right)ds. \end{aligned} \tag{2.2}$$

Proof For $t \in [0,t_1]$, integrating (2.1) from 0 to t, we have

$$\int_0^t (\phi_p(x'(\tau)))'\,d\tau = -\int_0^t y(\tau)\,d\tau.$$

That is,

$$\phi_p(x'(t)) = \phi_p(x'(0)) - \int_0^t y(\tau)\,d\tau, \tag{2.3}$$

which implies that

$$\phi_p\bigl(x'\bigl(t_1^-\bigr)\bigr) = \phi_p\bigl(x'(0)\bigr) - \int_0^{t_1} y(\tau)\,d\tau. \tag{2.4}$$

For $t \in [t_1, t_2]$, integrating (2.1) from t_1 to t, we have

$$\int_{t_1}^{t} \bigl(\phi_p\bigl(x'(\tau)\bigr)\bigr)'\,d\tau = -\int_{t_1}^{t} y(\tau)\,d\tau.$$

That is,

$$\phi_p\bigl(x'(t)\bigr) = \phi_p\bigl(x'\bigl(t_1^+\bigr)\bigr) - \int_{t_1}^{t} y(\tau)\,d\tau. \tag{2.5}$$

Adding (2.4) and (2.5) together, we have

$$\phi_p\bigl(x'(t)\bigr) = \phi_p\bigl(x'(0)\bigr) - \int_0^{t} y(\tau)\,d\tau + \overline{I}_1\bigl(x(t_1)\bigr). \tag{2.6}$$

Repeating the previous process, for any $t \in [0, +\infty)$, we get that

$$\phi_p\bigl(x'(t)\bigr) = \phi_p\bigl(x'(0)\bigr) - \int_0^{t} y(\tau)\,d\tau + \sum_{t_k<t} \overline{I}_k\bigl(x(t_k)\bigr). \tag{2.7}$$

Taking limit for $t \to +\infty$, by the boundary condition, we have

$$x'(t) = \phi_p^{-1}\biggl(\int_t^{+\infty} y(\tau)\,d\tau - \sum_{t_k\geq t} \overline{I}_k\bigl(x(t_k)\bigr)\biggr). \tag{2.8}$$

For $t \in [0, t_1]$, integrating (2.8) from 0 to t, we have

$$x(t) = x(0) + \int_0^{t} \phi_p^{-1}\biggl(\int_s^{+\infty} y(\tau)\,d\tau - \sum_{t_k\geq s} \overline{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds, \tag{2.9}$$

which implies that

$$x\bigl(t_1^-\bigr) = x(0) + \int_0^{t_1} \phi_p^{-1}\biggl(\int_s^{+\infty} y(\tau)\,d\tau - \sum_{t_k\geq s} \overline{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds. \tag{2.10}$$

For $t \in [t_1, t_2]$, integrating (2.8) from t_1 to t, we have

$$x(t) = x\bigl(t_1^+\bigr) + \int_{t_1}^{t} \phi_p^{-1}\biggl(\int_s^{+\infty} y(\tau)\,d\tau - \sum_{t_k\geq s} \overline{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds. \tag{2.11}$$

Adding (2.10) and (2.11) together, we have

$$\begin{aligned} x(t) = x(0) &+ \int_0^{t_1} \phi_p^{-1}\biggl(\int_s^{+\infty} y(\tau)\,d\tau - \sum_{t_k\geq s} \overline{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds \\ &+ \int_{t_1}^{t} \phi_p^{-1}\biggl(\int_s^{+\infty} y(\tau)\,d\tau - \sum_{t_k\geq s} \overline{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds + I_1\bigl(x(t_1)\bigr). \end{aligned} \tag{2.12}$$

Repeating the previous process, for any $t \in [0, +\infty)$, we get that

$$x(t) = x(0) + \sum_{t_i<t} \int_{t_{i-1}}^{t_i} \phi_p^{-1}\left(\int_s^{+\infty} y(\tau)\,d\tau - \sum_{t_k \geq s} \bar{I}_k\bigl(x(t_k)\bigr) \right) ds + \int_{t_i}^{t} \phi_p^{-1}\left(\int_s^{+\infty} y(\tau)\,d\tau - \sum_{t_k \geq s} \bar{I}_k\bigl(x(t_k)\bigr) \right) ds + \sum_{t_i<t} I_i\bigl(x(t_i)\bigr). \tag{2.13}$$

By (2.13) and the boundary condition, for any $t \in [0, +\infty)$, we have

$$\begin{aligned} x(t) = {}& \frac{1}{1 - \int_\eta^{+\infty} g(t)\,dt} \Biggl[\sum_{t_i<t} \int_{t_{i-1}}^{t_i} \phi_p^{-1}\left(\int_s^{+\infty} y(\tau)\,d\tau - \sum_{t_k \geq s} \bar{I}_k\bigl(x(t_k)\bigr) \right) ds \\ &+ \int_\eta^{+\infty} g(t) \int_{t_i}^{t} \phi_p^{-1}\left(\int_s^{+\infty} y(\tau)\,d\tau - \sum_{t_k \geq s} \bar{I}_k\bigl(x(t_k)\bigr) \right) ds\,dt + \sum_{t_i<t} I_i\bigl(x(t_i)\bigr) \Biggr] \\ &+ \int_{t_i}^{t} \phi_p^{-1}\left(\int_s^{+\infty} y(\tau)\,d\tau - \sum_{t_k \geq s} \bar{I}_k\bigl(x(t_k)\bigr) \right) ds. \end{aligned}$$

This completes the proof. □

Define an integral operator $T : P \to E$ by

$$\begin{aligned} (Tx)(t) = {}& \frac{1}{1 - \int_\eta^{+\infty} g(t)\,dt} \\ &\times \Biggl[\sum_{t_i<t} \int_{t_{i-1}}^{t_i} \phi_p^{-1}\left(\int_s^{+\infty} a(\tau) f\bigl(\tau, x(\tau), x'(\tau)\bigr)\,d\tau - \sum_{t_k \geq s} \bar{I}_k\bigl(x(t_k)\bigr) \right) ds \\ &+ \int_\eta^{+\infty} g(t) \int_{t_i}^{t} \phi_p^{-1}\left(\int_s^{+\infty} a(\tau) f\bigl(\tau, x(\tau), x'(\tau)\bigr)\,d\tau - \sum_{t_k \geq s} \bar{I}_k\bigl(x(t_k)\bigr) \right) ds\,dt \\ &+ \sum_{t_i<t} I_i\bigl(x(t_i)\bigr) \Biggr] \\ &+ \int_{t_i}^{t} \phi_p^{-1}\left(\int_s^{+\infty} a(\tau) f\bigl(\tau, x(\tau), x'(\tau)\bigr)\,d\tau - \sum_{t_k \geq s} \bar{I}_k\bigl(x(t_k)\bigr) \right) ds. \end{aligned} \tag{2.14}$$

Obviously, T is well defined and $x \in PC(J, R)$ is a solution of BVP (1.1) if and only if x is a fixed point of T.

Lemma 2.8 *Assume that* (H_1)-(H_3) *hold. Then the operator* T *maps* P *into* P, *and*

$$\|Tx\| \leq A\|x\| + B, \quad \forall x \in P. \tag{2.15}$$

Moreover, for $x, y \in P$ *with* $x(t) \leq y(t)$, $x'(t) \leq y'(t)$, *for all* $t \in J$, *and one has*

$$(Tx)(t) \leq (Ty)(t), \qquad (Tx)'(t) \leq (Ty)'(t), \quad \forall t \in J, \tag{2.16}$$

where

$$
\begin{aligned}
A &= \frac{1}{1-\int_\eta^\infty g(t)\,dt}\left[a^* + \left(2+\int_\eta^\infty (t-1)g(t)\,dt\right)\Lambda_1\right], \\
B &= \frac{1}{1-\int_\eta^\infty g(t)\,dt}\left[b^* + \left(2+\int_\eta^\infty (t-1)g(t)\,dt\right)\Lambda_2\right], \\
\Lambda_1 &= \begin{cases} \phi_p^{-1}(p^*+q^*+c^*), & 1<p<2, \\ 2^{\frac{1}{p-1}}\phi_p^{-1}(p^*+q^*+c^*), & p\ge 2, \end{cases} \\
\Lambda_2 &= \begin{cases} \phi_p^{-1}(r^*+d^*), & 1<p<2, \\ 2^{\frac{1}{p-1}}(\phi_p^{-1}(r^*+d^*)), & p\ge 2. \end{cases}
\end{aligned} \tag{2.17}
$$

Proof Let $x \in P$. From the definition of T, (H_1)-(H_3) and (2.15), we can obtain that T is an operator from P to P, and

$$
\begin{aligned}
\frac{|(Tx)(t)|}{1+t} &= \frac{1}{1+t}\frac{1}{1-\int_\eta^{+\infty} g(t)\,dt} \\
&\quad\times \left|\sum_{t_i<t}\int_{t_{i-1}}^{t_i}\phi_p^{-1}\left(\int_s^{+\infty} a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau - \sum_{t_k\ge s}\bar{I}_k\bigl(x(t_k)\bigr)\right)ds\right. \\
&\quad + \int_\eta^{+\infty} g(t)\int_{t_i}^t \phi_p^{-1}\left(\int_s^{+\infty} a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau - \sum_{t_k\ge s}\bar{I}_k\bigl(x(t_k)\bigr)\right)ds\,dt \\
&\quad \left.+ \sum_{t_i<t} I_i\bigl(x(t_i)\bigr)\right| \\
&\quad + \frac{1}{1+t}\left|\int_{t_i}^t \phi_p^{-1}\left(\int_s^{+\infty} a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau - \sum_{t_k\ge s}\bar{I}_k\bigl(x(t_k)\bigr)\right)ds\right| \\
&\le \frac{1}{1-\int_\eta^{+\infty} g(t)\,dt} \\
&\quad\times \left|\frac{1}{1+t}\int_0^t \phi_p^{-1}\left(\int_0^{+\infty} a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau + \sum_{k=1}^\infty \bar{I}_k\bigl(x(t_k)\bigr)\right)ds\right. \\
&\quad + \int_\eta^{+\infty} g(t)\int_0^t \phi_p^{-1}\left(\int_0^{+\infty} a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau + \sum_{k=1}^\infty \bar{I}_k\bigl(x(t_k)\bigr)\right)ds\,dt \\
&\quad \left.+ \sum_{i=1}^\infty I_i\bigl(x(t_i)\bigr)\right| \\
&\quad + \frac{1}{1+t}\left|\int_{t_i}^t \phi_p^{-1}\left(\int_s^{+\infty} a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau - \sum_{t_k\ge s}\bar{I}_k\bigl(x(t_k)\bigr)\right)ds\right| \\
&\le \frac{1}{1-\int_\eta^{+\infty} g(t)\,dt}\left[\left(2+\int_\eta^{+\infty}(t-1)g(t)\,dt\right)\right. \\
&\quad \left.\times \phi_p^{-1}\left(\int_0^{+\infty} a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau + \sum_{k=1}^\infty \bar{I}_k\bigl(x(t_k)\bigr)\right) + \sum_{i=1}^\infty I_i\bigl(x(t_i)\bigr)\right].
\end{aligned} \tag{2.18}
$$

Therefore,

$$\begin{aligned}\frac{|(Tx)(t)|}{1+t} &\le \frac{1}{1-\int_\eta^{+\infty} g(t)\,dt}\left[\left(2+\int_\eta^{+\infty}(t-1)g(t)\,dt\right)\right.\\ &\quad \times \phi_p^{-1}\big(p^*\phi_p\big(\|x\|_1\big)+q^*\phi_p\big(\|x'\|_\infty\big)+r^*+c^*\phi_p\big(\|x\|_1\big)+d^*\big)\\ &\quad \left.+a^*\big(\|x\|_1\big)+b^*\right]\\ &\le A\|x\|+B, \quad \forall t\in J.\end{aligned}$$

Direct differentiation of T implies, for $t\neq t_k$,

$$(Tx)'(t)=\phi_p^{-1}\left(\int_t^{+\infty}a(\tau)f\big(\tau,x(\tau),x'(\tau)\big)\,d\tau-\sum_{t_k\ge t}\bar{I}_k\big(x(t_k)\big)\right).$$

Thus, we have

$$\big|(Tx)'(t)\big|\le\phi_p^{-1}\left(\int_0^{+\infty}a(\tau)f\big(\tau,x(\tau),x'(\tau)\big)\,d\tau+\sum_{i=1}^{\infty}\bar{I}_k\big(x(t_k)\big)\right)\le A\|x\|+B, \quad \forall t\in J.$$

It follows that (2.15) is satisfied and equation (2.16) is easily obtained by (H_3). □

Lemma 2.9 *Let* (H_2), (H_4), *and* (H_5) *hold. Then* $T:P\to P$ *is completely continuous.*

Proof For any $x\in P$, by (2.14), we have

$$\phi_p\big((Tx)'(t)\big)=\int_t^{+\infty}a(\tau)f\big(\tau,x(\tau),x'(\tau)\big)\,d\tau-\sum_{t_k\ge t}\bar{I}_k\big(x(t_k)\big), \tag{2.19}$$

$$\big(\phi_p\big((Tx)'(t)\big)\big)'=-a(t)f\big(t,x(t),x'(t)\big). \tag{2.20}$$

It follows from (2.14), (2.19) and (H_4) that

$$(Tx)(t)\ge 0, \qquad (Tx)'(t)\ge 0, \qquad (Tx)''(t)\le 0,$$

that is, $T(P)\subset P$. Next, we divide the proof into two steps.

Step 1. We prove that T is continuous.

Let $x_n\to x$ as $n\to\infty$ in P, then there exists r_0 such that $\sup_{n\in N\setminus\{0\}}\|x\|<r_0$. Set $B_{r_0}=\sup\{f(t,(1+t)u,v),(t,u,v)\in J\times[0,r_0]^2\}$, and we have

$$\int_0^{+\infty}a(\tau)\big|f\big(\tau,x_n(\tau),x_n'(\tau)\big)-f\big(\tau,x(\tau),x'(\tau)\big)\big|\,d\tau\le 2B_{r_0}\int_0^{+\infty}a(\tau)\,d\tau<+\infty. \tag{2.21}$$

Therefore, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}&\big|\phi_p\big((Tx_n)'(t)\big)-\phi_p\big((Tx)'(t)\big)\big|\\ &\quad=\left|\int_t^{+\infty}a(\tau)f\big(\tau,x_n(\tau),x_n'(\tau)\big)\,d\tau-\sum_{t_k\ge t}\bar{I}_k(x_n(t_k)\right.\end{aligned}$$

$$
\begin{aligned}
&- \int_t^{+\infty} a(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\, d\tau + \sum_{t_k \geq t} \bar{I}_k\big(x(t_k)\big) \bigg| \\
&\leq \int_t^{+\infty} a(\tau) \big| f\big(\tau, x_n(\tau), x_n'(\tau)\big) - f\big(\tau, x(\tau), x'(\tau)\big) \big|\, d\tau + \sum_{t_k \geq t} \big| \bar{I}_k(x_n(t_k) - \bar{I}_k(x(t_k) \big| \\
&\to 0 \quad (n \to \infty).
\end{aligned}
\tag{2.22}
$$

From above and (H_4), (H_5), we get

$$
\begin{aligned}
&\left| \frac{(Tx_n)(t)}{1+t} - \frac{(Tx_n)(t)}{1+t} \right| \\
&= \frac{1}{1+t} \bigg| \frac{1}{1 - \int_\eta^{+\infty} g(t)\, dt} \\
&\quad \times \bigg[\sum_{t_i < t} \int_{t_{i-1}}^{t_i} \phi_p^{-1} \bigg(\int_s^{+\infty} a(\tau) f\big(\tau, x_n(\tau), x_n'(\tau)\big)\, d\tau - \sum_{t_k \geq s} \bar{I}_k\big(x_n(t_k)\big) \bigg)\, ds \\
&\quad - \sum_{t_i < t} \int_{t_{i-1}}^{t_i} \phi_p^{-1} \bigg(\int_s^{+\infty} a(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\, d\tau - \sum_{t_k \geq s} \bar{I}_k\big(x(t_k)\big) \bigg)\, ds \\
&\quad + \int_\eta^{+\infty} g(t) \int_{t_i}^{t} \phi_p^{-1} \bigg(\int_s^{+\infty} a(\tau) f\big(\tau, x_n(\tau), x_n'(\tau)\big)\, d\tau - \sum_{t_k \geq s} \bar{I}_k\big(x_n(t_k)\big) \bigg)\, ds\, dt \\
&\quad - \int_\eta^{+\infty} g(t) \int_{t_i}^{t} \phi_p^{-1} \bigg(\int_s^{+\infty} a(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\, d\tau - \sum_{t_k \geq s} \bar{I}_k\big(x(t_k)\big) \bigg)\, ds\, dt \\
&\quad + \sum_{t_i < t} I_i\big(x_n(t_i)\big) - \sum_{t_i < t} I_i\big(x(t_i)\big) \bigg] \\
&\quad + \int_{t_i}^{t} \phi_p^{-1} \bigg(\int_s^{+\infty} a(\tau) f\big(\tau, x_n(\tau), x_n'(\tau)\big)\, d\tau - \sum_{t_k \geq s} \bar{I}_k\big(x_n(t_k)\big) \bigg)\, ds \\
&\quad - \int_{t_i}^{t} \phi_p^{-1} \bigg(\int_s^{+\infty} a(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\, d\tau - \sum_{t_k \geq s} \bar{I}_k\big(x(t_k)\big) \bigg)\, ds \bigg| \\
&\leq \frac{1}{1 - \alpha \int_\eta^{+\infty} g(t)\, dt} \bigg[\sum_{t_i < t} \int_{t_{i-1}}^{t_i} \bigg| \phi_p^{-1} \bigg(\int_s^{+\infty} a(\tau) f\big(\tau, x_n(\tau), x_n'(\tau)\big)\, d\tau - \sum_{t_k \geq s} \bar{I}_k\big(x_n(t_k)\big) \bigg) \\
&\quad - \phi_p^{-1} \bigg(\int_s^{+\infty} a(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\, d\tau - \sum_{t_k \geq s} \bar{I}_k\big(x(t_k)\big) \bigg) \bigg|\, ds \\
&\quad + \int_\eta^{+\infty} g(t) \int_{t_i}^{t} \bigg| \phi_p^{-1} \bigg(\int_s^{+\infty} a(\tau) f\big(\tau, x_n(\tau), x_n'(\tau)\big)\, d\tau - \sum_{t_k \geq s} \bar{I}_k\big(x_n(t_k)\big) \bigg) \\
&\quad - \phi_p^{-1} \bigg(\int_s^{+\infty} a(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\, d\tau - \sum_{t_k \geq s} \bar{I}_k\big(x(t_k)\big) \bigg) \bigg|\, ds\, dt \\
&\quad + \sum_{t_i < t} \big| I_i\big(x_n(t_i)\big) - I_i\big(x(t_i)\big) \big| \bigg] \\
&\quad + \int_{t_i}^{t} \bigg| \phi_p^{-1} \bigg(\int_s^{+\infty} a(\tau) f\big(\tau, x_n(\tau), x_n'(\tau)\big)\, d\tau - \sum_{t_k \geq s} \bar{I}_k\big(x_n(t_k)\big) \bigg)
\end{aligned}
$$

$$
\begin{aligned}
&- \phi_p^{-1}\left(\int_s^{+\infty} a(\tau)f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau - \sum_{t_k\geq s}\bar{I}_k\big(x(t_k)\big)\right)\Bigg|\,ds \\
&\to 0 \quad (n\to\infty),
\end{aligned}
$$

and

$$
\begin{aligned}
&\big|(Tx_n)'(t) - (Tx)'(t)\big| \\
&\quad = \left|\phi_p^{-1}\left(\int_t^{+\infty} a(\tau)f\big(\tau, x_n(\tau), x_n'(\tau)\big)\,d\tau - \sum_{t_k\geq t}\bar{I}_k\big(x_n(t_k)\big)\right)\right. \\
&\qquad \left. - \phi_p^{-1}\left(\int_t^{+\infty} a(\tau)f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau - \sum_{t_k\geq t}\bar{I}_k\big(x(t_k)\big)\right)\right| \\
&\quad \to 0 \quad (n\to\infty).
\end{aligned}
$$

We can easily get $\|Tx_n - Tx\| \to 0$ $(n\to\infty)$. Hence, T is continuous.

Step 2. We prove that T is compact provided that it maps bounded sets into relatively compact sets.

First, let Ω be a bounded subset of P, then there exists $r > 0$ such that $\|x\| < r$ for all $x \in \Omega$. By (2.18), we have

$$
\begin{aligned}
\frac{|(Tx)(t)|}{1+t} \leq{}& \frac{1}{1-\int_\eta^{+\infty} g(t)\,dt} \\
&\times \left[\left(2 + \int_\eta^{+\infty}(t-1)g(t)\,dt\right)\phi_p^{-1}\left(B_r\int_0^{+\infty} a(\tau)\,d\tau + c^*\phi_p(r) + d^*\right)\right. \\
&\left. + a^*r + b^*\right] := R_1,
\end{aligned}
$$

and

$$
\begin{aligned}
\big|(Tx)'(t)\big| &= \left|\phi_p^{-1}\left(\int_s^{+\infty} a(\tau)f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau - \sum_{t_k\geq s}\bar{I}_k\big(x(t_k)\big)\right)\right| \\
&\leq \phi_p^{-1}\left(\int_0^{+\infty} a(\tau)f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau + \sum_{k=1}^{\infty}\bar{I}_k\big(x(t_k)\big)\right) \\
&\leq \phi_p^{-1}\left(B_r\int_0^{+\infty} a(\tau)\,d\tau + c^*\phi_p(r) + d^*\right) := R_0,
\end{aligned}
$$

where $R_1 = \frac{1}{1-\int_\eta^{+\infty} g(t)\,dt}[(2+\alpha\int_\eta^{+\infty} g(t)\,dt)R_0 + a^*r + b^*]$. Hence, $\|Tx\| \leq \max\{R_0, R_1\}$. So $T\Omega$ is bounded.

Second, for any $L \in (0, +\infty)$ and $t', t'' \in J_k \cap [0, L]$ with $t' < t''$, we have

$$
\begin{aligned}
&\left|\frac{(Tx)(t')}{1+t'} - \frac{(Tx)(t'')}{1+t''}\right| \\
&\quad = \left|\frac{1}{1+t'}\frac{1}{1-\int_\eta^{+\infty} g(t)\,dt}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times \biggl[\sum_{t_i<t'}\int_{t_{i-1}}^{t_i}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds\\
&+\int_\eta^{+\infty}g(t)\int_{t_i}^{t'}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds\,dt\\
&+\sum_{t_i<t'}I_i\bigl(x(t_i)\bigr)\biggr]\\
&-\frac{1}{1+t''}\frac{1}{1-\int_\eta^{+\infty}g(t)\,dt}\\
&\times \biggl[\sum_{t_i<t''}\int_{t_{i-1}}^{t_i}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds\\
&+\int_\eta^{+\infty}g(t)\int_{t_i}^{t''}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds\,dt\\
&+\sum_{t_i<t''}I_i\bigl(x(t_i)\bigr)\biggr]\\
&+\frac{1}{1+t'}\int_{t_i}^{t'}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds\\
&-\frac{1}{1+t''}\int_{t_i}^{t''}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds\biggr|\\
\leq{}&\frac{1}{1-\int_\eta^{+\infty}g(t)\,dt}\\
&\times \biggl[\sum_{t_i<t''}\int_{t_{i-1}}^{t_i}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds\\
&+\sum_{t_i<t''}I_i\bigl(x(t_i)\bigr)\biggr]\biggl|\frac{1}{1+t'}-\frac{1}{1+t''}\biggr|\\
&+\frac{1}{(1+t')(1-\alpha\int_\eta^{+\infty}g(t)\,dt)}\\
&\times\int_\eta^{+\infty}g(t)\int_{t'}^{t''}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds\,dt\\
&+\biggl|\frac{1}{1+t'}\int_{t_i}^{t'}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds\\
&-\frac{1}{1+t''}\int_{t_i}^{t''}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds\biggr|\\
\leq{}&\frac{1}{1-\int_\eta^{+\infty}g(t)\,dt}\\
&\times \biggl[\sum_{t_i<t''}\int_{t_{i-1}}^{t_i}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{t_i<t''} I_i\bigl(x(t_i)\bigr)\Biggr]\left|\frac{1}{1+t'}-\frac{1}{1+t''}\right|+\frac{1}{1-\alpha\int_\eta^{+\infty}g(t)\,dt}\\
&\times\int_\eta^{+\infty}g(t)\int_{t'}^{t''}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds\,dt\\
&+\int_0^{t''}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds\left|\frac{1}{1+t'}-\frac{1}{1+t''}\right|\\
\leq{}&\left[\frac{1}{1-\int_\eta^{+\infty}g(t)\,dt}\bigl(t''R_0+a^*r+b^*\bigr)+t''R_0\right]\left|\frac{1}{1+t'}-\frac{1}{1+t''}\right|\\
&+\frac{R_0}{1-\int_\eta^{+\infty}g(t)\,dt}\int_\eta^{+\infty}\bigl(t'-t''\bigr)g(t)\,dt\to 0,\quad \text{uniformly as } t'\to t'',
\end{aligned}
$$

and

$$
\begin{aligned}
&\bigl|\phi_p(Tx)'\bigl(t'\bigr)-\phi_p(Tx)'\bigl(t''\bigr)\bigr|\\
&=\biggl|\int_{t'}^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq t'}\bar{I}_k\bigl(x(t_k)\bigr)\\
&\quad-\int_{t''}^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau+\sum_{t_k\geq t''}\bar{I}_k\bigl(x(t_k)\bigr)\biggr|\\
&\leq\int_{t'}^{t''}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau+\sum_{t'\leq t_k\leq t''}\bar{I}_k\bigl(x(t_k)\bigr)\\
&\leq B_r\int_{t'}^{t''}a(\tau)\,d\tau+\sum_{t'\leq t_k\leq t''}\bigl(c_k\phi_p(r)+d_k\bigr)\to 0,\quad \text{uniformly as } t'\to t'',
\end{aligned}
$$

for all $x\in\Omega$. So $T\Omega$ is equicontinuous on any compact interval of J_k $(k=1,2,\ldots)$.

Third, we prove that for any $\varepsilon>0$, $x\in\Omega$, there exists sufficiently large $N>0$ such that

$$
\left|\frac{(Tx)(t')}{1+t'}-\frac{(Tx)(t'')}{1+t''}\right|<\varepsilon,\qquad \bigl|(Tx)'\bigl(t'\bigr)-(Tx)'\bigl(t'\bigr)\bigr|<\varepsilon,\quad \forall t',t''>N.
$$

For any $x\in\Omega$, we have

$$
\begin{aligned}
&\lim_{t\to+\infty}\frac{1}{1+t}\int_{t_i}^{t}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds\\
&=\lim_{t\to+\infty}\phi_p^{-1}\biggl(\int_t^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq t}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\\
&=0
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{t\to+\infty}\frac{1}{1+t}\frac{1}{1-\int_\eta^{+\infty}g(t)\,dt}\\
&\quad\times\Biggl[\sum_{t_i<t}\int_{t_{i-1}}^{t_i}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\geq s}\bar{I}_k\bigl(x(t_k)\bigr)\biggr)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_\eta^{+\infty} g(t)\int_{t_i}^t \phi_p^{-1}\left(\int_s^{+\infty} a(\tau)f\left(\tau,x(\tau),x'(\tau)\right)d\tau-\sum_{t_k\geq s}\bar{I}_k\left(x(t_k)\right)\right)ds\,dt\\
&+\sum_{t_i<t} I_i\left(x(t_i)\right)\Bigg]\\
\leq{}& \lim_{t\to+\infty}\frac{1}{1+t}\frac{1}{1-\int_\eta^{+\infty}g(t)\,dt}\\
&\times\left[\int_{t_i}^t \phi_p^{-1}\left(\int_s^{+\infty} a(\tau)f\left(\tau,x(\tau),x'(\tau)\right)d\tau-\sum_{t_k\geq s}\bar{I}_k\left(x(t_k)\right)\right)ds\right.\\
&\left.+R_0\int_\eta^{+\infty} tg(t)\,dt+a^*r+b^*\right]\\
={}&0.
\end{aligned}
$$

Hence, we obtain that

$$\lim_{t\to+\infty}\left|\frac{(Tx)(t)}{1+t}\right|=0,$$

and

$$
\begin{aligned}
&\lim_{t\to+\infty}\left|(Tx)'(t)\right|\\
&\quad=\lim_{t\to+\infty}\phi_p^{-1}\left(\int_t^{+\infty} a(\tau)f\left(\tau,x(\tau),x'(\tau)\right)d\tau-\sum_{t_k\geq t}\bar{I}_k\left(x(t_k)\right)\right)ds=0.
\end{aligned}
$$

So $T\Omega$ is equiconvergent at infinity. By Lemma 2.4, we obtain $T\Omega$ is relatively compact, that is, T is a compact operator.

Therefore, $T:P\to P$ is completely continuous. The proof is complete. □

Remark 2.10 Similarly, we may prove that when (H_1) and (H_2) hold, then $T:P\to P$ is completely continuous.

3 Main result

Theorem 3.1 *Let conditions* (H_1)-(H_3) *be satisfied. Suppose further that* $A<1$. *Then IBVP* (1.1) *has the minimal nonnegative solution* $\overline{x}$ *with* $\|\overline{x}\|\leq\frac{B}{1-A}$, *where A and B are defined as in Lemma* 2.8. *Moreover, if we let* $x_0(t)=0$, $x_n(t)=(Tx_{n-1}(t))$ *for all* $t\in J$ $(n=1,2,\ldots)$, *then* $x_n(t)\in P$ *with*

$$0=x_0(t)\leq x_1(t)\leq\cdots\leq x_n(t)\leq\overline{x}(t),\quad\forall t\in J, \tag{3.1}$$

$$0=x_0'(t)\leq x_1'(t)\leq\cdots\leq x_n'(t)\leq\overline{x}'(t),\quad\forall t\in J, \tag{3.2}$$

and $\{x_n(t)\}$ *and* $\{x_n'(t)\}$ *converge uniformly to* $\overline{x}(t)$ *and* $\overline{x}'(t)$ *on* J_i $(i=1,2,\ldots)$, *respectively.*

Proof By Lemma 2.8 and the definition of operator T, we have $x_n(t)\in P$ and

$$\|x_n\|\leq A\|x_{n-1}\|+B,\quad n=1,2,\ldots, \tag{3.3}$$

$$0 = x_0(t) \le x_1(t) \le \cdots \le x_n(t) \le \cdots, \quad \forall t \in J, \tag{3.4}$$

$$0 = x_0'(t) \le x_1'(t) \le \cdots \le x_n'(t) \le \cdots, \quad \forall t \in J. \tag{3.5}$$

By (3.3), we can get

$$\begin{aligned} \|x_n\| &\le A\|x_{n-1}\| + B \le A\big(A\|x_{n-2}\| + B\big) + B = A^2\|x_{n-2}\| + AB + B \\ &\le A^2\big(A\|x_{n-3}\| + B\big) + AB + B = A^3\|x_{n-3}\| + A^2B + AB + B \le \cdots \\ &\le A^n\|x_0\| + A^{n-1}B + A^{n-2}B + \cdots + AB + B \\ &= \frac{B(1-A^n)}{1-A} \le \frac{B}{1-A} \quad (n = 1,2,\ldots). \end{aligned} \tag{3.6}$$

From (3.4)-(3.6), we know that $\lim_{n\to+\infty} x_n(t)$ and $\lim_{n\to+\infty} x_n'(t)$ exist. Suppose that

$$\lim_{n\to+\infty} x_n(t) = \overline{x}(t), \qquad \lim_{n\to+\infty} x_n'(t) = h(t), \quad \forall t \in J. \tag{3.7}$$

According to the definition of $x_n(t)$, we have

$$\begin{aligned} & x_n'(t) = \phi_p^{-1}\left(\phi_p(x_\infty) + \int_t^{+\infty} f\big(\tau, x_{n-1}(\tau), x_{n-1}'(\tau)\big)\,d\tau - \sum_{t_k \ge t} \overline{I}_k\big(x(t_k)\big)\right), \\ & \quad \forall t \in J, t \ne t_k\ (n = 1,2,\ldots), \end{aligned} \tag{3.8}$$

$$\big(\phi_p\big(x_n'(t)\big)\big)' = -a(t)f\big(t, x_{n-1}(t), x_{n-1}'(t)\big), \quad \forall t \in J, t \ne t_k\ (n = 1,2,\ldots). \tag{3.9}$$

From (3.6), we obtain

$$\frac{|x_n(t)|}{1+t} \le \frac{B}{1-A}, \qquad \big|x_n'(t)\big| \le \frac{B}{1-A}, \quad \forall t \in J, t \ne t_k\ (n = 1,2,\ldots).$$

It follows that $x_n(t)$ is equicontinuous on every J_i $(i = 0,1,2,\ldots)$. Combining this with the Ascoli-Arzela theorem and diagonal process, there exists a subsequence which converges uniformly to $\overline{x}$ on J_i $(i = 0,1,2,\ldots)$, which together with (3.4) imply that $x_n(t)$ converges uniformly to $\overline{x}(t)$ on J_i $(i = 0,1,2,\ldots)$, and $\overline{x}(t) \in PC[J,R]$, $\|\overline{x}\|_1 \le \frac{B}{1-A}$. On the other hand, by (H_1), (3.6) and (3.9), we have

$$\begin{aligned} \big|x_n''(t)\big| &\le a(\tau)\big(p(t)(t+1)^{p-1}\|\overline{x}_{n-1}\|_1 + q(t)\big\|\overline{x}_{n-1}'\big\|_\infty + r(t)\big) \\ &\le a(\tau)\left(p(t)(t+1)^{p-1}\frac{B}{1-A} + q(t)\frac{B}{1-A} + r(t)\right) \\ &= s(t) \in C(J, J_+), \quad \forall t \in J'\ (n = 1,2,\ldots). \end{aligned}$$

Since $s(t)$ is bounded on $[0,M]$ (M is a finite positive number), $x_n'(t)$ is equicontinuous on every J_i $(i = 0,1,2,\ldots)$. Combining this with the Ascoli-Arzela theorem and diagonal process, there exists a subsequence which converges uniformly to $h(t)$ on J_i $(i = 0,1,2,\ldots)$, which together with (3.5) imply that $x_n'(t)$ converges uniformly to $h(t)$ on J_i $(i = 0,1,2,\ldots)$, and $h(t) \in PC[J,R]$, $\|h\|_\infty \le \frac{B}{1-A}$. From above, we know that $\overline{x}'(t)$ exists and $\overline{x}'(t) = h(t)$ for all $t \in J$. It follows that $\overline{x} \in P$ and $\|\overline{x}\| \le \frac{B}{1-A}$. Now taking limits from two sides of

$x_n(t) = (Tx_{n-1})(t)$, we have $\overline{x}(t) = (T\overline{x})(t)$, that is, T has a fixed point. By Lemma 2.7, $\overline{x}(t)$ is a nonnegative solution of IBVP (1.1).

Suppose that $x \in P \cap C^2[J,R]$ is an arbitrary nonnegative solution of IBVP (1.1). Then $x(t) = (Tx)(t)$. It is clear that $x(t) \geq 0$, $x'(t) \geq 0$, $\forall t \in J$. Suppose that $x(t) \geq x_{n-1}(t)$, $x'(t) \geq x'_{n-1}(t)$ for $t \in J$. By (2.14), we have $(Tx)(t) \geq (Tx_{n-1})(t)$, $(Tx)'(t) \geq (Tx_{n-1})'(t)$ for all $t \in J$. This means that $x(t) \geq x_n(t)$, $x'(t) \geq x'_n(t)$ for all $t \in J$ $(n = 1, 2, \ldots)$. Taking limit, we have $x(t) \geq \overline{x}(t)$, $x'(t) \geq \overline{x}'(t)$ for all $t \in J$. The proof of Theorem 3.1 is complete. □

Next, for notational convenience, we denote that

$$m = 2^{\frac{2}{p-1}} \frac{1}{1 - \int_\eta^\infty g(t)\,dt} \left(2 + \int_\eta^\infty (t-1)g(t)\,dt\right) \phi_p^{-1}\left(\int_0^\infty a(\tau)\,d\tau\right), \tag{3.10}$$

$$m' = \frac{1}{1 - \int_\eta^\infty g(t)\,dt} \left(2 + \int_\eta^\infty (t-1)g(t)\,dt\right) \phi_p^{-1}\left(\int_0^\infty a(\tau)\,d\tau\right), \tag{3.11}$$

$$n = 2^{\frac{2}{p-1}} \frac{1}{1 - \int_\eta^\infty g(t)\,dt} \left(2 + \int_\eta^\infty (t-1)g(t)\,dt\right), \tag{3.12}$$

$$n' = \frac{1}{1 - \int_\eta^\infty g(t)\,dt} \left(2 + \int_\eta^\infty (t-1)g(t)\,dt\right), \tag{3.13}$$

$$\Lambda = \max\left\{\frac{b^*}{1 - \int_\eta^\infty g(t)\,dt - 3a^*}, \frac{n\phi_p^{-1}(d^*)}{1 - 3n\phi_p^{-1}(c^*)}\right\}, \tag{3.14}$$

$$\Lambda' = \max\left\{\frac{b^*}{1 - \int_\eta^\infty g(t)\,dt - 3a^*}, \frac{n'\phi_p^{-1}(d^*)}{1 - 3n'\phi_p^{-1}(c^*)}\right\}. \tag{3.15}$$

Theorem 3.2 *Assume that* (H_2)-(H_5) *hold, and there exists*

$$d \geq \begin{cases} 3\Lambda, & p \geq 2, \\ 3\Lambda', & 1 < p < 2, \end{cases}$$

such that

(A_1)
$$f\bigl(t, (1+t)u, v\bigr) \leq \begin{cases} \phi_p(\frac{d}{3m}), & p \geq 2, \\ \phi_p(\frac{d}{3m'}), & 1 < p < 2, \end{cases} \quad \text{for } (t,u,v) \in [0,+\infty) \times [0,d] \times [0,d].$$

Then IBVP (1.1) *admits positive, nondecreasing on* $[0, +\infty)$ *and concave solutions* w^* *and* v^* *such that* $0 < \|w^*\| \leq d$, *and* $\lim_{n\to+\infty} w_n = \lim_{n\to+\infty} A^n w_0 = w^*$, *where*

$$w_0(t) = d + dt, \quad t \in J, \tag{3.16}$$

and $0 < \|v^*\| \leq d$, *and* $\lim_{n\to+\infty} v_n = \lim_{n\to+\infty} A^n v_0 = v^*$, *where* $v_0(t) = 0$, $t \in J$.

Proof We only prove the case that $p \geq 2$, another case can be proved in a similar way. By Lemma 2.9, we know that $T : P \to P$ is completely continuous. From the definition of T and (H_3), we can easily get that $Tx_1 \leq Tx_2$ for any $x_1, x_2 \in P$ with $x_1 \leq x_2$, $x'_1 \leq x'_2$. Denote that

$$\overline{P}_d = \bigl\{x \in P \mid \|x\| \leq d\bigr\}. \tag{3.17}$$

In what follows, we first prove that $T:\overline{P}_d \to \overline{P}_d$. If $x \in \overline{P}_d$, then $\|x\| \le d$. By (1.3), (2.18), (3.10), (3.12) and (3.14), we get that

$$\begin{aligned}
\frac{|(Tx)(t)|}{1+t} &\le \frac{1}{1-\int_\eta^{+\infty} g(t)\,dt}\biggl[2^{\frac{2}{p-1}}\biggl(2+\int_\eta^{+\infty}(t-1)g(t)\,dt\biggr) \\
&\quad\times\biggl(\phi_p^{-1}\biggl(\phi_p\biggl(\frac{d}{3m}\biggr)\biggr)\phi_p^{-1}\biggl(\int_0^{+\infty}a(\tau)\,d\tau\biggr)+\phi_p^{-1}\bigl(c^*\bigr)d+\phi_p^{-1}\bigl(d^*\bigr)\biggr) \\
&\quad+\bigl(a^*d+b^*\bigr)\biggr] \\
&\le \frac{d}{3}+\frac{d}{3}+\frac{d}{3}=d,
\end{aligned}$$

and

$$\begin{aligned}
\bigl|(Tx)'(t)\bigr| &= \biggl|\phi_p^{-1}\biggl(\int_t^{+\infty}a(\tau)f\bigl(\tau,x(\tau),x'(\tau)\bigr)\,d\tau-\sum_{t_k\ge t}\overline{I}_k\bigl(x(t_k)\bigr)\biggr)\biggr| \\
&\le 2^{\frac{2}{p-1}}\biggl(\phi_p^{-1}\biggl(\phi_p\biggl(\frac{d}{3m}\biggr)\biggr)+\phi_p^{-1}\bigl(c^*\bigr)d+\phi_p^{-1}\bigl(d^*\bigr)\biggr)<d.
\end{aligned}$$

Thus, we get that $\|Tx\| \le d$. Hence, we have proved that $T:\overline{P}_d \to \overline{P}_d$.

Let $w_0(t)=d+dt$, $0\le t<+\infty$, then $w_0(t)\in\overline{P}_d$. Let $w_1(t)=Tw_0(t)$, $w_2(t)=T^2w_0(t)$, then by Lemma 2.9, we have $w_1(t)\in\overline{P}_d$ and $w_2(t)\in\overline{P}_d$. Denote that

$$w_{n+1}(t)=Tw_n(t)=T^{n+1}w_0(t),\quad n=0,1,2,\ldots. \tag{3.18}$$

Since $T:\overline{P}_d\to\overline{P}_d$, we have that

$$w_n(t)\in T(\overline{P}_d)\subset\overline{P}_d,\quad n=1,2,\ldots. \tag{3.19}$$

It follows from the complete continuity of T that $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$, and there exists $w^*\in\overline{P}_d$ such that $w_{n_k}\to w^*$.

By (3.18), (H_3) and (A_1), we get that

$$\begin{aligned}
w_1(t) &= \frac{1}{1-\int_\eta^{+\infty}g(t)\,dt} \\
&\quad\times\biggl[\sum_{t_i<t}\int_{t_{i-1}}^{t_i}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,w_0(\tau),w_0'(\tau)\bigr)\,d\tau-\sum_{t_k\ge s}\overline{I}_k\bigl(w_0(t_k)\bigr)\biggr)\,ds \\
&\quad+\int_\eta^{+\infty}g(t)\int_{t_i}^{t}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,w_0(\tau),w_0'(\tau)\bigr)\,d\tau-\sum_{t_k\ge s}\overline{I}_k\bigl(w_0(t_k)\bigr)\biggr)\,ds\,dt \\
&\quad+\sum_{t_i<t}I_i\bigl(w_0(t_i)\bigr)\biggr] \\
&\quad+\int_{t_i}^{t}\phi_p^{-1}\biggl(\int_s^{+\infty}a(\tau)f\bigl(\tau,w_0(\tau),w_0'(\tau)\bigr)\,d\tau-\sum_{t_k\ge s}\overline{I}_k\bigl(w_0(t_k)\bigr)\biggr)\,ds \\
&\le \frac{1}{1-\int_\eta^{+\infty}g(t)\,dt}\biggl[\biggl(t+\int_\eta^{+\infty}tg(t)\,dt\biggr)
\end{aligned}$$

$$
\begin{aligned}
& \times \phi_p^{-1}\left(\int_0^{+\infty} a(\tau) f\left(\tau, w_0(\tau), w_0'(\tau)\right) d\tau + \sum_{k=1}^{\infty} \bar{I}_k\left(w_0(t_k)\right)\right)\Bigg] \\
& + t\phi_p^{-1}\left(\int_0^{+\infty} a(\tau) f\left(\tau, w_0(\tau), w_0'(\tau)\right) d\tau + \sum_{k=1}^{\infty} \bar{I}_k\left(w_0(t_k)\right)\right) \\
& + \frac{1}{1-\int_\eta^{+\infty} g(t)\,dt}\left(a^* d + b^*\right) \\
\leq {} & \frac{1}{1-\int_\eta^{+\infty} g(t)\,dt}\Bigg[\left(2t + \int_\eta^{+\infty} (t-1) g(t)\,dt\right) \\
& \times \phi_p^{-1}\left(\int_0^{+\infty} a(\tau)\phi_p\left(\frac{d}{3m}\right) d\tau + c^*\phi_p(d) + d^*\right)\Bigg] \\
& + \frac{1}{1-\alpha\int_\eta^{+\infty} g(t)\,dt}\left(a^* d + b^*\right) \\
\leq {} & \frac{1}{1-\int_\eta^{+\infty} g(t)\,dt}\Bigg[2^{\frac{2}{p-1}}\left(2t + \int_\eta^{+\infty} (t-1) g(t)\,dt\right) \\
& \times \left(\phi_p^{-1}\left(\phi_p\left(\frac{d}{3m}\right)\right)\phi_p^{-1}\left(\int_0^{+\infty} a(\tau)\,d\tau\right) + \phi_p^{-1}\left(c^*\right) d + \phi_p^{-1}\left(d^*\right)\right)\Bigg] \\
& + \frac{1}{1-\int_\eta^{+\infty} g(t)\,dt}\left(a^* d + b^*\right) \\
\leq {} & d + dt = w_0(t),
\end{aligned}
$$

and

$$
\begin{aligned}
w_1'(t) &= (Tw_0)'(t) \\
&= \phi_p^{-1}\left(\int_t^{+\infty} a(\tau) f\left(\tau, w_0(\tau), w_0'(\tau)\right) d\tau - \sum_{t_k \geq t} \bar{I}_k\left(w_0(t_k)\right)\right) \\
&\leq 2^{\frac{2}{p-1}}\left(\phi_p^{-1}\left(\phi_p\left(\frac{d}{3m}\right)\right)\phi_p^{-1}\left(\int_0^{+\infty} a(\tau)\,d\tau\right)\phi_p^{-1}\left(c^*\right) d + \phi_p^{-1}\left(d^*\right)\right) \\
&\leq d = w_0'(t), \quad 0 \leq t < +\infty.
\end{aligned}
$$

So, by (3.18), (H_3) and (A_1), we have

$$
w_2(t) = (Tw_1)(t) \leq (Tw_0)(t) = w_1(t), \quad 0 \leq t < +\infty, \tag{3.20}
$$

$$
w_2'(t) = (Tw_1)'(t) \leq (Tw_0)'(t) = (w_1)'(t), \quad 0 \leq t < +\infty. \tag{3.21}
$$

By induction, we see

$$
w_{n+1}(t) \leq w_n(t), \qquad w_{n+1}'(t) \leq (w_n)'(t), \quad 0 \leq t < +\infty, n = 0, 1, 2, \ldots. \tag{3.22}
$$

Hence, we claim that $w_n \to w^*$ as $n \to \infty$. Applying the continuity of T and $w_{n+1}(t) = Tw_n(t)$, we know $Tw^* = w^*$. Let $v_0 = 0$, $0 \leq t < +\infty$, then $v_0(t) \in \overline{P}_d$. Let $v_1 = Tv_0$, $v_2 = T^2 v_0$. By Lemma 2.9, we get $v_1 \in \overline{P}_d$ and $v_2 \in \overline{P}_d$. Denote

$$
v_{n+1} = Tv_n = T^{n+1} v_0, \quad n = 0, 1, 2, \ldots. \tag{3.23}
$$

Since $T : \overline{P}_d \to \overline{P}_d$, we have $v_n \in T(\overline{P}_d) \subset \overline{P}_d$, $n = 1, 2, \ldots$. It follows from the complete continuity of T that $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set. Furthermore, we assert that $\{v_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$, and there exists $v^* \in \overline{P}_d$ such that $v_{n_k} \to v^*$.

For $v_1 = Tv_0 \in \overline{P}_d$, we obtain

$$v_1(t) = (Tv_0)(t) = (T_0)(t) \geq 0, \quad 0 \leq t < +\infty, \tag{3.24}$$

$$v_1'(t) = (Tv_0)'(t) = (T_0)' \geq 0, \quad 0 \leq t < +\infty. \tag{3.25}$$

By (H_3) and (A_1), we have

$$v_2(t) = (Tv_1)(t) \geq (Tv_0)(t) = v_1(t), \quad 0 \leq t < +\infty, \tag{3.26}$$

$$v_2'(t) = (Tv_1)'(t) \geq (Tv_0)'(t) = (v_1)'(t), \quad 0 \leq t < +\infty. \tag{3.27}$$

By induction, we see

$$v_{n+1}(t) \geq v_n(t), \qquad v_{n+1}'(t) \geq (v_n)'(t), \quad 0 \leq t < +\infty, n = 0, 1, 2, \ldots. \tag{3.28}$$

Hence, we claim that $v_n \to v^*$ as $n \to \infty$. Applying the continuity of T and $v_{n+1}(t) = Tv_n(t)$, we know $Tv^* = v^*$.

Since $f(t, 0, 0) \not\equiv 0$, $0 \leq t < +\infty$, then the zero function is not the solution of IBVP (1.1). Thus, v^* is a positive solution of IBVP (1.1). By Lemma 2.7, we know that w^* and v^* are positive, nondecreasing on $[0, \infty)$ and concave solutions of IBVP (1.1).

We can easily get that the theorem holds for $1 < p < 2$ in a similar way. □

Theorem 3.3 *Assume that* (H_2)-(H_5) *hold, and there exists*

$$d_n > d_{n-1} > \cdots > d_1 \geq \begin{cases} 3\Lambda, & p \geq 2, \\ 3\Lambda', & 1 < p < 2, \end{cases}$$

such that

(A_2)

$$f\bigl(t, (1+t)u, v\bigr) \leq \begin{cases} \phi_p(\frac{d_k}{3m}), & p \geq 2, \\ \phi_p(\frac{d_k}{3m'}), & 1 < p < 2, \end{cases}$$

for $(t, u, v) \in [0, +\infty) \times [0, d_k] \times [0, d_k]$.

Then the boundary value problem (1.3) *admits positive, nondecreasing on* $[0, +\infty)$ *and concave solutions* w_k^* *and* v_k^* *such that* $0 < \|w_k^*\| \leq d_k$, *and* $\lim_{n\to+\infty} w_{kn} = \lim_{n\to+\infty} A^n w_{k0} = w_k^*$, *where*

$$w_0(t) = d_k + d_k t, \quad t \in J, \tag{3.29}$$

and $0 < \|v_k^*\| \leq d$, *and* $\lim_{n\to+\infty} v_{kn} = \lim_{n\to+\infty} A^n v_{k0} = v_k^*$, *where* $v_0(t) = 0$, $t \in J$.

Proof It is similar to the proof of Theorem 3.2. □

4 Example

Example 4.1 Consider the following IBVP for double impulsive differential equation with p-Laplacian on an infinite interval:

$$\begin{cases} (|x'|x')' + e^{-t}[\frac{\ln(1+\phi_p(x))}{100(1+t)^2} + \frac{e^t \arctan(\phi_p(x'))}{100(1+t^2)} + 1] = 0, & t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = \frac{1}{4^k}(x(t_k)+1)^{\frac{3}{16}}, & t_k = 2^k, k = 1,2,\ldots, \\ \Delta \phi_p(x')|_{t=t_k} = \frac{1}{5^k}(\frac{\ln(\phi_p(x(t_k)))}{25(1+2^k)^2} + 1), & t_k = 2^k, k = 1,2,\ldots, \\ x(0) = \int_1^{+\infty} \frac{1}{2}e^{-2t}x(t)\,dt, & x'(+\infty) = 0. \end{cases} \tag{4.1}$$

Here, $p = 3$, $a(t) = e^{-t}$, $f(t,x(t),x'(t)) = \frac{\ln(1+\phi_p(x))}{100(1+t)^2} + \frac{e^t \arctan(\phi_p(x'))}{100(1+t^2)} + 1$, $I_k(x(t_k)) = \frac{1}{4^k}(x(t_k)+1)^{\frac{3}{16}}$, $\overline{I}_k(x(t_k)) = \frac{1}{5^k}(\frac{\ln(\phi_p(x(t_k)))}{25(1+2^k)^2} + 1)$, $g(t) = \frac{1}{2}e^{-2t}$, $\eta = 1$. Evidently, $x(t) = 0$ is not the solution of IBVP (4.1).

It is clear that $\int_1^{+\infty} \frac{1}{2}e^{-2t}\,dt < 1$ and (H_3) is satisfied. Since

$$\begin{aligned} &f\bigl(t,x(t),x'(t)\bigr) \leq \frac{1}{100(1+t)^2}\phi_p(x) + \frac{e^t}{100(1+t^2)}\phi_p\bigl(x'\bigr) + 1, \\ &I_k(x) \leq \frac{3}{16\cdot 4^k}x + \frac{1}{4^k}, \qquad \overline{I}_k(x) \leq \frac{1}{25(1+2^k)^2 5^k}\phi_p(x) + \frac{1}{5^k}, \quad k = 1,2,\ldots. \end{aligned}$$

So we have

$$\begin{aligned} &p(t) = \frac{1}{100(1+t)^2}, \qquad q(t) = \frac{e^t}{100(1+t^2)}, \qquad r(t) = 1, \\ &a_k = \frac{3}{16\cdot 4^k}, \qquad b_k = \frac{1}{4^k}, \qquad c_k = \frac{1}{25(1+2^k)^2 5^k}, \qquad d_k = \frac{1}{5^k}. \end{aligned}$$

Then we easily obtain that

$$\begin{aligned} &a^* = \sum_{k=1}^{\infty}(t_k+1)a_k = \frac{1}{4}, \qquad b^* = \sum_{k=1}^{\infty} b_k = \frac{1}{3}, \qquad c^* = \sum_{k=1}^{\infty}(t_k+1)^2 c_k = \frac{1}{100}, \\ &d^* = \sum_{k=1}^{\infty} d_k = \frac{1}{4}, \qquad p^* = \int_0^{+\infty} a(t)p(t)(1+t)^{p-1}\,dt = \frac{1}{100}, \\ &q^* = \int_0^{+\infty} a(t)q(t)\,dt = \frac{\pi}{200}, \qquad r^* = \int_0^{+\infty} a(t)r(t)\,dt = 1, \\ &\int_\eta^{+\infty} g(t)\,dt = \frac{1}{4e^2}, \qquad \int_\eta^{+\infty} (t-1)g(t)\,dt = \frac{1}{8e^2}. \end{aligned}$$

Thus, (H_1) and (H_2) are satisfied. Clearly, $A \approx 0.8166 < 1$. By Theorem 3.1, we obtain that BVP (4.1) has a minimal positive solution $\overline{x}$ and $\|\overline{x}\| \leq 19.9189$.

Example 4.2 Consider the following IBVP for double impulsive differential equation with p-Laplacian on an infinite interval:

$$\begin{cases} (|x'|x')' + e^{-4t}f(t,x(t),x'(t)) = 0, & t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = \frac{1}{2^k}(\frac{1}{2^k}x(t_k)+1)^{\frac{1}{12}}, & t_k = 2^k, k = 1,2,\ldots, \\ \Delta \phi_p(x')|_{t=t_k} = \frac{1}{52\cdot 14^k}(\frac{\arctan(\phi_p(x(t_k)))}{(1+2^k)^2} + 1), & t_k = 2^k, k = 1,2,\ldots, \\ x(0) = \int_1^{+\infty} \frac{4}{(1+t)^4}x(t)\,dt, & x'(+\infty) = 0. \end{cases} \tag{4.2}$$

Here,

$$f(t,u,v)=\begin{cases}\frac{1}{25}|\sin(t)|+\frac{1}{100}(\frac{u}{1+t})^4+\frac{v}{100}, & u\le 4,\\ \frac{1}{25}|\sin(t)|+\frac{1}{100}(\frac{1}{1+t})^4+\frac{v}{100}, & u\ge 4.\end{cases} \tag{4.3}$$

It is clear that (H_2) and (H_4) hold for $p=3$, $a(t)=e^{-4t}$, $g(t)=4/(1+t)^4$. By direct computation, we obtain that

$$\int_0^{+\infty} a(t)\,dt=\frac{1}{4},\qquad \int_0^{+\infty}\phi_p\left(\int_t^{+\infty}a(s)\,ds\right)dt=\frac{1}{4},$$

which implies that (H_5) holds.

Obviously,

$$a_k=\frac{1}{12\cdot 4^k},\qquad b_k=\frac{1}{2^k},\qquad c_k=\frac{1}{52(1+2^k)^2 14^k},\qquad d_k=\frac{1}{52\cdot 14^k}.$$

Hence, we can obtain that

$$\begin{aligned}&a^*=\sum_{k=1}^{\infty}(t_k+1)a_k=\frac{1}{9},\qquad b^*=\sum_{k=1}^{\infty}b_k=1,\qquad c^*=\sum_{k=1}^{\infty}(t_k+1)^2c_k=\frac{1}{676},\\&d^*=\sum_{k=1}^{\infty}d_k=\frac{1}{676},\qquad \int_\eta^{+\infty}g(t)\,dt=\frac{1}{6},\qquad \int_\eta^{+\infty}(t-1)g(t)\,dt=\frac{1}{6},\\&m=\frac{13}{5},\qquad n=\frac{26}{5},\qquad \Lambda=2.\end{aligned}$$

Take $d=13$. In this case, we have

$$\phi_p\left(\frac{d}{3m}\right)=\phi_p\left(\frac{5}{3}\right)=\frac{25}{9}.$$

On the other hand, nonlinear term f satisfies

$$f\bigl(t,(1+t)u,v\bigr)\le\frac{1}{25}+\frac{256}{100}+\frac{13}{100}=\frac{273}{100}<\phi_p\left(\frac{d}{3m}\right),\quad t\in J, u,v\in[0,13],$$

which means that (A_2) holds. Thus, we have checked that all the conditions of Theorem 3.2 are satisfied. Therefore, we obtain that IBVP (4.2) has two iteration positive solutions.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

Each of the authors, CY, JW and YG, contributed to each part of this work equally and read and approved the final version of the manuscript.

Acknowledgements

Supported by the Natural Science Foundation of China (11201112), the Natural Science Foundation of Hebei Province (A2013208147), (A2014208152) and (A2015208114), the Foundation of Hebei Education Department (Z2014062) and (QN2015175).

Received: 7 May 2015 Accepted: 10 August 2015 Published online: 28 August 2015

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