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# Universal attractor for nonlinear one-dimensional compressible and radiative MHD flow

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**Abstract**

This paper is concerned with the existence of universal attractors in $H_+^i$ ($i = 1, 2$) for one-dimensional compressible and radiative magnetohydrodynamics equations in a bounded domain $\Omega := (0, 1)$. In this paper, the author extends the results in (Qin *et al.* in J. Differ. Equ. 253:1439-1488, 2012).

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**Keywords:** magnetohydrodynamics (MHD); thermal radiation; absorbing set; universal attractor

## 1 Introduction

In this paper, we study the existence of universal attractors to the one-dimensional compressible thermally radiative magnetohydrodynamic equations.

Magnetohydrodynamics (MHD) is concerned with the study of the interaction between magnetic fields and fluid conductors of electricity. The applications of magnetohydrodynamics cover a very wide range of physical areas from liquid metals to cosmic plasmas, for example, the intensely heated and ionized fluids in an electromagnetic field in astrophysics, geophysics, high-speed aerodynamics, and plasma physics. In addition to these situations, we also take into account the effect of the radiation field. The motion mentioned above is described by the following equations in the Lagrangian coordinate system:

$$\tau_t - u_x = 0, \tag{1.1}$$

$$u_t + \left(p + \frac{1}{2}|\mathbf{b}|^2\right)_x = \left(\frac{\lambda u_x}{\tau}\right)_x, \tag{1.2}$$

$$\mathbf{w}_t - \mathbf{b}_x = \left(\frac{\mu \mathbf{w}_x}{\tau}\right)_x, \tag{1.3}$$

$$(\tau \mathbf{b})_t - \mathbf{w}_x = \left(\frac{\nu \mathbf{b}_x}{\tau}\right)_x, \tag{1.4}$$

$$E_t + \left(u\left(p + \frac{1}{2}|\mathbf{b}|^2\right) - \mathbf{w}\cdot\mathbf{b}\right)_x = \left(\frac{\lambda u u_x + \mu \mathbf{w}\cdot\mathbf{w}_x + \nu \mathbf{b}\cdot\mathbf{b}_x + \kappa\theta_x}{\tau}\right)_x, \tag{1.5}$$

here $\tau = \frac{1}{\rho}$ denotes the specific volume, $u \in \mathbf{R}$ the longitudinal velocity, $\mathbf{w} \in \mathbf{R}^2$ the transverse velocity, $\mathbf{b} \in \mathbf{R}^2$ the transverse magnetic field, and $\theta$ the temperature, $p = p(\tau,\theta)$ the pressure, and $e = e(\tau,\theta)$ the internal energy; $\lambda$ and $\mu$ are the bulk and the shear viscosity coefficients, respectively, $\nu$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, $k = k(\tau,\theta)$ is the heat conductivity, $E$ is given by

$$E = e + \frac{1}{2}\left(u^2 + |\mathbf{w}|^2\right) + \frac{1}{2}\tau|\mathbf{b}|^2.$$

For the constitutive relations, we consider (see, *e.g.*, [1]) the Stefan-Boltzmann model, *i.e.*, the pressure $p(\tau,\theta)$, internal energy $e(\tau,\theta)$, and the thermo-radiative flux $Q(\tau,\theta)$ take the following forms, respectively:

$$p(\tau,\theta) = \frac{R\theta}{\tau} + \frac{a}{3}\theta^4, \qquad e(\tau,\theta) = C_v\theta + a\tau\theta^4, \qquad Q(\tau,\theta) = Q_F + Q_R = -\kappa\theta_x, \tag{1.6}$$

where $R > 0$ is the perfect gas constant, $C_v > 0$ is the specific heat at constant volume, $a > 0$ is a constant and the heat conductivity $\kappa(\tau,\theta) > 0$ is a function of $\tau$ and $\theta$. As initial and boundary conditions, we consider

$$(\tau,u,\mathbf{w},\mathbf{b},\theta)|_{t=0} = (\tau_0,u_0,\mathbf{w}_0,\mathbf{b}_0,\theta_0)(x), \quad x \in \overline{\Omega} = [0,1], \tag{1.7}$$

$$(u,\mathbf{w},\mathbf{b},\theta_x)|_{\partial\Omega} = 0. \tag{1.8}$$

Before starting and proving our results, let us first recall the related results in the literature. For the one-dimensional ideal gas, *i.e.*,

$$e = C_v\theta, \qquad \sigma = -\frac{R\theta}{\tau} + \frac{\mu}{\tau}u_x, \qquad Q = -\kappa\frac{\theta_x}{\tau}, \qquad \mathbf{w} = \mathbf{b} \equiv \mathbf{0}, \tag{1.9}$$

with suitable positive constants $C_v$, $R$, Kazhikhov [2, 3], Kazhikhov and Shelukhin [4], Kawashima and Nishida [5] established the existence of global smooth solutions. Zheng and Qin [6] proved the existence of maximal attractors in $H^i$ ($i = 1,2$). However, under very high temperatures and densities, the constitutive relations (1.9) become inadequate. Thus a more realistic model would be a linearly viscous gas (or Newtonian fluid),

$$\sigma(\tau,\theta,u_x) = -p(\tau,\theta) + \frac{\mu(\tau,\theta)}{\tau}u_x, \tag{1.10}$$

satisfying Fourier's law of heat flux,

$$Q(\tau,\theta,\theta_x) = -\frac{\kappa(\tau,\theta)}{\tau}\theta_x, \tag{1.11}$$

whose internal energy $e$ and pressure $p$ are coupled by the standard thermodynamical relation

$$e_\tau(\tau,\theta) = -p(\tau,\theta) + \theta p_\theta(\tau,\theta). \tag{1.12}$$

In this case, Kawohl [7] and Jiang [8] obtained the existence of global solutions to $1D$ viscous heat-conductive real gas with different growth assumptions on the pressure $p$, internal energy $e$, and heat conductivity $\kappa$ in terms of temperature. Qin [9] established the regularity and asymptotic behavior of global solutions with more general growth assumptions on $p$, $e$, $\kappa$ than those in [7, 8].

For the radiative and reactive gas, Ducomet [1] established the global existence and exponential decay in $H^1$ of smooth solutions. Umehara and Tani [10] and Qin *et al.* [11] proved the global existence of smooth solutions for a self-gravitating radiative and reactive gas.

For the non-radiative MHD flows (*i.e.*, $a \equiv 0$), there have been a number of studies under various conditions by several authors (see, *e.g.*, [12–18]). The existence and uniqueness of local smooth solutions was first obtained in [17], moreover, the existence of global smooth solutions with small smooth initial data was shown in [19]. Under the technical condition that $\kappa(\rho,\theta)$ satisfies

$$0 < C^{-1}\left(1+\theta^q\right) \le \kappa(\rho,\theta) \le C\left(1+\theta^q\right)$$

for $q \ge 2$, Chen and Wang [12] proved the existence and continuous dependence of global strong solutions with large initial data satisfying

$$0 < \inf \rho_0 \le \rho_0(x) \le \sup \rho_0 < \infty, \qquad \rho_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0 \in H^1(\Omega), \qquad \theta_0(x) > 0.$$

Chen and Wang [13] also investigated a free boundary problem with general large initial data. Wang [18] established the existence of large solutions to the initial-boundary value problem for planar magnetohydrodynamics. Under the technical condition upon $\kappa(\rho)$

$$\kappa(\rho,\theta) \equiv \kappa(\rho) > \frac{C}{\rho},$$

Fan, Jiang and Nakamura [15] investigated the uniqueness of the weak solutions of MHD with Lebesgue initial data. Fan, Jiang and Nakamura [16] also considered a one-dimensional plane MHD compressible flow, and proved that as the shear viscosity goes to zero, global weak solutions converge to a solution of the original equations with zero shear viscosity. The uniqueness and continuous dependence of weak solutions for the Cauchy problem have been proved by Hoff and Tsyganov [14].

For compressible and radiative MHD flow (*i.e.*, $a > 0$), the author and his colleagues [20, 21] established the global existence and exponential stability of solutions. For compressible and radiative MHD flow (*i.e.*, $a > 0$) with self-gravitation, Ducomet and Feireisl [22] proved the existence of global-in-time solutions of this problem with arbitrarily large initial data and conservative boundary conditions on a bounded spatial domain in $\mathbf{R}^3$. Under the technical condition that $\kappa(\rho,\theta)$ satisfies

$$k_1\left(1+\theta^q\right) \le \kappa(\rho,\theta), \qquad \left|\kappa_\rho(\rho,\theta)\right| \le k_2\left(1+\theta^q\right),$$

for some $q > \frac{5}{2}$, Zhang and Xie [23] investigated the existence of global smooth solutions to this problem. However, the large-time behavior is still open even for the non-self-gravitative case, *i.e.*, (1.1)-(1.8). In this paper, we obtained the existence of universal attractors in $H_+^i$ ($i = 1,2$) (see below for their definitions) for one-dimensional compressible and radiative magnetohydrodynamics equations. Before starting the research, let us first explain some mathematical difficulties in studying this problem.

Firstly, for physical reasons, the special volume $\tau$ and the absolute temperature $\theta$ should be positive for all time. These constraints give rise to some severe mathematical difficulties. For instance, we must work on incomplete metric spaces $H_+^1$ and $H_+^2$, $H_+^2 \subset H_+^1$, which are the usual Sobolev spaces with these constrains.

Secondly, since the universal attractor is just the $\omega$-limit set of an absorbing set in weak topology, the requirement of completeness of the spaces is needed. To overcome this severe mathematical difficulty, we restrict ourselves to a sequence of closed subspaces $H^1_+$ and $H^2_+$. It turns out that it is very crucial to prove that the orbit starting from any bounded set of this closed subspace will re-enter this subspace and stay there after a finite time, which should be uniform with respect to all orbits starting from a bounded set, otherwise, there is no ground to talk about the existence of an absorbing set and a (maximal) universal attractor in this subspace. The proof of the above fact becomes an essential part of this paper and it will be done by use of delicate *a priori* estimates.

Thirdly, the total mass and the total energy are conserved. These conservations indicate that there can be no absorbing set for initial data varying in the whole space. Instead, we should rather consider the dynamics in a sequence of closed subspaces defined by some parameters. In this regard, the situation is quite similar to those encountered for the single Cahn-Hilliard equation in the isothermal case (see Temam [24], Zheng and Qin [6] and Qin [9]). Therefore, one of the key issues is how to choose these closed subspaces.

Fourthly, (1.1)-(1.8) is a hyperbolic-parabolic coupled system. It turns out that in general the orbit is not compact. In order to prove the existence of a maximal attractor by the theory presented by Temam in [24], we have either to show the uniform compactness of the orbit of semigroup $S(t)$ for large time or to show that one can decompose $S(t)$ into two parts, $S_1(t)$ and $S_2(t)$, with $S_1(t)$ being uniformly compact for large time and $S_2(t)$ going to zero uniformly. Moreover, since our system is quasilinear, the usual way of decomposition of $S(t)$ into two parts for a semilinear system does not seem feasible. To overcome this difficulty, we will adopt an approach motivated by the ideas in Ghidaglia [25] (see also, Lemma 2.1).

Finally, (1.1)-(1.8) are complicated, it turns out that very delicate estimates are needed.

We define two spaces as follows:

$$\begin{aligned}
H^1_+ = \bigl\{&(\tau,u,\mathbf{w},\mathbf{b},\theta)\in\bigl(H^1[0,1]\bigr)^7 : \tau(x)>0,\theta(x)>0,x\in[0,1],\\
&u(0)=u(1)=0,\mathbf{w}(0)=\mathbf{w}(1)=\mathbf{b}(0)=\mathbf{b}(1)=\mathbf{0}\bigr\},\\
H^2_+ = \bigl\{&(\tau,u,\mathbf{w},\mathbf{b},\theta)\in\bigl(H^2[0,1]\bigr)^7 : \tau(x)>0,\theta(x)>0,x\in[0,1],\\
&u(0)=u(1)=0,\mathbf{w}(0)=\mathbf{w}(1)=\mathbf{b}(0)=\mathbf{b}(1)=\mathbf{0},\\
&\theta'(0)=\theta'(1)=0\bigr\},
\end{aligned}$$

which become two metric spaces when equipped with the metrics induced from the usual normal. In the above, $H^1$, $H^2$ are the usual Sobolev spaces.

Let

$$\begin{aligned}
H^i_\delta = \biggl\{&(\tau,u,\mathbf{w},\mathbf{b},\theta)\in H^i_+ : \delta_2\le\int_0^1\tau(x,t)\,dx=\int_0^1\tau_0(x)\,dx\le\delta_3,\\
&\delta_6\le\int_0^1 E(x,t)\,dx\le\delta_7,\delta_4\le\theta(x,t)\le\delta_5,\delta_2/2\le\tau(x,t)\le 2\delta_3,\\
&\Phi(t)+\int_0^t V(s)\,ds\le\delta_1\biggr\},\quad i=1,2,
\end{aligned}$$

where

$$\Phi(t) = \int_0^1 \big[C_v(\theta - \log\theta - 1) + R(\tau - \log\tau - 1)\big](x,t)\,dx,$$

$$V(t) = \int_0^1 \left(\frac{\kappa\theta_x^2}{\tau\theta^2} + \frac{\lambda u_x^2 + \mu|\mathbf{w}_x|^2 + \nu|\mathbf{b}_x|^2}{\tau\theta}\right)(x,t)\,dx,$$

$$E(x,t) = e(\tau,\theta) + \frac{1}{2}\big(u^2 + |\mathbf{w}|^2\big) + \frac{1}{2}\tau|\mathbf{b}|^2 = C_v\theta + a\tau\theta^4 + \frac{1}{2}\big(u^2 + |\mathbf{w}|^2\big) + \frac{1}{2}\tau|\mathbf{b}|^2,$$

$$\delta_1 \in \mathbf{R}^3, \qquad 0 < \delta_2 < \delta_3, \qquad 0 < \delta_4 < \delta_5, \qquad 0 < \delta_6 < \delta_7.$$

The notation in this paper will be chosen as follows:

$L^p$, $1 \le p \le +\infty$, $W^{m,p}$, $m \in N$, $H^1 = W^{1,2}$, $H_0^1 = W_0^{1,2}$ denote the usual (Sobolev) spaces on $[0,1]$. In addition, $\|\cdot\|_B$ denotes the norm in the space $B$, we also put $\|\cdot\| = \|\cdot\|_{L^2[0,1]}$. The constants $C_i$ ($i = 1,2$) stand for universal positive constants depending only on the $H^i$ norm of the initial data, $\min_{x\in[0,1]}\theta_0$ and $\min_{x\in[0,1]}\tau_0$. $C_\delta$ stands for the universal positive constant, but independent of any length of time. $C_{B_i,\delta}$ denotes the universal positive constant depending only on $\delta_i$ ($i = 1,\dots,7$), the $H_+^i$ norm of the initial data $(\tau_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0)$, $\min_{x\in[0,1]}\theta_0$, and $\min_{x\in[0,1]}\tau_0$.

Now we are in a position to state our main results.

**Theorem 1.1** *Assume that e, p, and Q are $C^2$ functions satisfying* (1.6) *on* $0 < \tau < \infty$ *and* $0 \le \theta < \infty$. *Then the unique generalized global solution* $(\tau, u, \mathbf{w}, \mathbf{b}, \theta)$ *to problem* (1.1)-(1.8) *defines a nonlinear $C_0$-semigroup $S(t)$ on $H_+^1$. Moreover, for any* $\delta_i$ ($i = 1,\dots,7$), *it possesses in $H_\delta^1$ a universal* (*maximal*) *attractor $\mathcal{A}_{1,\delta}$.*

**Theorem 1.2** *Assume that e, p, and Q are $C^3$ functions satisfying* (1.6) *on* $0 < \tau < \infty$ *and* $0 \le \theta < \infty$. *Then the unique generalized global solution* $(\tau, u, \mathbf{w}, \mathbf{b}, \theta)$ *to problem* (1.1)-(1.8) *defines a nonlinear $C_0$-semigroup $S(t)$ on $H_+^2$. Moreover, for any* $\delta_i$ ($i = 1,\dots,7$), *it possesses in $H_\delta^2$ a universal* (*maximal*) *attractor $\mathcal{A}_{2,\delta}$.*

**Remark 1.1** See Ghidaglia [25] and Qin [9] for the definition of universal (maximal) attractor.

## 2 An absorbing set in $H^1$

In this section we will prove the existence of an absorbing ball in $H_\delta^1$. Throughout this section we assume that the initial data belong to a bounded set of $H_\delta^1$. First, we have to prove that the orbit starting from any bounded set of $H_\delta^1$ will re-enter $H_\delta^1$ and stay there after a finite time, which should be a uniform with respect to all orbits starting from that bounded set.

**Lemma 2.1** *Let $H_1$, $H_2$, $H_3$ be three Banach spaces verifying the following conditions*:

1. *the embeddings $H_3 \to H_2$ and $H_2 \to H_1$ are compact*;
2. *there are $C_0$-semigroup $S(t)$ on $H_2$ and $H_3$ which map $H_2$, $H_3$ into $H_2$, $H_3$, respectively, and for any $t > 0$, $S(t)$ are continuous* (*nonlinear*) *operators on $H_2$, $H_3$ respectively*;
3. *the semigroup $S(t)$ on $H_3$ possesses a bounded absorbing set in $H_3$; then there is a weak universal attractor $\mathcal{A}_3$ in $H_3$.*

*If, further, the following conditions are valid*:

(4) *the semigroup* $S(t)$ *on* $H_2$ *possesses a bounded absorbing set in* $H_2$;

(5) *for any* $t > 0$, $S(t)$ *is continuous on bounded sets of* $H_2$ *for the topology of the norm of* $H_1$, *then there is a weak universal attractor* $\mathcal{A}_2$ *in* $H_2$.

*Proof* See, *e.g.*, Ghidaglia [25]. □

**Lemma 2.2** *Assume that the initial data* $(\tau_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0) \in H_+^1$, *and compatibility conditions are satisfied, and the heat conductivity* $\kappa$ *is a* $C^2$ *function on* $0 < \tau < \infty$ *and* $0 \leq \theta < \infty$ *and satisfies the growth condition*

$$k_1\left(1+\theta^q\right) \leq \kappa(\tau,\theta) \leq k_2\left(1+\theta^q\right), \qquad |\kappa_\tau| + |\kappa_{\tau\tau}| \leq k_2\left(1+\theta^q\right), \quad q > 2, \tag{2.1}$$

*with positive constants* $k_1 \leq k_2$, *and there exists a constant* $\varepsilon_0 > 0$ *such that* $\overline{\tau} = \int_0^1 \tau_0\,dx \leq \varepsilon_0$. *Then problem* (1.1)-(1.8) *admits a unique global solution* $(\tau, u, \mathbf{w}, \mathbf{b}, \theta) \in H_+^1$ *verifying*

$$0 < C_1^{-1} \leq \tau(x,t) \leq C_1, \qquad 0 < C_1^{-1} \leq \theta(x,t) \leq C_1, \quad \forall (x,t) \in [0,1] \times [0,\infty) \tag{2.2}$$

*and for any* $t > 0$

$$\begin{aligned}
&\left\|\tau(t) - \overline{\tau}\right\|_{H^1}^2 + \left\|u(t)\right\|_{H^1}^2 + \left\|\mathbf{w}(t)\right\|_{H^1}^2 + \left\|\mathbf{b}(t)\right\|_{H^1}^2 + \left\|\theta(t) - \overline{\theta}\right\|_{H^1}^2 \\
&\quad + \int_0^t \left(\|\tau - \overline{\tau}\|_{H^1}^2 + \|u\|_{H^2}^2 + \|\mathbf{w}\|_{H^2}^2 + \|\mathbf{b}\|_{H^2}^2 + \|\theta - \overline{\theta}\|_{H^2}^2 + \|u_t\|^2 \right. \\
&\quad \left. + \|\mathbf{w}_t\|^2 + \|\mathbf{b}_t\|^2 + \|\theta_t\|^2\right)(s)\,ds \leq C_1,
\end{aligned} \tag{2.3}$$

*where* $\overline{\tau} = \int_0^1 \tau\,dx = \int_0^1 \tau_0\,dx$, *constant* $\overline{\theta} > 0$ *is determined by*

$$e(\overline{\tau}, \overline{\theta}) = E_0 \equiv \int_0^1 \left(\frac{1}{2}\left(u_0^2 + |\mathbf{w}_0|^2 + \tau_0|\mathbf{b}_0|^2\right) + e(\tau_0, \theta_0)\right) dx.$$

*Proof* See, *e.g.*, Qin *et al.* [20, 21]. □

**Lemma 2.3** *Assume that the initial data* $(\tau_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0) \in H_+^2$ *and compatibility conditions are satisfied, the heat conductivity* $\kappa$ *is a* $C^3$ *function satisfying* (2.1) *on* $0 < \tau < \infty$ *and* $0 \leq \theta < \infty$, *and there exists a constant* $\varepsilon_0 > 0$ *such that* $\overline{\tau} = \int_0^1 \tau_0\,dx \leq \varepsilon_0$. *Then problem* (1.1)-(1.8) *admits a unique global solution* $(\tau, u, \mathbf{w}, \mathbf{b}, \theta) \in H_+^2$ *verifying that, for any* $t > 0$,

$$\begin{aligned}
&\left\|\tau(t) - \overline{\tau}\right\|_{H^2}^2 + \left\|u(t)\right\|_{H^2}^2 + \left\|\mathbf{w}(t)\right\|_{H^2}^2 + \left\|\mathbf{b}(t)\right\|_{H^2}^2 + \left\|\theta(t) - \overline{\theta}\right\|_{H^2}^2 + \left\|u_t(t)\right\|^2 + \left\|\mathbf{w}_t(t)\right\|^2 \\
&\quad + \left\|\mathbf{b}_t(t)\right\|^2 + \left\|\theta_t(t)\right\|^2 + \int_0^t \left(\|\tau - \overline{\tau}\|_{H^2}^2 + \|u\|_{H^3}^2 + \|\mathbf{w}\|_{H^3}^2 + \|\mathbf{b}\|_{H^3}^2 + \|\theta - \overline{\theta}\|_{H^3}^2 \right. \\
&\quad \left. + \|u_{tx}\|^2 + \|\mathbf{w}_{tx}\|^2 + \|\mathbf{b}_{tx}\|^2 + \|\theta_{tx}\|^2\right)(s)\,ds \leq C_2.
\end{aligned} \tag{2.4}$$

*Proof* See, *e.g.*, Qin *et al.* [20, 21]. □

**Lemma 2.4** *The unique generalized global solution* $(\tau, u, \mathbf{w}, \mathbf{b}, \theta)$ *in* $H_+^i$ $(i = 1, 2)$ *to problem* (1.1)-(1.8) *defines a nonlinear* $C_0$*-semigroup* $S(t)$ *on* $H_+^i$.

*Proof* By Lemmas 2.2-2.3, we know that, for any $t > 0$, the operator $S(t) : (\tau_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0) \in H_+^i \to (\tau, u, \mathbf{w}, \mathbf{b}, \theta) \in H_+^i$ $(i = 1, 2)$ exists and, by the uniqueness of generalized global solutions, satisfies on $H_+^1$, for any $t_1, t_2 \in [0, \infty)$,

$$S(t_1 + t_2) = S(t_1)S(t_2) = S(t_2)S(t_2). \qquad \square$$

**Lemma 2.5** *If* $(\tau_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0) \in H_\delta^1$, *then the following estimates hold for any* $t > 0$:

$$\delta_2 \le \int_0^1 \tau(x,t)\,dx = \int_0^1 \tau_0(x)\,dx \le \delta_3, \tag{2.5}$$

$$\delta_6 \le \int_0^1 E(x,t)\,dx = \int_0^1 E(x,0)\,dx \equiv E_0 \le \delta_7, \tag{2.6}$$

$$\Phi(t) + \int_0^t V(s)\,ds \le \delta_1. \tag{2.7}$$

*Proof* We integrate (1.1) with respect to $x$ and $t$ and exploit the boundary conditions (1.8), we will end up with (2.5). Integrating (1.5) over $Q_t := (0,1) \times (0,t)$ and noting (1.8), we get (2.6). The conservation law of total energy, (1.5), can be rewritten as

$$e_t + pu_x = \left(\frac{\kappa\theta_x}{\tau}\right)_x + \frac{\lambda u_x^2 + \mu|\mathbf{w}_x|^2 + \nu|\mathbf{b}_x|^2}{\tau}, \tag{2.8}$$

*i.e.*,

$$C_v\theta_t + 4a\tau\theta^3\theta_t + \frac{R\theta\tau_t}{\tau} + \frac{4a}{3}\tau_t\theta^4 = \left(\frac{\kappa\theta_x}{\tau}\right)_x + \frac{\lambda u_x^2 + \mu|\mathbf{w}_x|^2 + \nu|\mathbf{b}_x|^2}{\tau}. \tag{2.9}$$

Multiplying (2.9) by $\theta^{-1}$, and integrating the resulting equation over $Q_t$, we get (2.7). $\square$

**Lemma 2.6** *If* $(\tau_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0) \in H_\delta^1$, *then the following estimates hold for any* $t > 0$:

$$0 < C_\delta^{-1} \le \int_0^1 \theta\,dx \le C_\delta. \tag{2.10}$$

*Proof* It follows from (2.7) and the convexity of the function $-\log y$ that

$$\int_0^1 \theta\,dx - \log\int_0^1 \theta\,dx - 1 \le \int_0^1 (\theta - \log\theta - 1)\,dx \le C_\delta, \quad \forall t > 0,$$

which implies that there exist $b(t) \in [0,1]$ and two positive constants $r_1$, $r_2$ such that

$$0 < r_1 \le \int_0^1 \theta\,dx = \theta\bigl(b(t),t\bigr) \le r_2, \tag{2.11}$$

where $r_i = r_i(\delta)$ $(i = 1,2)$ are two positive roots of the equation $y - \log y - 1 = C_\delta$. Thus (2.10) follows from (2.11). The proof is complete. $\square$

**Lemma 2.7** *If* $(\tau_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0) \in H_\delta^1$, *then the following estimates hold for any* $t > 0$:

$$0 \le C_\delta^{-1} \le \tau(x,t) \le C_\delta, \quad \forall (x,t) \in [0,1] \times [0,+\infty). \tag{2.12}$$

*Proof* See, *e.g.*, Qin *et al.* [20, 21]. □

**Lemma 2.8** *If* $(\tau_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0) \in H_\delta^1$, *then the following estimates hold for any* $t > 0$:

$$\theta_* \le \bar{\theta} \le \theta^*, \tag{2.13}$$

$$0 \le C_\delta^{-1} \le \theta(x,t), \quad \forall (x,t) \in [0,1] \times [0,+\infty), \tag{2.14}$$

*where* $\theta_* = \min_{\tau\in[\delta_2,\delta_3],e\in[\delta_6,\delta_7]} \hat{\theta}(\tau,e)$, $\theta^* = \max_{\tau\in[\delta_2,\delta_3],e\in[\delta_6,\delta_7]} \hat{\theta}(\tau,e)$.

*Proof* We first show that, for boundary conditions (1.8),

$$\min_{\tau\in[\delta_2,\delta_3],e\in[\delta_6,\delta_7]} \le \bar{\theta} \le \max_{\tau\in[\delta_2,\delta_3],e\in[\delta_6,\delta_7]} \hat{\theta}(\tau,e). \tag{2.15}$$

In fact, it follows from (2.5)-(2.6) that

$$\delta_6 \le \bar{e} := e(\bar{\tau},\bar{\theta}) \le \delta_7, \qquad \delta_2 \le \bar{\tau} \le \delta_3,$$

which implies that $\bar{\theta} = \hat{\theta}(\bar{\tau},\bar{\theta})$ and (2.15) holds.

We derive from Lemma 2.2 that there exists a large time $t_0 > 0$ such that

$$\theta(x,t) \ge \frac{1}{2}\overline{\theta} > 0, \quad \forall t \ge t_0. \tag{2.16}$$

On the other hand, we put $\omega := \frac{1}{\theta}$, (2.8) becomes

$$e_\theta \omega_t = \left(\frac{\kappa\omega_x}{\tau}\right)_x + \frac{\tau p_\theta^2}{4\lambda} - \left[\frac{2\kappa\omega_x^2}{\tau\omega} + \frac{\omega^2}{\tau}\left(\mu|\mathbf{w}_x|^2 + \nu|\mathbf{b}_x|^2\right) + \frac{\lambda\omega^2}{\tau}\left(u_x - \frac{\tau p_\theta}{2\lambda\omega}\right)^2\right],$$

which with (2.3) and (2.12) implies that there exists a positive constant $C_1$ such that

$$\omega_t \le \frac{1}{e_\theta}\left(\frac{\kappa\omega_x}{\tau}\right)_x + C_1.$$

Defining $\tilde{\omega}(x,t) := C_1 t + \max_{[0,1]} \frac{1}{\theta_0(x)} - \omega(x,t)$ and a parabolic operator $\mathcal{L} := -\frac{\partial}{\partial t} + \frac{1}{e_\theta}\frac{\partial}{\partial x}(\frac{\kappa}{\tau}\frac{\partial}{\partial x})$, we have a system

$$\mathcal{L}\tilde{\omega} \le 0, \quad \text{on } Q_T = [0,1] \times [0, t_0 + 1],$$

$$\tilde{\omega}|_{t=0} \ge 0, \quad \text{on } [0,1],$$

$$\tilde{\omega}_x|_{x=0,1} = 0, \quad \text{on } [0, t_0 + 1].$$

The standard comparison argument implies

$$\min_{(x,t)\in\overline{Q}_T} \tilde{\omega}(x,t) \ge 0,$$

which gives, for any $(x,t) \in \overline{Q}_T$,

$$\theta(x,t) \ge \left(C_1 t + \max_{x\in[0,1]} \frac{1}{\theta_0(x)}\right)^{-1}.$$

Thus,

$$\theta(x,t) \geq \left( C_1 t_0 + \max_{x\in[0,1]} \frac{1}{\theta_0(x)} \right)^{-1} \geq C_\theta^{-1}, \quad 0 \leq t \leq t_0$$

which, together with (2.12) and (2.16), gives (2.14). □

**Lemma 2.9** *For initial data belonging to an arbitrary fixed bounded set $\mathcal{B}$ of $H_\delta^1$, there is $t_0 > 0$ depending only on the boundedness of this bounded set $\mathcal{B}$ such that, for all $t \geq t_0$, $x \in [0,1]$,*

$$\delta_4 \leq \theta(x,t) \leq \delta_5, \qquad \delta_2/2 \leq \tau(x,t) \leq 2\delta_3. \tag{2.17}$$

*Proof* Suppose that the assertion in Lemma 2.9 is not true. Then there is a sequence $t_n \to +\infty$, such that, for all $x \in [0,1]$,

$$\sup \theta(x,t_n) > \delta_5, \tag{2.18}$$

where sup is taken for all initial data in a given bounded set $\mathcal{B}$ of $H_\delta^1$. Then there exists $(\tau_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0) \in \mathcal{B}$ such that, for the corresponding solution $(\tau, u, \mathbf{w}, \mathbf{b}, \theta)$, we have

$$\theta(x,t_n) > \delta_5, \quad x \in [0,1],$$

which yields

$$\bar{\theta} \geq \delta_5. \tag{2.19}$$

This contradicts (2.13). Similarly, we can prove other parts of (2.17). The proof is complete. □

**Remark 2.10** It follows from Lemma 2.5 and Lemma 2.9 that, for initial data belonging to a given bounded set $\mathcal{B}$ of $H_\delta^1$, the orbit will re-enter $H_\delta^1$ and stay there after a finite time.

In the sequel, we shall prove the existence of an absorbing ball in $H_\delta^1$. Since we assume that the initial data $(\tau_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0)$ belong to an arbitrary bounded set $\mathcal{B}$ of $H_\delta^1$, there is a positive constant $B$ such that $\|(\tau_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0)\|_{H^1} \leq B$. We use $C_{B,\delta}$ to denote generic positive constants depending on $B$ and $\delta_i$ ($i = 1, \ldots, 7$).

**Lemma 2.11** *For any initial data $(\tau_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0) \in H_\delta^1$, the unique generalized global solution $(\tau, u, \mathbf{w}, \mathbf{b}, \theta)$ to problem* (1.1)-(1.8) *satisfies the estimate*:

$$\begin{aligned}
&\frac{1}{2}\left(u^2 + |\mathbf{w}|^2 + \tau|\mathbf{b}|^2\right) + C_{B,\delta}^{-1}\left(|\tau - \overline{\tau}|^2 + |\eta - \overline{\eta}|^2\right) \\
&\quad \leq \mathcal{E}(\tau, u, \mathbf{w}, \mathbf{b}, \theta) \\
&\quad \leq \frac{1}{2}\left(u^2 + |\mathbf{w}|^2 + \tau|\mathbf{b}|^2\right) + C_{B,\delta}\left(|\tau - \overline{\tau}|^2 + |\eta - \overline{\eta}|^2\right).
\end{aligned} \tag{2.20}$$

*Proof* See, *e.g.*, Qin *et al.* [20, 21]. □

**Lemma 2.12** *There are positive constants $\gamma_1' = \gamma_1'(C_{B,\delta}) > 0$ such that, for any fixed $\gamma \in (0, \gamma_1']$ and for any $t > 0$, we have*

$$e^{\gamma t}\big(\|\tau(t)-\overline{\tau}\|^2 + \|u(t)\|^2 + \|\mathbf{w}(t)\|^2 + \|\mathbf{b}(t)\|^2 + \|\theta(t)-\overline{\theta}\|^2 + \|\tau_x(t)\|^2 + \|\rho_x(t)\|^2\big) + \int_0^t e^{\gamma s}\big(\|\rho_x\|^2 + \|u_x\|^2 + \|\mathbf{w}_x\|^2 + \|\mathbf{b}_x\|^2 + \|\theta_x\|^2 + \|\tau_x\|^2\big)(s)\,ds \le C_{B,\delta}. \tag{2.21}$$

*Proof* See, *e.g.*, Qin *et al.* [20, 21]. □

**Lemma 2.13** *There exists a positive constant $\gamma_1 = \gamma_1(C_{B,\delta}) \le \gamma_1'$ such that, for any $t > 0$ and any fixed $\gamma \in (0, \gamma_1']$, the following estimate holds*:

$$e^{\gamma t}\big(\|u_x(t)\|^2 + \|\mathbf{w}_x(t)\|^2 + \|\mathbf{b}_x(t)\|^2 + \|\theta_x(t)\|^2\big) + \int_0^t e^{\gamma s}\big(\|u_{xx}\|^2 + \|\mathbf{w}_{xx}\|^2 + \|\mathbf{b}_{xx}\|^2 + \|\theta_{xx}\|^2 + \|u_t\|^2 + \|\mathbf{w}_t\|^2 + \|\mathbf{b}_t\|^2 + \|\theta_t\|^2\big)(s)\,ds \le C_{B,\delta}, \tag{2.22}$$

*which with Lemma* 2.12 *implies that, for any fixed $\gamma \in (0, \gamma_1']$,*

$$\big\|\big(\tau(t)-\bar{\tau}, u(t), \mathbf{w}(t), \mathbf{b}(t), \theta(t)-\bar{\theta}\big)\big\| \le C_{B,\delta}e^{-\gamma t}. \tag{2.23}$$

*Proof* See, *e.g.*, Qin *et al.* [20, 21]. □

Thus the following results on the existence of an absorbing set in $H_\delta^1$ follow from Lemma 2.13.

**Lemma 2.14** *Let $R_1 = R_1(\delta) = 2\sqrt{\delta_3^2 + (\theta^*)^2}$ and*

$$B_1 = \big\{\big(\tau(t), u(t), \mathbf{w}(t), \mathbf{b}(t), \theta(t)\big) \in H_\delta^1, \big\|\big(\tau(t), u(t), \mathbf{w}(t), \mathbf{b}(t), \theta(t)\big)\big\|_{H_+^1} \le R_1\big\}.$$

*Then $B_1$ is an absorbing ball in $H_\delta^1$, i.e., there exists some*

$$t_1 = t_1(C_{B,\delta}) = \max\big\{-\gamma_1^{-1}\log\big[2\big(\delta_3^2 + (\theta^*)^2\big)/C_{B,\delta}\big], t_0\big\} \ge t_0$$

*such that when $t \ge t_1$, $\|(\tau(t), u(t), \mathbf{w}(t), \mathbf{b}(t), \theta(t))\|_{H_+^1}^2 \le R_1^2$.*

## 3 An absorbing set in $H^2$

In this section we are going to prove the existence of an absorbing set in $H_\delta^2$. Throughout this section we always assume that the initial data belonging to an arbitrary fixed bounded set $\mathcal{B}$ of $H_\delta^2$.

The next two lemmas concern the existence of an absorbing set in $H_\delta^2$.

**Lemma 3.1** *There exists a positive constant $\gamma_2' = \gamma_2'(C_{B,\delta}) \le \gamma_1$ such that, for any fixed $\gamma \in (0, \gamma_2']$, the following estimate holds*:

$$e^{\gamma t}\big(\|u_t(t)\|^2 + \|\mathbf{w}_t(t)\|^2 + \|\mathbf{b}_t(t)\|^2 + \|\theta_t(t)\|^2 + \|u(t)\|_{H^2}^2 + \|\mathbf{w}(t)\|_{H^2}^2 + \|\mathbf{b}(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2\big) + \int_0^t e^{\gamma s}\big(\|u_{tx}\|^2 + \|\mathbf{w}_{tx}\|^2 + \|\mathbf{b}_{tx}\|^2 + \|\theta_{tx}\|^2\big)(s)\,ds \le C_{B,\delta}, \quad \forall t > 0. \tag{3.1}$$

*Proof* See, *e.g.*, Qin *et al.* [20, 21]. □

**Lemma 3.2** *There exists a positive constant $\gamma_2 = \gamma_2(C_{B,\delta}) \le \gamma_2'$ such that, for any fixed $\gamma \in (0,\gamma_2]$, the following estimate holds*:

$$\|\tau(t) - \overline{\tau}\|_{H^2} \le C_{B,\delta} e^{-\gamma t}, \tag{3.2}$$

*which together with Lemma* 3.1 *implies that, for any $\gamma \in (0,\gamma_2]$ and $\forall t > 0$,*

$$\begin{aligned} &\|\tau(t)\|_{H^2}^2 + \|u(t)\|_{H^2}^2 + \|\mathbf{w}(t)\|_{H^2}^2 + \|\mathbf{b}(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 \\ &\quad \le 2\left(\delta_3^2 + \left(\theta^*\right)^2\right) + C_{B,\delta} e^{-\gamma t}. \end{aligned} \tag{3.3}$$

*Proof* See, *e.g.*, Qin *et al.* [20, 21]. □

Now if we define $t_2 = t_2(C_{B,\delta}) \ge \max(t_1(C_{B,\delta}, -\gamma_2^{-1}) \log(2(\delta_3^2 + (\theta^*)^2)/C_{B,\delta}))$, then estimate (3.3) implies that, for any $t \ge t_2(C_{B,\delta})$,

$$\|\tau(t)\|_{H^2}^2 + \|u(t)\|_{H^2}^2 + \|\mathbf{w}(t)\|_{H^2}^2 + \|\mathbf{b}(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 \le 4\left(\delta_3^2 + \left(\theta^*\right)^2\right)).$$

Taking $R_2 = 2\sqrt{\delta_3^2 + (\theta^*)^2}$, we immediately infer the following theorem.

**Theorem 3.3** *The ball $B_2 = \{(\tau(t), u(t), \mathbf{w}(t), \mathbf{b}(t), \theta(t)) \in H_\delta^2, \|(\tau(t), u(t), \mathbf{w}(t), \mathbf{b}(t), \theta(t))\|_{H_+^2}^2 \le R_2^2\}$ is an absorbing ball in $H_\delta^2$, i.e., when $t \ge t_2$, we have*

$$\|\left(\tau(t), u(t), \mathbf{w}(t), \mathbf{b}(t), \theta(t)\right)\|_{H_+^2}^2 \le R_2^2.$$

## 4 Universal attractor in $H^1$ and $H^2$

In this section we finish the proof of Theorems 1.1 and 1.2. Having proved the existence of absorbing balls in $H_\delta^2$ and $H_\delta^1$, we can use the abstract framework established in [25] by Ghidaglia (see also Lemma 2.1) to conclude that

**Lemma 4.1** *The set*

$$\omega(B_2) = \bigcap_{s \ge 0} \overline{\bigcup_{t \ge s} S(t) B_2}, \tag{4.1}$$

*where the closures are taken with respect to the weak topology of $H_+^2$, is included in $B_2$ and is nonempty. It is invariant by $S(t)$, i.e.,*

$$S(t)\omega(B_2) = \omega(B_2), \quad \forall t > 0. \tag{4.2}$$

**Lemma 4.2** *The set*

$$\mathcal{A}_{2,\delta} = \omega(B_2) \tag{4.3}$$

*satisfies*

$$\mathcal{A}_{2,\delta} \textit{ is bounded and weakly closed in } H_\delta^2, \tag{4.4}$$

$$S(t)\mathcal{A}_{2,\delta} = \mathcal{A}_{2,\delta}, \quad \forall t \geq 0, \tag{4.5}$$

*for every bounded set $\mathcal{B}$ in $H^2_\delta$,*

$$\lim_{t\to+\infty} d^\omega\big(S(t)\mathcal{B}, \mathcal{A}_{2,\delta}\big) = 0. \tag{4.6}$$

*Moreover, it is the maximal set in the sense of an inclusion that satisfies* (4.4), (4.5) *and* (4.6).

*Proof* The proofs of Lemmas 4.1 and 4.2 follow from Lemma 2.1, using the facts that $S(t)$ is continuous on $H^2_\delta$ and $H^1_\delta$, respectively, $H^2_\delta$ is compactly embedded in $H^1_\delta$, $B_2$, and $B_1$ are absorbing balls in $H^2_\delta$ and $H^1_\delta$, respectively.

Following [25], we also call $\mathcal{A}_{2,\delta}$ the universal attractor of $S(t)$ in $H^2_\delta$. In order to discuss the existence of a universal attractor in $H^1_\delta$, we need to prove the following lemma.

**Lemma 4.3** *For every $t \geq 0$, the mapping $S(t)$ is continuous on bounded sets of $H^1_\delta$ for the topology induced by the norm $(L^2)^7$.*

*Proof* We suppose that $(\tau_{0j}, u_{0j}, \mathbf{w}_{0j}, \mathbf{b}_{0j}, \theta_{0j}) \in H^1_\delta$, $\|(\tau_{0j}, u_{0j}, \mathbf{w}_{0j}, \mathbf{b}_{0j}, \theta_{0j})\|_{H^1} \leq R$, $(\tau_j, u_j, \mathbf{w}_j, \mathbf{b}_j, \theta_j) = S(t)(\tau_{0j}, u_{0j}, \mathbf{w}_{0j}, \mathbf{b}_{0j}, \theta_{0j})$, and $(\tau, u, \mathbf{w}, \mathbf{b}, \theta) = (\tau_1, u_1, \mathbf{w}_1, \mathbf{b}_1, \theta_1) - (\tau_2, u_2, \mathbf{w}_2, \mathbf{b}_2, \theta_2)$.

Subtracting the corresponding equations (1.1)-(1.5) satisfied by $(\tau_1, u_1, \mathbf{w}_1, \mathbf{b}_1, \theta_1)$ and $(\tau_2, u_2, \mathbf{w}_2, \mathbf{b}_2, \theta_2)$, we obtain

$$\tau_t = u_x, \tag{4.7}$$

$$\begin{aligned} u_t = {} & -\frac{1}{2}|\mathbf{b}|^2_x - (\mathbf{b}_1\mathbf{b}_{2x} + \mathbf{b}_2\mathbf{b}_{1x}) + R\left[\frac{\theta\tau_{1x}}{\tau_1^2} + \left(\frac{\theta_2\tau}{\tau_1\tau_2}\right)_x - \frac{\theta_x}{\tau_1}\right] \\ & - \lambda\left[\frac{u_x\tau_{1x}}{\tau_1^2} + \left(\frac{u_{2x}\tau}{\tau_1\tau_2}\right)_x - \frac{u_{xx}}{\tau_1}\right] - \frac{4a}{3}\big(\theta_1^3\theta_{1x} - \theta_2^3\theta_{2x}\big)\theta_x^4, \end{aligned} \tag{4.8}$$

$$\mathbf{w}_t = -\mathbf{b}_x - \mu\left[\frac{\mathbf{w}_x\tau_{1x}}{\tau_1^2} + \left(\frac{\mathbf{w}_{2x}\tau}{\tau_1\tau_2}\right)_x - \frac{\mathbf{w}_{xx}}{\tau_1}\right], \tag{4.9}$$

$$\tau\mathbf{b}_t = -u_x\mathbf{b} - \mathbf{w}_x - \nu\left[\frac{\mathbf{b}_x\tau_{1x}}{\tau_1^2} + \left(\frac{\mathbf{b}_{2x}\tau}{\tau_1\tau_2}\right)_x - \frac{\mathbf{b}_{xx}}{\tau_1}\right], \tag{4.10}$$

$$\begin{aligned} C_\nu\theta_t = {} & -\kappa\left[\frac{\theta_x\tau_{1x}}{\tau_1^2} + \left(\frac{\theta_{2x}\tau}{\tau_1\tau_2}\right)_x - \frac{\theta_{xx}}{\tau_1}\right] + \frac{\lambda u_{1x}^2 + \mu|\mathbf{w}_{1x}|^2 + \nu|\mathbf{b}_{1x}|^2}{\tau_1} \\ & - \frac{\lambda u_{2x}^2 + \mu|\mathbf{w}_{2x}|^2 + \nu|\mathbf{b}_{2x}|^2}{\tau_2} - 4a\tau_1\theta_1^3\theta_{1t} + \frac{R\theta_1\tau_{1t}}{\tau_1} + \frac{4a}{3}\tau_{1t}\theta_1^4 \\ & + 4a\tau_2\theta_2^3\theta_{2t} + \frac{R\theta_2\tau_{2t}}{\tau_2} + \frac{4a}{3}\tau_{2t}\theta_2^4. \end{aligned} \tag{4.11}$$

By Lemma 2.3, we know that, for any $t > 0$ and $j = 1, 2$,

$$\begin{aligned} & \big\|\tau_j(t), u_j(t), \mathbf{w}_j(t), \mathbf{b}_j(t), \theta_j(t)\big\|^2_{H^1} + \int_0^t \big(\|\tau_{jx}\|^2 + \|u_{jx}\|^2_{H^2} + \|\mathbf{w}_{jx}\|^2_{H^1} + \|\mathbf{b}_{jx}\|^2_{H^1} \\ & \quad + \|\theta_{jx}\|^2_{H^1} + \|u_{jt}\|^2 + \|\mathbf{w}_{jt}\|^2 + \|\mathbf{b}_{jt}\|^2 + \|\theta_{jt}\|^2\big)(s)\,ds \leq C_{R,\delta}, \end{aligned} \tag{4.12}$$

where $C_{R,\delta} > 0$ is a constant depending only on $R$ and $\delta$.

Multiplying (1.1)-(1.5) by $\tau$, $u$, $\mathbf{w}$, $\mathbf{b}$, and $\theta$, respectively, adding them up and integrating the result over $[0,1]$, and using Lemmas 2.2-2.3, the Cauchy inequality, the embedding theorem, the mean value theorem, and the inequalities

$$\|\theta\|_{L^\infty}^2 \le C\big(\|\theta\|\|\theta_x\| + \|\theta\|^2\big), \qquad \|\tau\|_{L^\infty}^2 \le C\|\tau_x\|,$$

we deduce that, for any small $\epsilon > 0$,

$$\begin{aligned}&\frac{d}{dt}\big(\|\tau\|^2 + \|u\|^2 + \|\mathbf{w}\|^2 + \|\mathbf{b}\|^2 + \|\theta\|^2\big) + \int_0^1 \left(\frac{\kappa\theta_x^2}{\tau\theta^2} + \frac{\lambda u_x^2 + \mu|\mathbf{w}_x|^2 + \nu|\mathbf{b}_x|^2}{\tau\theta}\right)(x,t)\,dx \\ &\quad\le \epsilon\big(\|u_x\|^2 + \|\mathbf{w}_x\|^2 + \|\mathbf{b}_x\|^2 + \|\theta_x\|^2\big) \\ &\qquad + C_{R,\delta}(\epsilon)H(t)\big(\|\tau\|^2 + \|u\|^2 + \|\mathbf{w}\|^2 + \|\mathbf{b}\|^2 + \|\theta\|^2\big)\end{aligned}$$

which, together with Lemmas 2.5-2.9, yields

$$\frac{d}{dt}\big(\|\tau\|^2 + \|u\|^2 + \|\mathbf{w}\|^2 + \|\mathbf{b}\|^2 + \|\theta\|^2\big) + C_\delta^{-1}\big(\|u_x\|^2 + \|\mathbf{w}_x\|^2 + \|\mathbf{b}_x\|^2 + \|\theta_x\|^2\big) \tag{4.13}$$

$$\le C_{R,\delta}H(t)\big(\|\tau\|^2 + \|u\|^2 + \|\mathbf{w}\|^2 + \|\mathbf{b}\|^2 + \|\theta\|^2\big), \tag{4.14}$$

where by (4.12), $H(t) = \|\theta_{1t}\|^2 + \|\theta_{2t}\|^2 + \|u_{1xx}\|^2 + \|u_{2xx}\|^2\| + \|\mathbf{w}_{1xx}\|^2 + \|\mathbf{w}_{2xx}\|^2\| + \|\mathbf{b}_{1xx}\|^2 + \|\mathbf{b}_{2xx}\|^2\| + \|\theta_{1xx}\|^2 + \|\theta_{2xx}\|^2 + 1$ satisfies, for any $t > 0$,

$$\int_0^t H(s)\,ds \le C_{R,\delta}(1+t). \tag{4.15}$$

Therefore the assertion of this lemma follows from Gronwall's inequality, (4.13), and (4.14). The proof is complete. □

Now we can again use Lemma 2.1 to obtain the following result on existence of a universal attractor in $H_\delta^1$.

**Lemma 4.4** *The set*

$$\mathcal{A}_{1,\delta} = \bigcap_{s\ge 0}\overline{\bigcup_{t\ge s} S(t)B_1}, \tag{4.16}$$

*where the closures are taken with respect to the weak topology of* $H_+^1$ *is the* (*maximal*) *universal attractor in* $H_\delta^1$.

**Remark 4.5** Since $\mathcal{A}_{2,\delta}$ is bounded in $H_+^2$ and is bounded in $H_+^1$ and by the invariance property (4.5), we have

$$\mathcal{A}_{2,\delta} \subset \mathcal{A}_{1,\delta}. \tag{4.17}$$

On the contrary if we knew that $\mathcal{A}_{1,\delta}$ is bounded in $H_+^2$, then the opposite inclusion would hold.

*Proof of Theorems* 1.1 *and* 1.2 Combing Lemmas 2.1-2.4 and Lemmas 4.1-4.4, we easily complete the proofs of Theorems 1.1 and 1.2. □

**Competing interests**
The author declares to have no competing interests.

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