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On analyticity rate estimates to the magneto-hydrodynamic equations in Besov-Morrey spaces

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Abstract

In this article, we establish higher-order regularizing rate estimates of solutions to generalized magneto-hydrodynamic equations in Morrey spaces with initial data (u_0, d_0) in Besov-Morrey spaces $\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty} \times \dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}$, where $n \geq 2$, $1 \leq r < \infty$, $0 \leq \lambda < n$, $r > n - \lambda$, $\frac{1}{2} + \frac{n-\lambda}{4r} < \sigma < 1 + \frac{n-\lambda}{4r}$, and $s = 2\sigma - 1 - \frac{n-\lambda}{r}$, for which under some smallness condition, the solution of the Cauchy problem is analytic in the spatial variable. Our class of initial data contains strongly singular functions and measures and extends the ones in early work.

MSC: 76B03; 76D03

Keywords: magneto-hydrodynamic equations; Besov-Morrey spaces; regularizing rate estimates; Morrey spaces

1 Introduction and main results

In this article, we investigate the generalized magneto-hydrodynamic equations in the whole space $\mathbb{R}^n$,

$$\begin{cases} u_t + u \cdot \nabla u + (-\Delta)^\sigma u - d \cdot \nabla d + \nabla p = 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty), \\ \nabla \cdot u = 0, \qquad \nabla \cdot d = 0, \\ d_t + u \cdot \nabla d + (-\Delta)^\sigma d - d \cdot \nabla u = 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty), \\ (u, d)|_{t=0} = (u_0, d_0). \end{cases} \tag{1.1}$$

Here u is the velocity field of the flow, $d(\cdot, t)$ is the magnetic field. $p(\cdot, t) : \mathbb{R}^n \to \mathbb{R}$ represents the pressure function. $\nabla \cdot u = 0$ and $\nabla \cdot d = 0$ represent the incompressible conditions. (u_0, d_0) is for given initial data with $\nabla \cdot u_0 = 0$ and $\nabla \cdot d_0 = 0$ in the distribution sense.

When $\sigma = 1$, the equations of system (1.1) become the usual MHD equations, which govern the dynamics of the velocity and magnetic fields in electrically conducting fluids. The system plays a fundamental role in applied sciences such as astrophysics, geophysics, and plasma physics. The first equation of system (1.1) reflects the conservation of momentum, the third equation of system (1.1) is the magnetic induction equation and the second equation of system (1.1) specifies the conservation of mass.

For general σ, system (1.1) is a generalization of the usual incompressible MHD system. As observed in [1], a fractional power of Laplacian can, in principle, be used as a mild dissipation in MHD equations. Besides their physical applications, system (1.1) is also mathematically significant.

According to Duhamel's principle, the mild solution (u,d) for system (1.1) can be represented as

$$\begin{cases} u = e^{-t\mathfrak{L}}u_0 - \int_0^t e^{-(t-s)\mathfrak{L}}\mathbb{P}\nabla\cdot(u\otimes u - d\otimes d)(\cdot,s)\,ds, \\ d = e^{-t\mathfrak{L}}d_0 - \int_0^t e^{-(t-s)\mathfrak{L}}\mathbb{P}\nabla\cdot(u\otimes d - d\otimes u)(\cdot,s)\,ds. \end{cases} \tag{1.2}$$

Here $\mathbb{P}$ is the Leray projection operator, which can be expressed as an $n\times n$ matrix: $\mathbb{P} = \{\mathbb{P}_{j,k}\}_{1\le j,k\le n} = \{\delta_{j,k} + \mathbb{R}_j\mathbb{R}_k\}_{1\le j,k\le n}$ with $\delta_{j,k}$ being the Kronecker symbol, $\mathbb{R}_j = \partial_j(-\Delta)^{-\frac{1}{2}}$ being the Riesz transform. $\mathfrak{L} := (-\Delta)^\sigma$ denotes the fractional Laplacian, which is defined as $[\widehat{(-\Delta)^\sigma f}](\xi) = |\xi|^{2\sigma}\hat{f}(\xi)$.

To give a clearer introduction to our results in this article, we first note that system (1.1) enjoys scaling properties. Clearly, if $(u(x,t),d(x,t))$ is a solution to system (1.1), then $(u^\lambda(x,t),d^\lambda(x,t))$ is also a solution of (1.1) corresponding to the initial data $(u_0^\lambda,d_0^\lambda)$, where

$$\begin{aligned} &u^\lambda(x,t) := \lambda^{2\sigma-1}u(\lambda x,\lambda^{2\sigma}t), \qquad d^\lambda(x,t) := \lambda^{2\sigma-1}d(\lambda x,\lambda^{2\sigma}t), \\ &u_0^\lambda(x) := \lambda^{2\sigma-1}u_0(\lambda x), \qquad d_0^\lambda(x) := \lambda^{2\sigma-1}d_0(\lambda x). \end{aligned} \tag{1.3}$$

We say that the solution (u,d) is self-similar for system (1.1), if $(u^\lambda,d^\lambda) = (u,d)$ for each $\lambda > 0$.

A function space $\mathbb{Y}$ is called a critical space for (1.1) if it satisfies invariance under the scaling $\|u(\cdot,t)\|_{\mathbb{Y}} = \|u^\lambda(\cdot,t)\|_{\mathbb{Y}}$ for all $u\in\mathbb{Y}$.

Before going further, we recall the functional spaces we are going to use. Let $\mathscr{S}$ be the Schwartz class of rapidly decreasing functions and $\mathscr{S}'$ be the space of tempered distributions. Here $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transforms of L^1 functions, respectively, defined by $\mathcal{F}f = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)\,dx$ and $\mathcal{F}^{-1}f = \check{f}(x) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}f(\xi)\,d\xi$. More generally, the Fourier transform of any $f\in\mathscr{S}'$ is given by $(\mathcal{F}f,g) = (f,\mathcal{F}g)$, for any $g\in\mathscr{S}$. First, we recall the definition of Morrey space introduced in [2]: for $1\le p<\infty$ and $0\le\lambda<n$, the Morrey space $M_{p,\lambda} = M_{p,\lambda}(\mathbb{R}^n)$ is defined as

$$M_{p,\lambda} := \{f\in L^p_{\mathrm{loc}}(\mathbb{R}^n), \|f\|_{p,\lambda}<\infty\},$$

with the norm given by

$$\|f\|_{p,\lambda} := \sup_{x_0\in\mathbb{R}^n}\sup_{r>0} r^{-\frac{\lambda}{p}}\left\{\int_{B(x_0,r)}|f(y)|^p\,dy\right\}^{\frac{1}{p}}, \tag{1.4}$$

where $B(x_0,r)$ denotes the ball in $\mathbb{R}^n$ with center x_0 and radius r. The space $M_{p,\lambda}$ endowed with the norm $\|\cdot\|_{p,\lambda}$ is a Banach space and has the following nice scaling property: for $\mu>0$,

$$\|f(\mu x)\|_{p,\lambda} = \mu^{-\frac{n-\lambda}{p}}\|f(x)\|_{p,\lambda}.$$

Set $S_h = \{\phi\in\mathscr{S}, \partial^\alpha\mathcal{F}f(0) = 0\}$ for any multi-index $\alpha\in\mathbb{N}_0 := \mathbb{N}^n\cup\{0\}$, $\mathbb{N}$ is the set of all positive integers. The dual space of S_h is given by $S_h' = \mathscr{S}/\mathcal{P}$, where $\mathcal{P}$ is the space of polynomials. We now introduce a dyadic partition of $\mathbb{R}^n$. Let $\varphi\in\mathscr{S}$ be a radially symmetric function with support in $\{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and such that

$$\sum_{k=-\infty}^{\infty} \varphi_k(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \\ 0, & \text{if } \xi = 0. \end{cases}$$

Furthermore, we define $\varphi_k = \varphi(2^{-k}\xi)$ for every $k \in \mathbb{Z}$.

For any $f \in S'_h$, setting $\Delta_k f = (\varphi_k \hat{f})\check{}$, $k = 0, \pm1, \pm2, \ldots$, and $S_j f = \sum_{k\leq j-1} \Delta_k f$. We have the Littlewood-Paley decomposition,

$$f = \sum_{k=-\infty}^{\infty} \Delta_k f.$$

In [3], Kozono and Yamazaki introduced the homogeneous Besov-Morrey space $\dot{\mathbf{N}}^s_{p,\lambda,q}$. Recall that the space $\dot{\mathbf{N}}^s_{p,\lambda,q}$ is defined by

$$\dot{\mathbf{N}}^s_{p,\lambda,q} = \bigl\{ f \in S'_h(\mathbb{R}^n) : \|f\|_{\dot{\mathbf{N}}^s_{p,\lambda,q}} < \infty \bigr\},$$

where

$$\|f\|_{\dot{\mathbf{N}}^s_{p,\lambda,q}} = \begin{cases} (\sum_{k\in\mathbb{Z}} (2^{ks} \|\Delta_k f\|_{p,\lambda})^q)^{\frac{1}{q}}, & \text{if } 1 \leq p \leq \infty, 1 \leq q < \infty, s \in \mathbb{R}, \\ \sup_{k\in\mathbb{Z}} 2^{ks} \|\Delta_k f\|_{p,\lambda}, & \text{if } 1 \leq p \leq \infty, q = \infty, s \in \mathbb{R}. \end{cases}$$

When $\lambda = 0$, $\dot{\mathbf{N}}^s_{p,0,q} = \dot{\mathbf{B}}^s_{p,q}$, where $\dot{\mathbf{B}}^s_{p,q}$ is the homogeneous Besov space (see [4]).

If $\sigma = 1$, $d = 0$, system (1.1) is the well-known Navier-Stokes equations (NS), Foias and Temam [5] proved spatial analyticity for solutions in Sobolev spaces of periodical functions in an elementary way. The analyticity of solutions in L^p for NS was first shown by Grujič, and Kukavica [6] and Lemarié-Rieusset [7] gave a different approach based on multilinear singular integrals. In a very interesting paper [8], Kahane established the spatial analyticity of weak solutions in Serrin's class $L^p_t L^q_x$ with $n/q + 2/p < 1$. In cylindrical domains, Komatsu [9] showed that the solutions have global spatial analyticity up to the boundary. Using iterative derivative estimates, in [10], Giga and Sawada considered the regularizing rates of the higher-order derivatives and analyticity for the NS for the initial velocity in L^n. Similar results for the Navier-Stokes equations have been established by Sawada [11] when initial value $u_0 \in \dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)$ and by Miura and Sawada [12] when $u_0 \in BMO^{-1}$. Recently, Bae *et al.* [13] obtained the analyticity of the solutions of NS for the sufficiently small initial data in critical Besov spaces $\dot{\mathbf{B}}^{-1+3/p}_{p,q}$, and Huang and Wang [14] showed the analyticity of the local solutions of NS with large initial data in critical Besov spaces $\dot{\mathbf{B}}^{-1+n/p}_{p,q}$ and modulation spaces $M^{-1}_{p,1}$.

For general σ and $d = 0$, the equations of system (1.1) reduce to generalized Navier-Stokes equations (GNS). Dong and Li [15] showed that solutions are analytic in space variables for $1/2 < \sigma < 1$ with initial data in $L^{n/(2\sigma-1)}$. Huang and Wang [14] showed the analyticity of the solutions of GNS in critical Besov spaces $\dot{\mathbf{B}}^{1-2\sigma+n/p}_{p,q}$ and modulation spaces $M^{1-2\sigma}_{p,1}$ for $1/2 < \sigma < 1$. When $\sigma = \frac{1}{2}$, Huang and Wang [14] showed the analyticity of the solutions of GNS in critical Besov spaces $\dot{\mathbf{B}}^{n/p}_{p,1}$ and modulation spaces $M^0_{\infty,1} \cap \dot{\mathbf{B}}^0_{\infty,1}$.

When $\sigma = 1$, Liu and Cui in [16] show the analyticity of the usual MHD with initial data in L^n, $\dot{H}^{\frac{n}{2}-1}$ and BMO^{-1}. When $1/2 < \sigma < (n+2)/4$, Liu *et al.* in [17] show that the solution is analytic in the spatial variable of system (1.1) with the initial velocity in $PM^{n-2\sigma+1}$.

In [18], Yamamoto considered the regularizing rates and analyticity for the drift-diffusion equation for the initial data in $L^{\frac{n}{\theta}}$ $(1 < \theta \le n)$ and extended the results to Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces.

Inspired by the interesting work above, especially [10–12, 16–18] and motivated by the work of Mazzucato [19], and Kozono and Yamazaki [3] on the Navier-Stokes equations and a particular class of semi-linear heat equations with initial data in a certain Besov-Morrey space, our goal in the present article is to establish regularizing decay rate estimates and show space analyticity of mild solutions of system (1.1) with initial data in certain Besov-Morrey spaces. For more information on Besov-Morrey spaces, we also refer to [20–22]. The question of the largest Besov-type spaces on initial data for which the solutions of (1.1) have well-posedness and analyticity is still open.

We give our main results in the following theorem.

Theorem 1.1 *Let* $n \ge 2$, $1 \le r < \infty$, $0 \le \lambda < n$, $r > n-\lambda$, $\frac{1}{2} + \frac{n-\lambda}{4r} < \sigma < 1 + \frac{n-\lambda}{4r}$, $s = 2\sigma - 1 - \frac{n-\lambda}{r}$, $\alpha = \frac{2\sigma-1}{\sigma} - \frac{n-\lambda}{2r\sigma}$, $\nabla \cdot u_0 = 0$, $\nabla \cdot d_0 = 0$, $(u_0, d_0) \in \dot{\mathbf{N}}^{-s}_{r,\lambda,\infty} \times \dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}$, $q \in [r, \infty]$. *Assume further that there exist positive constants* M_1 *and* M_2, *such that the solutions* (u, d) *to system* (1.1) *exist globally in time and satisfy*

$$\left\|(u_0, d_0)\right\|_{\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}} \le M_1 < +\infty, \qquad \sup_{t>0} t^{\frac{\alpha}{2}} \left\|(u, d)\right\|_{2r,\lambda} \le M_2 < +\infty \tag{1.5}$$

for any $t > 0$ *and* M_1 *sufficiently small. Then there exist positive constants* K_1, K_2 *such that*

$$\left\|\left(\nabla^m u, \nabla^m d\right)\right\|_{q,\lambda} \le K_1 \left(K_2 |\tilde{\beta}|\right)^{2m} t^{-\frac{m}{2\sigma} - \frac{1}{2\sigma}(2\sigma - 1 - \frac{n-\lambda}{q})}, \tag{1.6}$$

where $\tilde{\beta} \in \mathbb{N}_0^n$ *is a multi-index and* $|\tilde{\beta}| = m$.

Remarks (I) The assumptions in Theorem 1.1 are natural. Indeed, let $1 \le r < \infty$, $0 \le \lambda < n$, $r > n - \lambda$, $\frac{1}{2} + \frac{n-\lambda}{4r} < \sigma < 1 + \frac{n-\lambda}{4r}$, $\alpha = \frac{2\sigma-1}{\sigma} - \frac{n-\lambda}{2r\sigma}$, $s = 2\sigma - 1 - \frac{n-\lambda}{r}$. The Banach spaces E are defined by $E = \{u : \nabla \cdot u = 0, u \in BC((0,\infty), \dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}), t^{\frac{\alpha}{2}} u \in BC((0,\infty), M_{2r,\lambda})\}$, which are Banach spaces with norms given by $\|u\|_E = \sup_{t>0} \|u(t)\|_{\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}} + \sup_{t>0} t^{\frac{\alpha}{2}} \|u(t)\|_{2r,\lambda}$. Let u_0 and d_0 be divergence free vector fields and $(u_0, d_0) \in \dot{\mathbf{N}}^{-s}_{r,\lambda,\infty} \times \dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}$ with $\|(u_0, d_0)\|_{\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}}$ sufficiently small. Following a similar method to Theorems 3 and 4 on p.967 in [3] for Navier-Stokes equations, then there exists a globally in time solution $(u(x), d(x)) \in E \times E$ to (1.1) that satisfies (1.5). The proof of this is standard by making minor modifications with Theorems 3 and 4 on p.967 in [3], and we will outline its proof in the Appendix for completeness.

(II) When $\frac{1}{2} < \delta \le 1$, $K_2 \ge 2$, the estimate (1.6) is equivalent to (see [10])

$$\left\|\left(\nabla^m u, \nabla^m d\right)\right\|_{q,\lambda} \le K_1 \left(K_2 |\tilde{\beta}|\right)^{2m-\delta} t^{-\frac{m}{2\sigma} - \frac{1}{2\sigma}(2\sigma - 1 - \frac{n-\lambda}{q})}.$$

(III) From Remark 1.4 of [23], we get $PM^{n-2\sigma+1} \subset \dot{\mathbf{B}}^{1-2\sigma+\frac{n}{p}}_{p,\infty}$ for $\frac{n}{2\sigma-1} < p < \infty$. It follows (see [3], p.964) that the space $\dot{\mathbf{N}}^{-2\sigma+1+\frac{n-\lambda}{r}}_{r,\lambda,\infty}$ is strictly larger than $\dot{\mathbf{B}}^{1-2\sigma+\frac{n}{p}}_{p,\infty}$, when $p = \frac{nr}{n-\lambda}$, $\lambda > 0$. The pseudomeasure space PM^a $(a \ge 0)$ introduced in [24] is defined as $PM^a := \{f \in \mathscr{S}' : \hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^n), \|f\|_{PM^a} = \operatorname{ess\,sup}_{\xi \in \mathbb{R}^n} |\xi|^a |\hat{f}| < \infty\}$. In view of the continuous inclusions above, we see that the initial spaces $\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}$ $(r > \max\{\frac{n-\lambda}{2\sigma-1}, n - \lambda\}, \lambda > 0)$ defined in Theorem 1.1

is larger than pseudomeasure space $PM^{n-2\sigma+1}$ in [17]. In [17], the authors considered the regularizing rates of the higher-order derivatives for system (1.1) for the initial velocity in $PM^{n-2\sigma+1}$.

(IV) In particular, when $\sigma = 1$, $d(x,t) = 0$, system (1.1) becomes the usual Navier-Stokes equations. We also notice that BMO^{-1} may be regarded as the largest critical space for initial data, where well-posedness and spatial analyticity of the Navier-Stokes equations can be constructed (see [25]). In [12], Miura and Sawada considered the regularizing rates of the higher-order derivatives for the Navier-Stokes equations for the initial velocity $u_0 \in BMO^{-1}$. The space BMO^{-1} is the space of tempered distributions that can be written as divergence of a vector with components in BMO, where BMO is the space of functions of bounded mean oscillations. The norm on BMO^{-1} is given by

$$\|f\|_{BMO^{-1}} := \sup_{x_0\in\mathbb{R}^n}\sup_{r>0}\left(r^{-n}\int_{B(x_0,r)}\int_0^{r^2}\left|e^{t\Delta}f(y)\right|^2\right)^{\frac{1}{2}}.$$

But the initial data $\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}$ ($\sigma = 1$) given in Theorem 1.1 is not included completely with the space BMO^{-1}. Using the characterization from Lemma 2.3 below, we obtain (see [19], p.1314)

$$\dot{\mathbf{N}}^{\frac{n-\lambda}{r}-1}_{r,\lambda,\infty} \subset BMO^{-1}, \quad r \geq 2, n \geq 2, 0 \leq \lambda < n, r > n-\lambda,$$

and, for $0 \leq \lambda < n$, $n \geq 2$,

$$\dot{\mathbf{N}}^{n-1-\lambda}_{1,\lambda,\infty} \nsubseteq BMO^{-1}, \qquad BMO^{-1} \nsubseteq \dot{\mathbf{N}}^{n-1-\lambda}_{1,\lambda,\infty}.$$

Thus we note that even for the Navier-Stokes equations, our result in Theorem 1.1 is also new.

Notation Throughout this article, we denote vector fields $u = (u_1, u_2, \dots, u_n)$, $d = (d_1, d_2, \dots, d_n)$. For a functional space X, we denote by $\|(u,d)\|_X$,

$$\|d\|_X := \sum_i^n \|d_i\|_X, \qquad \|u\|_X := \sum_i^n \|u_i\|_X, \qquad \left\|(u,d)\right\|_X := \|u\|_X + \|d\|_X.$$

We use $c > 0$ to denote a constant independent of the main variables, which may be different from line to line. We will employ the notation $a \lesssim b$ to mean that $a \leq cb$ for a universal constant $c > 0$ that only depends on the parameters coming from the problems.

2 Preliminaries

In this section, we prepare several tools from harmonic analysis to be used in the proof of Theorem 1.1.

Lemma 2.1 *Assume that* $1 \leq p_j, q \leq \infty$ *for all* $j = 1,2,3$, *and* $1 \leq p \leq \infty$, $1 \leq r < \infty$, $0 \leq \lambda$, $\lambda_i < n$ *for all* $i = 1,2,3$.

(1) *If* $p_1 > p_2$, $s_1 - \frac{n-\lambda}{p_1} = s_2 - \frac{n-\lambda}{p_2}$, $s_1, s_2 \in \mathbb{R}$, *then*

$$\dot{\mathbf{N}}^{s_2}_{p_2,\lambda,q} \hookrightarrow \dot{\mathbf{N}}^{s_1}_{p_1,\lambda,q} \quad \textit{and} \quad \dot{\mathbf{N}}^{0}_{r,\lambda,1} \hookrightarrow M_{r,\lambda} \hookrightarrow \dot{\mathbf{N}}^{0}_{r,\lambda,\infty}.$$

(2) *If* $1 \leq r \leq \tilde{r} \leq \infty$, $s \in \mathbb{R}$, *then* $\dot{\mathbf{N}}^{s}_{p,\lambda,r} \hookrightarrow \dot{\mathbf{N}}^{s}_{p,\lambda,\tilde{r}}$.

(3) *If* $1 \le p_1 \le p_2 \le \infty$, $0 \le \lambda_1, \lambda_2 < n$, $\frac{n-\lambda_1}{p_1} = \frac{n-\lambda_2}{p_2}$, *then* $M_{p_2,\lambda_2} \subset M_{p_1,\lambda_1}$.
(4) *If* $\frac{1}{p_3} = \frac{1}{p_2} + \frac{1}{p_1}$, $\frac{\lambda_3}{p_3} = \frac{\lambda_2}{p_2} + \frac{\lambda_1}{p_1}$, $h_i \in M_{p_i,\lambda_i}$ *for* $i = 1, 2$, *then* $\|h_1 h_2\|_{p_3,\lambda_3} \le \|h_1\|_{p_1,\lambda_1} \|h_2\|_{p_2,\lambda_2}$.

Proof For the proof of Lemma 2.1, we refer to [2, 3, 19, 26, 27]. □

From the Calderón-Zygmund operator theory, the Riesz transform $\mathbb{R}_j$ is continuous on $M_{r,\upsilon}$ for $1 < r < \infty$ and $0 \le \upsilon < n$, thus $\mathbb{P}$ is bounded on $M_{r,\upsilon}$. By the estimates for the multiplier operator, we can also see that $\mathbb{P}$ is bounded on $\dot{\mathbf{N}}^s_{p,\lambda,q}$ for $1 \le p, q \le \infty$, $0 \le \lambda < n$, and $s \in \mathbb{R}$.

Lemma 2.2 *Let* $\mu > 0$ *and* $\mathbb{N}_0^n \ni \alpha = (\alpha_1, \alpha_1, \dots, \alpha_n)$ *be a multi-index with* $|\alpha| = \mu$, $s_1 \le s_2$, $1 \le q \le \infty$, $1 \le p_1 \le p_2 \le \infty$, $0 \le \lambda < n$, *for all* $f \in \mathscr{S}'$, *then there exist* $c_0, c_1, \tilde{c}, \tilde{c}_0, \tilde{c}_1, c$, *and* $\bar{c}$ *depending only on* n *such that*

$$\left\|e^{-t\mathfrak{L}} f\right\|_{p_2,\lambda} \le \bar{c} t^{-\frac{1}{2\sigma}(\frac{n-\lambda}{p_1} - \frac{n-\lambda}{p_2})} \|f\|_{p_1,\lambda}, \tag{2.1}$$

$$\left\|\partial^\alpha e^{-t\mathfrak{L}} f\right\|_{p_2,\lambda} \le c_0 (c_1 \mu)^{\frac{\mu}{2\sigma}} t^{-\frac{\mu}{2\sigma} - \frac{1}{2\sigma}(\frac{n-\lambda}{p_1} - \frac{n-\lambda}{p_2})} \|f\|_{p_1,\lambda}, \tag{2.2}$$

$$\left\|e^{-t\mathfrak{L}} f\right\|_{\dot{\mathbf{N}}^{s_2}_{p_2,\lambda,q}} \le c t^{-\frac{s_2 - s_1}{2\sigma} - \frac{1}{2\sigma}(\frac{n-\lambda}{p_1} - \frac{n-\lambda}{p_2})} \|f\|_{\dot{\mathbf{N}}^{s_1}_{p_1,\lambda,q}}, \tag{2.3}$$

$$\left\|\partial^\alpha e^{-t\mathfrak{L}} f\right\|_{\dot{\mathbf{N}}^{s_2}_{p_2,\lambda,q}} \le \tilde{c}_0 (\tilde{c}_1 \mu)^{\frac{\mu}{2\sigma}} t^{-\frac{\mu + s_2 - s_1}{2\sigma} - \frac{1}{2\sigma}(\frac{n-\lambda}{p_1} - \frac{n-\lambda}{p_2})} \|f\|_{\dot{\mathbf{N}}^{s_1}_{p_1,\lambda,q}}. \tag{2.4}$$

Further, if $s < \rho$, *the estimate*

$$\left\|e^{-t\mathfrak{L}} f\right\|_{\dot{\mathbf{N}}^{\rho}_{r,\lambda,1}} \le \tilde{c} t^{\frac{s-\rho}{2\sigma}} \|f\|_{\dot{\mathbf{N}}^{s}_{r,\lambda,\infty}} \tag{2.5}$$

holds for every $t > 0$.

Lemma 2.2 still holds true with $(-\Delta)^\mu$ in place of ∂^α.

Proof We first prove (2.1) by proceeding in the following way. For all $1 \le p \le \infty$, $0 \le \lambda < n$, $g \in M_{p,\lambda}$, $\phi \in L^1$, in Morrey spaces we have

$$\|g * \phi\|_{p,\lambda} \le \|\phi\|_{L^1} \|g\|_{p,\lambda}. \tag{2.6}$$

Note that (2.6) implies

$$\left\|e^{-t\mathfrak{L}} f\right\|_{p_2,\lambda} = \|K_t * f\|_{p_2,\lambda} \le \|K_t\|_{L^1} \|f\|_{p_2,\lambda}.$$

According to Lemma 2.1 of [28], we have $K_t \in L^p$ for $1 \le p \le \infty$, where $K(x) := (\frac{1}{2\pi})^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-|\xi|^{2\sigma}}\, d\xi$ and $K_t := t^{-\frac{n}{2\sigma}} K(\frac{x}{t^{\frac{1}{2\sigma}}})$. Thus we get $\|e^{-t\mathfrak{L}} f\|_{p_2,\lambda} \le \bar{c} \|f\|_{p_2,\lambda}$.

From Lemma 2.1 of [28], we have the point-wise estimate $|K(x)| \le \bar{c}(1 + |x|)^{-n-2\sigma}$. Hölder's inequality yields $|e^{-t\mathfrak{L}} f|^{p_1} \le \bar{c} K_t * |f|^{p_1}$. Therefore, one has

$$\begin{aligned} \left|e^{-t\mathfrak{L}} f(x)\right|^{p_1} &\le \bar{c} \int_{\mathbb{R}^n} \left|f(y)\right|^{p_1} \left(1 + |x - y| t^{-\frac{1}{2\sigma}}\right)^{-n-2\sigma} dy \\ &\le \bar{c} t^{-\frac{n}{2\sigma}} \int_0^{+\infty} \int_{\partial B(x,r)} \left|f(y)\right|^{p_1} dS_y \left(1 + r t^{-\frac{1}{2\sigma}}\right)^{-n-2\sigma} r^{n-1}\, dr \end{aligned}$$

$$\begin{aligned}
&\leq \bar{c}t^{-\frac{n}{2\sigma}} \int_0^{+\infty} \bigl(1 + rt^{-\frac{1}{2\sigma}}\bigr)^{-n-2\sigma} \, d\rho(r) \\
&\leq \bar{c}t^{-\frac{n}{2\sigma}} \int_0^{+\infty} \bigl(-\rho(r)\bigr) \, d\bigl(1 + rt^{-\frac{1}{2\sigma}}\bigr)^{-n-2\sigma} \\
&\leq \bar{c}t^{-\frac{n}{2\sigma}} t^{-\frac{1}{2\sigma}} \|f\|_{p_1,\lambda}^{p_1} \int_0^{+\infty} r^{\lambda} \bigl(1 + rt^{-\frac{1}{2\sigma}}\bigr)^{-n-2\sigma-1} \, dr \\
&\leq \bar{c}t^{\frac{\lambda-n}{2\sigma}} \|f\|_{p_1,\lambda}^{p_1}.
\end{aligned} \tag{2.7}$$

Using the definition of Morrey spaces in (1.4), we get

$$\rho(r) := \int_{B(x,r)} \bigl|f(y)\bigr|^{p_1} \, dy \leq \|f\|_{p,\lambda}^{p_1} r^{\lambda}.$$

Then from the interpolation inequality it follows that

$$\begin{aligned}
\bigl\|e^{-t\mathfrak{L}}f\bigr\|_{L^{p_2}(B(x,r))} &\leq \bigl\|e^{-t\mathfrak{L}}f\bigr\|_{L^\infty}^{1-\frac{p_1}{p_2}} \bigl\|e^{-t\mathfrak{L}}f\bigr\|_{L^{p_1}(B(x,r))}^{\frac{p_1}{p_2}} \\
&\leq \bigl\|e^{-t\mathfrak{L}}f\bigr\|_{L^\infty}^{1-\frac{p_1}{p_2}} \bigl\|e^{-t\mathfrak{L}}f\bigr\|_{p,\lambda}^{\frac{p_1}{p_2}} r^{\frac{\lambda}{p_2}}.
\end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8) gives

$$\begin{aligned}
\bigl\|e^{-t\mathfrak{L}}f\bigr\|_{p_2,\lambda} &\leq \bigl\|e^{-t\mathfrak{L}}f\bigr\|_{L^\infty}^{1-\frac{p_1}{p_2}} \bigl\|e^{-t\mathfrak{L}}f\bigr\|_{p_1,\lambda}^{\frac{p_1}{p_2}} \\
&\leq \bar{c}t^{\frac{\lambda-n}{2\sigma}(\frac{1}{p_1}-\frac{1}{p_2})} \|f\|_{p_1,\lambda}.
\end{aligned}$$

Thus, we complete the estimate (2.1).

To estimate (2.2), application of the commutativity of the semigroup and derivatives gives the following estimate:

$$\partial^\alpha e^{-t\mathfrak{L}}f = \prod_{j=1}^n \bigl(\partial_j e^{-\frac{t}{2\mu}\mathfrak{L}}\bigr)^{\alpha_j} e^{-\frac{t}{2}\mathfrak{L}}f. \tag{2.9}$$

Then, by (2.6),

$$\bigl\|\partial^\alpha e^{-t\mathfrak{L}}f\bigr\|_{p_2,\lambda} \leq \prod_{j=1}^n \bigl\|\mathcal{F}^{-1}\bigl(i\xi_j e^{-\frac{t}{2\mu}(|\xi|)^{2\sigma}}\bigr)\bigr\|_{L^1}^{\alpha_j} \bigl\|e^{-\frac{t}{2}\mathfrak{L}}f\bigr\|_{p_2,\lambda}. \tag{2.10}$$

With the aid of the Hörmander-Mikhlin type estimate in [29], we obtain

$$\bigl\|\mathcal{F}^{-1}\bigl(i\xi_j e^{-\frac{t}{2\mu}(|\xi|)^{2\sigma}}\bigr)\bigr\|_{L^1} = (c_1\mu)^{\frac{1}{2\sigma}} t^{-\frac{1}{2\sigma}}. \tag{2.11}$$

Applying (2.10) and (2.11), we get

$$\bigl\|\partial^\alpha e^{-t\mathfrak{L}}f\bigr\|_{p_2,\lambda} \leq c_0(c_1\mu)^{\frac{\mu}{2\sigma}} t^{-\frac{\mu}{2\sigma}-\frac{1}{2\sigma}(\frac{n-\lambda}{p_1}-\frac{n-\lambda}{p_2})} \|f\|_{p_1,\lambda}. \tag{2.12}$$

Thus, one obtains the estimate of (2.2).

To estimate (2.3), we apply the frequency projection operator Δ_j to $e^{-t\mathfrak{L}}$ and take the $M_{p,\lambda}$ norm, then by (2.1)

$$\left\|\Delta_j e^{-t\mathfrak{L}} f\right\|_{p_2,\lambda} \leq c t^{-\frac{1}{2\sigma}(\frac{n-\lambda}{p_1}-\frac{n-\lambda}{p_2})}\left\|\Delta_j e^{-\frac{t}{2}\mathfrak{L}} f\right\|_{p_1,\lambda}. \tag{2.13}$$

For every $j \in \mathbb{Z}$, it follows from (2.13) that

$$\begin{aligned}\left\|\Delta_j e^{-t\mathfrak{L}} f\right\|_{p_2,\lambda} &\leq c t^{-\frac{1}{2\sigma}(\frac{n-\lambda}{p_1}-\frac{n-\lambda}{p_2})}\left\|\Delta_j e^{-\frac{t}{2}\mathfrak{L}} f\right\|_{p_1,\lambda} \\ &\leq c t^{-\frac{1}{2\sigma}(\frac{n-\lambda}{p_1}-\frac{n-\lambda}{p_2})} t^{-\frac{s_2-s_1}{2\sigma}} 2^{s_1-s_2}\|\Delta_j f\|_{p_1,\lambda}.\end{aligned} \tag{2.14}$$

By the definition of Besov-Morrey spaces, from (2.14) we get (2.3) immediately.

For (2.4), using the estimate of (2.2), we can prove (2.4) exactly in the same way as deriving (2.3). Here we omit the proof of (2.4).

Assume that $s < \rho$, applying (2.3) with $q = \infty$, we obtain

$$\left\|e^{-t\mathfrak{L}} f\right\|_{\dot{\mathbf{N}}^{2\rho-s}_{r,\lambda,\infty}} \leq \tilde{c} t^{\frac{s-\rho}{\sigma}}\|f\|_{\dot{\mathbf{N}}^{s}_{r,\lambda,\infty}} \tag{2.15}$$

and

$$\left\|e^{-t\mathfrak{L}} f\right\|_{\dot{\mathbf{N}}^{s}_{r,\lambda,\infty}} \leq \tilde{c}\|f\|_{\dot{\mathbf{N}}^{s}_{r,\lambda,\infty}}. \tag{2.16}$$

Using (2.15), (2.16), and the interpolation relation $(\dot{\mathbf{N}}^{2\rho-s}_{r,\lambda,\infty}, \dot{\mathbf{N}}^{s}_{r,\lambda,\infty})_{\frac{1}{2},1} = \dot{\mathbf{N}}^{\rho}_{r,\lambda,1}$ (see Proposition 2.12 of [3]), we get the desired estimate (2.5). Thus, we complete the proof of Lemma 2.2. □

Following the method used by [4], we give the proof of Lemma 2.3.

Lemma 2.3 *Suppose* $1 \leq p, q \leq \infty$, $s > 0$, *and* $0 < \sigma < \infty$, *then one has* $f \in \dot{\mathbf{N}}^{-2s}_{p,\lambda,q}$ *if and only if*

$$\begin{cases} (\int_0^\infty (t^{\frac{s}{\sigma}}\|e^{-t\mathfrak{L}} f\|_{p,\lambda})^q \frac{dt}{t})^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty, \\ \sup_{t>0} t^{\frac{s}{\sigma}}(\|e^{-t\mathfrak{L}} f\|_{p,\lambda}), & \text{if } q = \infty. \end{cases}$$

Proof Let $\mathbb{C} = \{\xi : 0 < r_1 \leq |\xi| \leq r_2, r_1 > 0, r_2 > 0\}$ be an annulus, there exists a positive constant $c > 0$, such that for any $1 \leq p \leq \infty$ and any couple (t, λ) of positive real numbers, from the same ideas from Lemma 2.4 of [4], we have

$$\operatorname{Supp} \hat{u} \subset \lambda\mathbb{C} \quad \Rightarrow \quad \left\|e^{-t\mathfrak{L}} u\right\|_{p,\lambda} \leq c e^{-ct\lambda^{2\sigma}}\|u\|_{p,\lambda}; \tag{2.17}$$

here, we omit the proof (2.17).

In the following, we only show the case $1 \leq q < \infty$. For $q = \infty$ we have the same process. Note that, by (2.17),

$$\left\|t^{\frac{s}{\sigma}}\Delta_j e^{-t\mathfrak{L}} f\right\|_{p,\lambda} \leq c t^{\frac{s}{\sigma}} e^{-ct2^{2j\sigma}}\|\Delta_j f\|_{p,\lambda}. \tag{2.18}$$

Then, in virtue of $f = \sum_{j\in\mathbb{Z}} \Delta_j f$, we deduce that

$$\bigl\|t^{\frac{s}{\sigma}} e^{-t\mathfrak{L}} f\bigr\|_{p,\lambda} \le c\|f\|_{\dot{\mathbf{N}}^{-2s}_{p,\lambda,q}} \sum t^{\frac{s}{\sigma}} 2^{2js} e^{-ct2^{2j\sigma}} c_{r,j}, \tag{2.19}$$

where $\{c_{r,j} = \frac{2^{-2js}\|\Delta_j f\|_{p,\lambda}}{\|f\|_{\dot{\mathbf{N}}^{-2s}_{p,\lambda,q}}}\}_{j\in\mathbb{Z}} \in l^q$. Note that $\|c_{r,j}\|_{l^q} = 1$, the change of variable $\tau = ct2^{2l\sigma}$ yields

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} t^{\frac{s}{\sigma}} 2^{2js} e^{-ct2^{2j\sigma}} &\le \int_{-\infty}^{+\infty} t^{\frac{s}{\sigma}} 2^{2ls} e^{-ct2^{2l\sigma}}\, dl \\
&= \int_0^{+\infty} t^{\frac{s}{\sigma}} \left(\frac{\tau}{ct}\right)^{\frac{s}{\sigma}} e^{-\tau} \frac{1}{ct2^{2\sigma l}(2\sigma \log 2)}\, d\tau \\
&\quad \left(\tau = ct2^{2l\sigma}, \frac{d\tau}{ct} = 2^{2l\sigma} \log\bigl(2^{2\sigma}\bigr)\, dl\right) \\
&= \int_0^{+\infty} t^{\frac{s}{\sigma}} \left(\frac{\tau}{ct}\right)^{\frac{s}{\sigma}} e^{-\tau} \frac{1}{\frac{\tau}{2^{2l\sigma}} 2^{2\sigma l}(2\sigma \log 2)}\, d\tau \quad \left(ct = \frac{\tau}{2^{2l\sigma}}\right) \\
&= \frac{1}{2\sigma c^{s/\sigma} \log 2} \int_0^{+\infty} t^{\frac{s}{\sigma}} \left(\frac{\tau}{t}\right)^{\frac{s}{\sigma}} e^{-\tau} \frac{1}{\tau}\, d\tau \\
&= \frac{1}{2\sigma c^{s/\sigma} \log 2} \int_0^{+\infty} \tau^{\frac{s}{\sigma}-1} e^{-\tau}\, d\tau \\
&= \frac{1}{2\sigma c^{s/\sigma} \log 2} \Gamma(s/\sigma),
\end{aligned} \tag{2.20}$$

which is based on a technique developed in [30] (see (2.59) on p.27 in [30]), where $\Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x}\, dx$ is the Γ function for $s > 0$.

Therefore, Hölder's inequality with weight $t^{\frac{s}{\sigma}} 2^{2js} e^{ct2^{2j\sigma}}$, Fubini's theorem, (2.19), and (2.20) imply that

$$\begin{aligned}
&\int_0^{+\infty} t^{\frac{s}{\sigma}} e^{-t\mathfrak{L}} \|f\|^q_{p,\lambda} \frac{dt}{t} \\
&\quad \le c\|f\|_{\dot{\mathbf{N}}^{-2s}_{p,\lambda,q}} \int_0^{+\infty} \left(\sum_{j\in\mathbb{Z}} t^{\frac{s}{\sigma}} 2^{2jse^{-ct2^{2j\sigma}}}\right)^{q-1} \left(\sum_{j\in\mathbb{Z}} t^{\frac{s}{\sigma}} 2^{2jse^{-ct2^{2j\sigma}}} c^q_{r,j}\right) \frac{dt}{t} \\
&\quad \le c\|f\|_{\dot{\mathbf{N}}^{-2s}_{p,\lambda,q}} \sum_{j\in\mathbb{Z}} c^q_{r,j} \int_0^{+\infty} t^{\frac{s}{\sigma}} 2^{2js} e^{-ct2^{2j\sigma}} \frac{dt}{t} \\
&\quad \le c\|f\|_{\dot{\mathbf{N}}^{-2s}_{p,\lambda,q}} \int_0^{+\infty} t^{\frac{s}{\sigma}-1} e^{-t}\, dt \\
&\quad \lesssim \|f\|_{\dot{\mathbf{N}}^{-2s}_{p,\lambda,q}}.
\end{aligned}$$

Since $\Gamma(\frac{s}{\sigma} + 1) = \int_0^{+\infty} t^{\frac{s}{\sigma}} e^{-t}\, dt$, by the definition of the Fourier transform, we thus get

$$\Delta_j f = \Gamma\left(\frac{s}{\sigma} + 1\right)^{-1} \int_0^{+\infty} t^{\frac{s}{\sigma}} (-\Delta)^{s+\sigma} e^{-t\mathfrak{L}} \Delta_j f\, dt. \tag{2.21}$$

Taking the $M_{p,\lambda}$ norm on (2.21), in view of (2.17), one easily sees that

$$\|\Delta_j f\|_{p,\lambda} \lesssim \Gamma\left(\frac{s}{\sigma}+1\right)^{-1} \int_0^{+\infty} t^{\frac{s}{\sigma}} 2^{2js+2j\sigma} e^{-\frac{ct}{2}2^{2j\sigma}} \left\|e^{-t\mathfrak{L}} f\right\|_{p,\lambda} dt. \tag{2.22}$$

The change of the variable $x = \frac{ct}{2}2^{2m\sigma}$ implies that

$$\begin{aligned}\sum_{j\in\mathbb{Z}} 2^{-2jsq} \|\Delta_j f\|_{p,\lambda}^q &\lesssim \int_0^{+\infty} t^{\frac{sq}{\sigma}} \left\|e^{-t\mathfrak{L}} f\right\|_{p,\lambda}^q \left(\sum_{j\in\mathbb{Z}} 2^{2j\sigma} e^{-\frac{ct}{2}2^{2j\sigma}} t\right)\frac{dt}{t} \\ &\lesssim \int_0^{+\infty} t^{\frac{sq}{\sigma}} \left\|e^{-t\mathfrak{L}} f\right\|_{p,\lambda}^q \left(\int_0^{+\infty} \frac{2x}{c} e^{-x} \frac{1}{\sigma x \ln 2}\, dx\right)\frac{dt}{t} \\ &\lesssim \int_0^{+\infty} t^{\frac{sq}{\sigma}} \left\|e^{-t\mathfrak{L}} f\right\|_{p,\lambda}^q \frac{dt}{t}.\end{aligned}$$

Thus, we complete the proof of Lemma 2.3. □

Lemma 2.4 *For all $\delta \in (\frac{1}{2},1)$, then there exists a constant $c > 0$ such that*

$$\sum_{\tilde{\alpha}\le\gamma} \binom{\gamma}{\tilde{\alpha}} |\tilde{\alpha}|^{|\tilde{\alpha}|-\delta} |\gamma-\tilde{\alpha}|^{|\gamma-\tilde{\alpha}|-\delta} \le c|\gamma|^{|\gamma|-\delta}$$

holds for every $\gamma \in \mathbb{Z}_+^n$, $(\tilde{\alpha}_1,\tilde{\alpha}_2,\dots,\tilde{\alpha}_n) = \tilde{\alpha} \le \gamma = (\gamma_1,\gamma_2,\dots,\gamma_n)$. Note that $\tilde{\alpha} \le \gamma$ means $\tilde{\alpha}_i \le \gamma_i$ for all $i = 1,2,\dots,n$ and $\binom{\gamma}{\tilde{\alpha}} = \prod_{j=1}^n \frac{\gamma_j!}{\tilde{\alpha}_j!(\gamma_j-\tilde{\alpha}_j)!}$.

Proof For the proof of Lemma 2.4, see [8]. □

Lemma 2.5 *Let ψ_0 be a measurable and locally bounded function in $(0,T)$. Let $\{\psi_j\}_{j=1}^\infty$ be a sequence of measurable functions in $(0,T)$. Assume that $\alpha \in \mathbb{R}$ and $\mu,\nu > 0$ satisfying $\mu + \nu = 1$. Let $B_\eta > 0$ be a number depending on $\eta \in (0,1)$, and assume that $B_\eta > 0$ is nonincreasing with respect to η. Assume that there is a positive constant θ such that*

$$0 \le \psi_0(t) \le B_\eta t^{-\alpha} + \theta \int_{(1-\eta)t}^t (t-s)^{-\mu}(s)^{-\nu}\psi_0(s)\, ds$$

and

$$0 \le \psi_{j+1}(t) \le B_\eta t^{-\alpha} + \theta \int_{(1-\eta)t}^t (t-s)^{-\mu}(s)^{-\nu}\psi_j(s)\, ds$$

for all $j \ge 0$, $t > 0$ and $\eta \in (0,1)$. Let η_0 be a unique positive number such that $I(\eta_0) = \min\{\frac{1}{2\theta}, I(1)\}$ with $I(\eta) = \int_{(1-\eta)t}^t (t-s)^{-\mu}(s)^{-\nu-\alpha}\, ds$. Then

$$\psi_j(t) \le 2B_{\tilde{\eta}} t^{-\alpha}$$

for all $j \ge 0$, $0 < \tilde{\eta} \le \eta_0$, and $0 < t < T$.

Proof For the proof of Lemma 2.5, we refer to [10]. □

3 Proof of Theorem 1.1

Before proving Theorem 1.1, we first follow the ideas from [10, 12] and prove a variant of Theorem 1.1 under some additional regularity assumptions.

Proposition 3.1 *Suppose that the assumptions of Theorem* 1.1 *are satisfied. Assume furthermore that*

$$t^{\frac{m}{2\sigma}+\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})}\big(\partial_x^{\tilde{\beta}}u,\partial_x^{\tilde{\beta}}d\big)\in L^\infty\big((0,\infty);M_{q,\lambda}\big) \tag{3.1}$$

for all $r\le q\le\infty$. *Then given* $\frac{1}{2}<\delta\le 1$, *there exist constants* $K_1>0$, $K_2>0$ (*depending only on* n, M_1, M_2, δ, *and* σ), *such that*

$$\big\|\big(\partial_x^{\tilde{\beta}}u,\partial_x^{\tilde{\beta}}d\big)\big\|_{q,\lambda}\le K_1(K_2m)^{2m-\delta}t^{-\frac{m}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})} \tag{3.2}$$

for all $r\le q\le\infty$, *where* $|\tilde{\beta}|=m$.

Proof For $1\le p\le\infty$ and $0\le\lambda<n$, by Lemma 2.2, note that (2.6), there exists a constant $c>0$ such that

$$\big\|e^{-t\mathfrak{L}}\mathbb{P}\nabla f\big\|_{p,\lambda}\le ct^{-\frac{1}{2\sigma}}\|f\|_{p,\lambda}. \tag{3.3}$$

In fact, the proof of (3.3) is essentially the same as the proof of $\|e^{-t\Delta}\mathbb{P}\nabla f\|_p\le ct^{-\frac{1}{2}}\|f\|_p$. The process of proving $\|e^{-t\Delta}\mathbb{P}\nabla f\|_p\le ct^{-\frac{1}{2\sigma}}\|f\|_p$ can be found in [2, 29].

Using Lemma 2.2 and (3.3), for $1\le p_1\le p_2\le\infty$ and $0\le\lambda<n$, a straightforward calculation yields the following elementary estimates:

$$\big\|(-\Delta)^{\frac{\mu}{2}}e^{-t\mathfrak{L}}\mathbb{P}\nabla f\big\|_{p_2,\lambda}\le c_0(c_1\mu)^{\frac{\mu}{2\sigma}}t^{-\frac{\mu+1}{2\sigma}-\frac{1}{2\sigma}(\frac{n-\lambda}{p_1}-\frac{n-\lambda}{p_2})}\|f\|_{p_1,\lambda}. \tag{3.4}$$

We use an induction argument with respect to m.

Step 1. We first shall prove (3.2) for $m=0$. Taking the $M_{q,\lambda}$ norm to the first term of (1.2), for some $\epsilon\in(0,1)$,

$$\begin{aligned}\big\|u(t)\big\|_{q,\lambda}\le{}&\big\|e^{-t\mathfrak{L}}u_0\big\|_{q,\lambda}+\int_0^{(1-\epsilon)t}\big\|e^{-(t-\tau)\mathfrak{L}}\mathbb{P}\nabla(u\otimes u-d\otimes d)\big\|_{q,\lambda}\,d\tau\\&+\int_{(1-\epsilon)t}^{t}\big\|e^{-(t-\tau)\mathfrak{L}}\mathbb{P}\nabla(u\otimes u-d\otimes d)\big\|_{q,\lambda}\,d\tau\\:={}&B_1+B_2+B_3.\end{aligned} \tag{3.5}$$

We shall estimate each term. To estimate the first term B_1 on the right side of (3.5), we note that, by (3.4),

$$\begin{aligned}B_1&\le t^{-\frac{n-\lambda}{2\sigma}(\frac{1}{r}-\frac{1}{q})}\big\|e^{-\frac{t}{2}\mathfrak{L}}u_0\big\|_{r,\lambda}\\&\le t^{-\frac{n-\lambda}{2\sigma}(\frac{1}{r}-\frac{1}{q})}\left(\frac{t}{2}\right)^{-\frac{s}{2\sigma}}\sup_{t>0}\left(\frac{t}{2}\right)^{\frac{s}{2\sigma}}\big\|e^{-\frac{t}{2}\mathfrak{L}}u_0\big\|_{r,\lambda}\\&\le Ct^{-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})}\|u_0\|_{\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}}\\&\le CM_1t^{-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})}.\end{aligned} \tag{3.6}$$

It follows from (4) of Lemma 2.1, (1.5), and (3.4) that

$$\begin{aligned} B_2 &\leq \int_0^{(1-\epsilon)t} (t-\tau)^{-\frac{1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{1}{r}-\frac{1}{q})} \left(\|u\|_{2r,\lambda}^2 + \|d\|_{2r,\lambda}^2 \right) d\tau \\ &\leq M_2^2 \int_0^{(1-\epsilon)t} (t-\tau)^{-\frac{1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{1}{r}-\frac{1}{q})} \tau^{-\alpha} \, d\tau \\ &\leq CM_2^2 t^{-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})} \epsilon^{-\frac{1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{1}{r}-\frac{1}{q})}. \end{aligned} \tag{3.7}$$

By Lemma 2.1, (1.5), and (3.4), similarly we can derive

$$\begin{aligned} B_3 &\leq \int_0^{(1-\epsilon)t} (t-\tau)^{-\frac{1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{q+2r}{2rq}-\frac{1}{q})} \|u \otimes u + d \otimes d\|_{\frac{2rq}{q+2r},\lambda} \, d\tau \\ &\leq \int_0^{(1-\epsilon)t} (t-\tau)^{-\frac{1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{q+2r}{2rq}-\frac{1}{q})} \left(\|u\|_{q,\lambda}\|u\|_{2r,\lambda} + \|d\|_{q,\lambda}\|d\|_{2r,\lambda} \right) d\tau \\ &\leq CM_2 \int_0^{(1-\epsilon)t} (t-\tau)^{-\frac{1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{q+2r}{2rq}-\frac{1}{q})} \tau^{-\frac{\alpha}{2}} \left\| (u,d) \right\|_{q,\lambda} d\tau. \end{aligned} \tag{3.8}$$

Note that $\alpha = \frac{2\sigma-1}{\sigma} - \frac{n-\lambda}{2r\sigma}$, denoting $B_\epsilon = CM_1 + CM_2^2 \epsilon^{-\frac{1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{1}{r}-\frac{1}{q})}$, and combining (3.6), (3.7), and (3.8), one obtains

$$\begin{aligned} \left\| u(t) \right\|_{q,\lambda} &\leq B_\epsilon t^{-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})} \\ &\quad + CM_2 \int_0^{(1-\epsilon)t} (t-\tau)^{-\frac{1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{q+2r}{2rq}-\frac{1}{q})} \tau^{-\frac{\alpha}{2}} \left\| (u,d) \right\|_{q,\lambda} d\tau. \end{aligned} \tag{3.9}$$

Similarly, we can get the desired estimate of $\|d\|_{q,\lambda}$,

$$\begin{aligned} \left\| d(t) \right\|_{q,\lambda} &\leq B_\epsilon t^{-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})} \\ &\quad + CM_2 \int_0^{(1-\epsilon)t} (t-\tau)^{-\frac{1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{q+2r}{2rq}-\frac{1}{q})} \tau^{-\frac{\alpha}{2}} \left\| (u,d) \right\|_{q,\lambda} d\tau. \end{aligned} \tag{3.10}$$

Thus, by (3.9) and (3.10),

$$\begin{aligned} \left\| \left(u(t), d(t) \right) \right\|_{q,\lambda} &\leq B_\epsilon t^{-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})} \\ &\quad + CM_2 \int_0^{(1-\epsilon)t} (t-\tau)^{-\frac{1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{q+2r}{2rq}-\frac{1}{q})} \tau^{-\frac{\alpha}{2}} \left\| (u,d) \right\|_{q,\lambda} d\tau. \end{aligned}$$

Therefore, according to Lemma 2.5, we get (3.2) for $|\tilde{\beta}| = m = 0$.

Step 2. We next consider the case $m = 1$. The proof of (3.2) is essentially contained in Step 3. Thus here we omit the details.

Step 3. Assume that $m \geq 2$. We suppose that (3.2) holds for $q \in [r, \infty]$ and all $|\tilde{\beta}| \leq m-1$. We need to prove that (3.2) holds for $|\tilde{\beta}| = m$. Then, for $|\tilde{\beta}| = m$ and some $\epsilon \in (0,1)$, we see that

$$\begin{aligned} \left\| \partial^\beta u(t) \right\|_{q,\lambda} &\leq \left\| \partial^{\tilde{\beta}} e^{-t\mathfrak{L}} u_0 \right\|_{q,\lambda} \\ &\quad + \int_0^{(1-\epsilon)t} \left\| \partial^{\tilde{\beta}} e^{-(t-\tau)\mathfrak{L}} \mathbb{P} \nabla (u \otimes u - d \otimes d) \right\|_{q,\lambda} d\tau \end{aligned}$$

$$\begin{aligned}&+\int_{(1-\epsilon)t}^{t}\big\|\partial^{\tilde{\beta}}e^{-(t-\tau)\mathfrak{L}}\mathbb{P}\nabla(u\otimes u-d\otimes d)\big\|_{q,\lambda}\,d\tau\\ &=:A_1+A_2+A_3.\end{aligned}$$

We shall estimate each of the above terms A_1, A_2, A_3 separately. Note that $\frac{m}{2\sigma}\le 2m-\delta$, since $m\ge 2$ and $0<\delta\le 1$. Observe that by (1.5), and Lemmas 2.2 and 2.3

$$\begin{aligned}A_1&\le t^{-\frac{n-\lambda}{2\sigma}(\frac{1}{r}-\frac{1}{q})}\big\|e^{-\frac{t}{2}\mathfrak{L}}u_0\big\|_{r,\lambda}\\ &\le\tilde{C}_1(\tilde{C}_2m)^{\frac{m}{2\sigma}}t^{-\frac{m}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{1}{r}-\frac{1}{q})}\left(\frac{t}{2}\right)^{-\frac{s}{2\sigma}}\sup_{t>0}\left(\frac{t}{2}\right)^{\frac{s}{2\sigma}}\big\|e^{-\frac{t}{2}\mathfrak{L}}u_0\big\|_{r,\lambda}\\ &\lesssim\tilde{C}_1(\tilde{C}_2m)^{\frac{m}{2\sigma}}t^{-\frac{m}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})}\|u_0\|_{\dot{\mathbf{N}}^{-\tilde{\beta}}_{r,\lambda,\infty}}\\ &\lesssim C_1M_1(C_2m)^{2m-\delta}t^{-\frac{m}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})}.\end{aligned}\tag{3.11}$$

To estimate the term A_2, we note that, by Lemma 2.1, (1.5), and (3.4),

$$\begin{aligned}A_2&\le\tilde{C}_3(\tilde{C}_4m)^{\frac{m}{2\sigma}}\int_0^{(1-\epsilon)t}(t-\tau)^{-\frac{m+1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{1}{r}-\frac{1}{q})}\|u\otimes u-d\otimes d\|_{r,\lambda}\,d\tau\\ &\le\tilde{C}_3(\tilde{C}_4m)^{\frac{m}{2\sigma}}M_2^2\int_0^{(1-\epsilon)t}(t-\tau)^{-\frac{m+1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{1}{r}-\frac{1}{q})}\tau^{-\alpha}\,d\tau\\ &\le C_3(C_4m)^{\frac{m}{2\sigma}}M_2^2t^{-\frac{m}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})}\epsilon^{-\frac{m+1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{1}{r}-\frac{1}{q})}\\ &\le C_3(C_4m)^{2m-\delta}M_2^2t^{-\frac{m}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})}\epsilon^{-\frac{m+1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{1}{r}-\frac{1}{q})}.\end{aligned}\tag{3.12}$$

We now calculate $\nabla^m(u\otimes u-d\otimes d)$ by Leibniz's rule. Lemma 2.1 and (3.4) yield

$$\begin{aligned}A_3&\le 2C\int_{(1-\epsilon)t}^{t}(t-\tau)^{-\frac{1}{2\sigma}}\left(\big\|\nabla^m u\big\|_{q,\lambda}\|u\|_{L^\infty}+\big\|\nabla^m d\big\|_{q,\lambda}\|d\|_{L^\infty}\right)d\tau\\ &\quad+C\int_{(1-\epsilon)t}^{t}(t-\tau)^{-\frac{1}{2\sigma}}\\ &\quad\times\max_{|\tilde{\beta}|=m}\sum_{0<\gamma<\tilde{\beta}}\binom{\tilde{\beta}}{\gamma}\left(\big\|\partial_x^\gamma u\big\|_{q,\lambda}\big\|\partial_x^{\tilde{\beta}-\gamma}u\big\|_{L^\infty}+\big\|\partial_x^\gamma d\big\|_{q,\lambda}\big\|\partial_x^{\tilde{\beta}-\gamma}d\big\|_{L^\infty}\right)d\tau\\ &=:A_{31}+A_{32}.\end{aligned}\tag{3.13}$$

Here, $\gamma<\tilde{\beta}$ means $\gamma_i\le\tilde{\beta}_i$ and $|\gamma|<|\tilde{\beta}|$ for the multi-indices $\tilde{\beta}=(\tilde{\beta}_1,\tilde{\beta}_2,\dots,\tilde{\beta}_n)$ and $\gamma=(\gamma_1,\gamma_2,\dots,\gamma_n)$, where $i=1,2,\dots,n$.

In order to estimate the first term on the right hand of (3.13), according to Step 1, we note that there exists $c_5>0$ such that $\|(u,d)\|_{\infty,\lambda}=\|(u,d)\|_{L^\infty}\le C_5K_1t^{\frac{1-2\sigma}{2\sigma}}$, then

$$\begin{aligned}A_{31}&\le C_5K_1\int_{(1-\epsilon)t}^{t}(t-\tau)^{-\frac{1}{2\sigma}}\tau^{\frac{1-2\sigma}{2\sigma}}\left(\big\|\nabla^m u\big\|_{q,\lambda}+\big\|\nabla^m d\big\|_{q,\lambda}\right)d\tau\\ &\le C_5K_1\int_{(1-\epsilon)t}^{t}(t-\tau)^{-\frac{1}{2\sigma}}\tau^{\frac{1-2\sigma}{2\sigma}}\big\|\left(\nabla^m u,\nabla^m d\right)\big\|_{q,\lambda}\,d\tau.\end{aligned}\tag{3.14}$$

By the assumption of the induction, we obtain

$$\begin{aligned} A_{32} &\le C_6 \int_{(1-\epsilon)t}^{t} (t-\tau)^{-\frac{1}{2\sigma}} \max_{|\tilde{\beta}|=m} \sum_{0<\gamma<\tilde{\beta}} \binom{\tilde{\beta}}{\gamma} K_1 \bigl(K_2|\tilde{\beta}-\gamma|\bigr)^{2|\tilde{\beta}-\gamma|-\delta} \tau^{-\frac{|\tilde{\beta}-\gamma|}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})} \\ &\quad \times K_1 \bigl(K_2|\gamma|\bigr)^{2|\gamma|-\delta} \tau^{-\frac{|\gamma|}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1)}\,d\tau \\ &\le C_6 K_1^2 K_2^{2m-2\delta} \sum_{0<\gamma<\tilde{\beta}} \binom{\tilde{\beta}}{\gamma} |\tilde{\beta}-\gamma|^{2|\tilde{\beta}-\gamma|-\delta} |\gamma|^{2|\gamma|-\delta} \\ &\quad \times \int_{(1-\epsilon)t}^{t} (t-\tau)^{-\frac{1}{2\sigma}} \tau^{-\frac{|\tilde{\beta}-\gamma|}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})} \tau^{-\frac{|\gamma|}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1)}\,d\tau \\ &\le C_6 K_1^2 K_2^{2m-2\delta} \sum_{0<\gamma<\tilde{\beta}} \binom{\tilde{\beta}}{\gamma} |\tilde{\beta}-\gamma|^{2|\tilde{\beta}-\gamma|-\delta} |\gamma|^{2|\gamma|-\delta} t^{-\frac{m}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})} \\ &\quad \times \int_{(1-\epsilon)t}^{1} (1-\tau)^{-\frac{1}{2\sigma}} \tau^{-\frac{m}{2\sigma}-\frac{1}{2\sigma}(4\sigma-2-\frac{n-\lambda}{q})}\,d\tau. \end{aligned}$$

Applying Lemma 2.4, it follows that

$$A_{32} \le C_6 K_1^2 K_2^{2m-2\delta} m^{2m-\delta} t^{-\frac{m}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})} I(\epsilon), \tag{3.15}$$

where $I(\epsilon) := \int_{1-\epsilon}^{1} (1-\tau)^{-\frac{1}{2\sigma}} \tau^{-\frac{m}{2\sigma}-\frac{1}{2\sigma}(4\sigma-2-\frac{n-\lambda}{q})}\,d\tau$.

Note that we set

$$\begin{aligned} b_\epsilon &= C_1 M_1 (C_2 m)^{2m-\delta} + C_3 (C_4 m)^{2m-\delta} M_2^2 \epsilon^{-\frac{m+1}{2\sigma}-\frac{n-\lambda}{2\sigma}(\frac{1}{r}-\frac{1}{q})} \\ &\quad + C_6 K_1^2 K_2^{2m-2\delta} m^{2m-\delta} I(\epsilon). \end{aligned} \tag{3.16}$$

Combining the above estimates for (3.11), (3.12), (3.13), (3.14), (3.15), and (3.16), we obtain

$$\bigl\|\nabla^m u(t)\bigr\|_{q,\lambda} \le b_\epsilon t^{-\frac{m}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})} + C_5 K_1 \int_{(1-\epsilon)t}^{t} (t-\tau)^{-\frac{1}{2\sigma}} \tau^{\frac{1-2\sigma}{2\sigma}} \bigl\|\bigl(\nabla^m u, \nabla^m d\bigr)\bigr\|_{q,\lambda}\,d\tau.$$

Similarly, from a computation it follows that

$$\bigl\|\nabla^m d(t)\bigr\|_{q,\lambda} \le b_\epsilon t^{-\frac{m}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})} + C_5 K_1 \int_{(1-\epsilon)t}^{t} (t-\tau)^{-\frac{1}{2\sigma}} \tau^{\frac{1-2\sigma}{2\sigma}} \bigl\|\bigl(\nabla^m u, \nabla^m d\bigr)\bigr\|_{q,\lambda}\,d\tau.$$

Thus, we have

$$\begin{aligned} &\bigl\|\bigl(\nabla^m u(t), \nabla^m d(t)\bigr)\bigr\|_{q,\lambda} \\ &\quad \le b_\epsilon t^{-\frac{m}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})} + C_5 K_1 \int_{(1-\epsilon)t}^{t} (t-\tau)^{-\frac{1}{2\sigma}} \tau^{\frac{1-2\sigma}{2\sigma}} \bigl\|\bigl(\nabla^m u, \nabla^m d\bigr)\bigr\|_{q,\lambda}\,d\tau. \end{aligned}$$

Applying Lemma 2.5, we see that there exists $\epsilon_{m_0} \in (0,1)$, such that for any $0 < \epsilon_m \le \epsilon_{m_0}$, we have

$$\bigl\|\bigl(\nabla^m u(t), \nabla^m d(t)\bigr)\bigr\|_{q,\lambda} \le 2 b_{\epsilon_m} t^{-\frac{m}{2\sigma}-\frac{1}{2\sigma}(2\sigma-1-\frac{n-\lambda}{q})},$$

where $I(\epsilon_{m_0}) = \min\{\frac{1}{2C_5K_1}, I(1)\}$. Let $\epsilon_m = \frac{1}{m^{2\sigma}}$, since $I(\epsilon)$ is nonincreasing with respect to ϵ. We can choose $m_0 > 2$ sufficiently large such that $I(\frac{1}{m^{2\sigma}}) \leq \frac{1}{2C_5K_1}$ for all $m \geq m_0$. Hence, we obtain

$$\left\| \left(\nabla^m u(t), \nabla^m d(t)\right) \right\|_{q,\lambda} \leq 2b_{\frac{1}{m^{2\sigma}}} t^{-\frac{m}{2\sigma} - \frac{1}{2\sigma}(2\sigma - 1 - \frac{n-\lambda}{q})}. \tag{3.17}$$

By (3.17), we can choose K_1 and K_2 sufficiently large such that (3.2) holds for all $|\tilde{\beta}| \leq m_0$. Finally, it is enough to show that $2b_{\frac{1}{m^{2\sigma}}} \leq K_1(K_2|\tilde{\beta}|)^{2|\tilde{\beta}|-\delta}$ for any $m > m_0 \geq 2$ with constants K_1 and K_2 sufficiently large.

Next, we compute $I(\frac{1}{m^{2\sigma}})$,

$$\begin{aligned}
I\left(\frac{1}{m^{2\sigma}}\right) &\leq \int_{1-\frac{1}{m^{2\sigma}}}^{1} (1-\tau)^{-\frac{1}{2\sigma}} \tau^{-\frac{m}{2\sigma} - \frac{1}{2\sigma}(4\sigma - 2 - \frac{n-\lambda}{q})} \, d\tau \\
&\leq \int_{1-\frac{1}{m^{2\sigma}}}^{1} (1-\tau)^{-\frac{1}{2\sigma}} \, d\tau \left(1 - \frac{1}{m^{2\sigma}}\right)^{-\frac{m}{2\sigma}} \left(1 - \frac{1}{m^{2\sigma}}\right)^{-\frac{1}{2\sigma}(4\sigma - 2 - \frac{n-\lambda}{q})} \\
&\leq \frac{2\sigma}{2\sigma - 1} m_0^{1-2\sigma} e^{\frac{m_0^{1-2\sigma}}{2\sigma}} \left(1 - \frac{1}{m_0^{2\sigma}}\right)^{-\frac{1}{2\sigma}(4\sigma - 2)} \\
&\leq \frac{2\sigma}{2\sigma - 1} 2^{1-2\sigma} e^{\frac{2^{1-2\sigma}}{2\sigma}} \left(1 - \frac{1}{2^{2\sigma}}\right)^{-\frac{1}{2\sigma}(4\sigma - 2)} \\
&\leq C(\sigma).
\end{aligned} \tag{3.18}$$

Since $\delta \leq 2m - \frac{m}{2\sigma}$ ($\frac{1}{2} < \delta \leq 1$, $m \geq 2$, $\frac{1}{2} < \sigma < 2$), $r > n - \lambda$, $\frac{1}{2} + \frac{n-\lambda}{4r} < \sigma < 1 + \frac{n-\lambda}{4r} < 2$, and $m^{\delta+2} \leq 8 \cdot 2^{m-\delta}\sqrt{m} \leq 8 \cdot 2^{2m-\delta}$, we thus have

$$\begin{aligned}
2b_{\frac{1}{m^{2\sigma}}} &\leq 2\left\{ C_1M_1C_2^{2m-\delta} m^{2m-\delta} + C_3M_2^2C_4^{\frac{m}{2\sigma}} m^{\frac{m}{2\sigma}} \left(\frac{1}{m^{2\sigma}}\right)^{-\frac{m+1}{2\sigma} - \frac{n-\lambda}{2\sigma}(\frac{1}{r} - \frac{1}{q})} \right. \\
&\quad \left. + C_6C(\sigma)K_1^2K_2^{2m-2\delta} m^{2m-\delta} \right\} \\
&\leq 2\left\{ C_1M_1C_2^{2m-\delta} m^{2m-\delta} + C_3M_2^2C_4^{\frac{m}{2\sigma}} m^{\frac{m}{2\sigma}} m^{m+1+\frac{n-\lambda}{r} - \frac{n-\lambda}{q}} \right. \\
&\quad \left. + C_6C(\sigma)K_1^2K_2^{2m-2\delta} m^{2m-\delta} \right\} \\
&\leq 2\left\{ C_1M_1C_2^{2m-\delta} m^{2m-\delta} + C_3M_2^2C_4^{2m-\delta} m^{2m-\delta} m^{\delta+2} + C_6C(\sigma)K_1^2K_2^{2m-2\delta} m^{2m-\delta} \right\} \\
&\leq 2\left\{ C_1M_1C_2^{2m-\delta} m^{2m-\delta} + C_3M_2^2C_4^{2m-\delta} m^{2m-\delta} 8 \cdot 2^{2m-\delta} \right. \\
&\quad \left. + C_6C(\sigma)K_1^2K_2^{2m-2\delta} m^{2m-\delta} \right\} \\
&\leq 2\left\{ C_1M_1C_2^{2m-\delta} + C_3M_2^2C_4^{2m-\delta} 2^{2m-\delta} + C_6C(\sigma)K_1^2K_2^{2m-2\delta} \right\} m^{2m-\delta}.
\end{aligned}$$

We choose the constants $K_1 := 4C_1M_1 + 4C_3M_2^2$. We take K_2 large enough, such that $K_2 \geq 2C_4 + C_2$ and $C_6C(\sigma)K_1K_2^{-\delta} < \frac{1}{2}$. Then we obtain (3.2) immediately. □

Proposition 3.2 *Suppose that the assumptions of Theorem* 1.1 *are satisfied. Then the mild solution* (u, d) *of* (1.2) *satisfies* (3.1), *and there exist constants* $\tilde{K}_1, \tilde{K}_2 > 0$ *such that*

$$\left\| \left(\partial_x^{\tilde{\beta}} u, \partial_x^{\tilde{\beta}} d\right) \right\|_{q,\lambda} \leq \tilde{K}_1(\tilde{K}_2 m)^{2m} t^{-\frac{m}{2\sigma} - \frac{1}{2\sigma}(2\sigma - 1 - \frac{n-\lambda}{q})} \tag{3.19}$$

for all $r \leq q \leq \infty$, *where* $|\tilde{\beta}| = m$.

Proof The proof is now standard, we refer the reader to [10–12]. □

Now Theorem 1.1 follows immediately from Proposition 3.1 and Proposition 3.2. We thus complete the proof of Theorem 1.1.

Appendix

In this Appendix, we will show the global existence of solution for system (1.1) mentioned in Theorem 1.1 or (I) of Remarks below Theorem 1.1. We note that, similarly to the Navier-Stokes equations ([3], Theorems 3 and 4, p.967), the proof of global existence can be obtained by making minor modifications to Theorems 3 and 4 on p.967 in [3]. Here, we give a brief argument of this proof for completeness and for convenience of the reader.

We say $(u,d) \in E^{\epsilon_0} \times E^{\epsilon_0}$ if $(u,d) \in E \times E$ and $\|(u,d)\|_{E^{\epsilon_0}} = \|(u,d)\|_E = \sup_{t>0} \|(u(t), d(t))\|_{\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}} + \sup_{t>0} t^{\frac{\alpha}{2}} \|(u(t),d(t))\|_{2r,\lambda} \le C\epsilon_0$. The definition of E can be found in (I) of Remarks below Theorem 1.1.

Lemma A.1 *Let* $n \ge 2, 1 \le r < \infty, 0 \le \lambda < n, r > n-\lambda, \frac{1}{2} + \frac{n-\lambda}{4r} < \sigma < 1 + \frac{n-\lambda}{4r}, s = 2\sigma - 1 - \frac{n-\lambda}{r}$, $\alpha = \frac{2\sigma-1}{\sigma} - \frac{n-\lambda}{2r\sigma}$, $\nabla \cdot u_0 = 0$, $\nabla \cdot d_0 = 0$, $(u_0,d_0) \in \dot{\mathbf{N}}^{-s}_{r,\lambda,\infty} \times \dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}$, $q \in [r,\infty]$. *There exists a constant* $M_1 > 0$, *such that* (u_0,d_0) *satisfies* (1.5), *then we have* $(\overline{u}_0, \overline{d}_0) \in E_{M_1} \times E_{M_1}$, *where* $\bar{u}_0 = e^{-t\mathfrak{L}} u_0$ *and* $\bar{d}_0 = e^{-t\mathfrak{L}} d_0$.

Proof From (2.3) of Lemma 2.2, we thus obtain

$$\|(\bar{u}_0, \bar{d}_0)\|_{\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}} \lesssim \|(u_0,d_0)\|_{\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}}. \tag{A.1}$$

Note that $\alpha\sigma + \frac{n-\lambda}{2r} = s + \frac{n-\lambda}{r}$, it follows from Lemma 2.3 and a Sobolev-type embedding of Lemma 2.1 that

$$\sup_{t>0} t^{\frac{\alpha}{2}} \|(\bar{u}_0, \bar{d}_0)\|_{2r,\lambda} \cong \|(u_0,d_0)\|_{\dot{\mathbf{N}}^{-\alpha\sigma}_{2r,\lambda,\infty}} \lesssim \|(u_0,d_0)\|_{\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}}. \tag{A.2}$$

Hence, the proof of Lemma A.1 is now completed. □

Define

$$\begin{cases} \Phi_1(u,d) = -\int_0^t e^{-(t-s)\mathfrak{L}} \mathbb{P}\nabla \cdot (u \otimes u - d \otimes d)(\cdot,s)\,ds, \\ \Phi_2(u,d) = -\int_0^t e^{-(t-s)\mathfrak{L}} \mathbb{P}\nabla \cdot (u \otimes d - d \otimes u)(\cdot,s)\,ds. \end{cases} \tag{A.3}$$

Lemma A.2 *Let* $n \ge 2, 1 \le r < \infty, 0 \le \lambda < n, r > n-\lambda, \frac{1}{2} + \frac{n-\lambda}{4r} < \sigma < 1 + \frac{n-\lambda}{4r}, s = 2\sigma - 1 - \frac{n-\lambda}{r}$, $\alpha = \frac{2\sigma-1}{\sigma} - \frac{n-\lambda}{2r\sigma}$. Φ_1 *and* Φ_2 *were defined by* (A.3), *respectively. It holds true that*

$$\sup_{t>0} \|(\Phi_1(u,d), \Phi_2(u,d))\|_{\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}} \lesssim \|(u,d)\|_E^2,$$

$$\sup_{t>0} t^{\frac{\alpha}{2}} \|(\Phi_1(u,d), \Phi_2(u,d))\|_{2r,\lambda} \lesssim \|(u,d)\|_E^2$$

for all $(u,d) \in E \times E$.

Proof From Lemmas 2.1 and 2.2, it follows that

$$\begin{aligned}
\|\Phi_1(u,d)\|_{\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}} &\lesssim \int_0^t (t-s)^{\frac{s-1}{2\sigma}} \|\mathbb{P}\nabla\cdot(u\otimes u-d\otimes d)(\cdot,s)\|_{\dot{\mathbf{N}}^{-1}_{r,\lambda,\infty}}\,ds \\
&\lesssim \int_0^t (t-s)^{\frac{s-1}{2\sigma}} \|(u\otimes u-d\otimes d)(\cdot,s)\|_{\dot{\mathbf{N}}^{0}_{r,\lambda,\infty}}\,ds \\
&\lesssim \int_0^t (t-s)^{\frac{s-1}{2\sigma}} \|(u\otimes u-d\otimes d)(\cdot,s)\|_{r,\lambda}\,ds \\
&\lesssim \int_0^t (t-s)^{\frac{s-1}{2\sigma}} \big(\|u(\cdot,s)\|^2_{2r,\lambda}+\|d(\cdot,s)\|^2_{2r,\lambda}\big)\,ds \\
&\lesssim \int_0^t (t-s)^{\frac{s-1}{2\sigma}} s^{-\alpha}\,ds\big(\|u\|_E^2+\|d\|_E^2\big) \\
&\lesssim \|(u,d)\|_E^2. \qquad\qquad (A.4)
\end{aligned}$$

By Lemmas 2.1, 2.2 and 2.3, we obtain the estimate $t^{\frac{\alpha}{2}}\|\Phi_1(u,d)\|_{2r,\lambda}\lesssim t^{\frac{\alpha}{2}}\|\Phi_1(u,d)\|_{\dot{\mathbf{N}}^{0}_{2r,\lambda,1}}\lesssim t^{\frac{\alpha}{2}}\|\Phi_1(u,d)\|_{\dot{\mathbf{N}}^{2\sigma-1-\alpha\sigma}_{r,\lambda,1}}$. We thus obtain

$$\begin{aligned}
t^{\frac{\alpha}{2}}\|\Phi_1(u,d)\|_{2r,\lambda} &\lesssim t^{\frac{\alpha}{2}}\int_0^t (t-s)^{\frac{\alpha-2}{2}} \|\mathbb{P}\nabla\cdot(u\otimes u-d\otimes d)(\cdot,s)\|_{\dot{\mathbf{N}}^{-1}_{r,\lambda,\infty}}\,ds \\
&\lesssim t^{\frac{\alpha}{2}}\int_0^t (t-s)^{\frac{\alpha-2}{2}} \|\mathbb{P}(u\otimes u-d\otimes d)(\cdot,s)\|_{\dot{\mathbf{N}}^{0}_{r,\lambda,\infty}}\,ds \\
&\lesssim t^{\frac{\alpha}{2}}\int_0^t (t-s)^{\frac{\alpha-2}{2}} \big(\|u(\cdot,s)\|^2_{2r,\lambda}+\|d(\cdot,s)\|^2_{2r,\lambda}\big)\,ds \\
&\lesssim t^{\frac{\alpha}{2}}\int_0^t (t-s)^{\frac{\alpha-2}{2}} s^{-\alpha}\,ds\big(\|u\|_E^2+\|d\|_E^2\big) \\
&\lesssim \|(u,d)\|_E^2. \qquad\qquad (A.5)
\end{aligned}$$

In the following, in a similar way to the derivation of (A.4) and (A.5), we have

$$\begin{aligned}
&\|\Phi_2(u,d)\|_{\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}} \lesssim \|u\|_E\|d\|_E \lesssim \|(u,d)\|_E^2, \\
&t^{\frac{\alpha}{2}}\|\nabla\Phi_2(u,d)\|_{2r,\lambda} \lesssim \|u\|_E\|d\|_E \lesssim \|(u,d)\|_E^2.
\end{aligned} \qquad (A.6)$$

Thus, we complete the proof of Lemma A.2. □

Lemma A.3 *Let* $n\ge 2$, $1\le r<\infty$, $0\le\lambda<n$, $r>n-\lambda$, $\frac{1}{2}+\frac{n-\lambda}{4r}<\sigma<1+\frac{n-\lambda}{4r}$, $s=2\sigma-1-\frac{n-\lambda}{r}$, $\alpha=\frac{2\sigma-1}{\sigma}-\frac{n-\lambda}{2r\sigma}$, $\nabla\cdot u_0=0$, $\nabla\cdot d_0=0$, $(u_0,d_0)\in\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}\times\dot{\mathbf{N}}^{-s}_{r,\lambda,\infty}$, $q\in[r,\infty]$. *Given a constant* $M_2>0$ *small enough, let* $(\hat{u},\hat{d})\in E_{M_2}\times E_{M_2}$, *and* (u_0,d_0) *satisfy* (1.5), *then* $(u,d)\in E_{M_2}\times E_{M_2}$, *where*

$$\begin{aligned}
u &= e^{-t\mathfrak{L}}u_0-\int_0^t e^{-(t-s)\mathfrak{L}}\mathbb{P}\nabla\cdot(\hat{u}\otimes\hat{u}-\hat{d}\otimes\hat{d})(\cdot,s)\,ds, \\
d &= e^{-t\mathfrak{L}}d_0-\int_0^t e^{-(t-s)\mathfrak{L}}\mathbb{P}\nabla\cdot(\hat{u}\otimes\hat{d}-\hat{d}\otimes\hat{u})(\cdot,s)\,ds.
\end{aligned}$$

Proof We will prove $(u,d) \in E_{M_2} \times E_{M_2}$. Due to Lemmas A.1 and A.2, one thus has

$$\begin{aligned}\|(u,d)\|_E &\lesssim \|(e^{-t\mathfrak{L}}u_0, e^{-t\mathfrak{L}}d_0)\|_E + \sup_{t>0}\|(\Phi_1(\hat{u},\hat{d}), \Phi_2(\hat{u},\hat{d}))\|_{\dot{\mathbf{N}}^{-\beta}_{r,\lambda,\infty}} \\ &\quad + \sup_{t>0} t^{\frac{\alpha}{2}}\|(\Phi_1(\hat{u},\hat{d}), \Phi_2(\hat{u},\hat{d}))\|_{2r,\lambda} \\ &\le C_1M_1 + C_2M_2^2 \\ &\le CM_2, \end{aligned} \tag{A.7}$$

provided $0 < M_1 \le M_2$ is chosen to be sufficiently small, where we have used the estimate

$$C_1M_1 + C_2M_2^2 \le M_2(C_1 + C_1M_2) \le M_2(C_1 + C_2) = CM_2$$

in the last step. Therefore, we obtain $(u,d) \in E_{M_2} \times E_{M_2}$.

Hence, the proof of Lemma A.3 is finished. □

Lemma A.4 *For all $M_2 > 0$ small enough, let $(u,d) \in E_{M_2} \times E_{M_2}$ and $(\tilde{u},\tilde{d}) \in E_{M_2} \times E_{M_2}$ with the same initial data (u_0,d_0), then $\Phi = [\Phi_1, \Phi_2]$ defined in* (A.3) *is a contractive map.*

Proof Let $\mathfrak{u} = u - \tilde{u}$ and $\mathfrak{d} = d - \tilde{d}$, repeating the proof as Lemma A.2, it holds true that

$$\begin{aligned}\|\Phi_1(u,d) - \Phi_1(\tilde{u},\tilde{d})\|_{E_{M_2}} &\lesssim (\|\tilde{u}\|_{E_{M_2}} + \|u\|_{E_{M_2}})\|\mathfrak{u}\|_{E_{M_2}} + (\|\tilde{d}\|_E + \|d\|_{E_{M_2}})\|\mathfrak{d}\|_{E_{M_2}} \\ &\le CM_2(\|\mathfrak{u}\|_{E_{M_2}} + \|\mathfrak{d}\|_{E_{M_2}}).\end{aligned}$$

Meanwhile, similar to the proof of Lemma A.2, we have

$$\begin{aligned}\|\Phi_2(u,d) - \Phi_2(\tilde{u},\tilde{d})\|_{E_{M_2}} &\lesssim (\|\tilde{u}\|_{E_{M_2}} + \|u\|_{E_{M_2}})\|\mathfrak{d}\|_{E_{M_2}} + (\|\tilde{d}\|_{E_{M_2}} + \|d\|_{E_{M_2}})\|\mathfrak{u}\|_{E_{M_2}} \\ &\le CM_2(\|\mathfrak{u}\|_{E_{M_2}} + \|\mathfrak{d}\|_{E_{M_2}}).\end{aligned}$$

Taking $M_2 > 0$ small enough, there exists $0 < \theta < \frac{1}{2}$, such that

$$\|\Phi_1(u,d) - \Phi_1(\tilde{u},\tilde{d})\|_{E_{M_2}} + \|\Phi_2(u,d) - \Phi_2(\tilde{u},\tilde{d})\|_{E_{M_2}} \le \theta(\|d - \tilde{d}\|_{E_{M_2}} + \|u - \tilde{u}\|_{E_{M_2}}).$$

Therefore, $\Phi = [\Phi_1, \Phi_2]$ is a contractive map and we complete the proof of Lemma A.4. □

Applying Banach's fixed pointed theorem, we finish the proof of global existence, it following directly from Lemmas A.1, A.2, A.3, and A.4.

Competing interests

The author declares to have no competing interests.

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