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Blow-up and nonexistence of solutions of some semilinear degenerate parabolic equations

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Abstract
In this paper we study a class of semilinear degenerate parabolic equations arising in mathematical finance and in the theory of diffusion processes. We show that blow-up of spatial derivatives of smooth solutions in finite time occurs to initial boundary value problems for a class of degenerate parabolic equations. Furthermore, nonexistence of nontrivial global weak solutions to initial value problems is studied by choosing a special test function. Finally, the phenomenon of blow-up is verified by a numerical experiment.

MSC: 35K57; 35K65; 35K70

Keywords: degenerate parabolic equation; blow-up; nonexistence

1 Introduction

In this paper, we consider the equation

$$u_{xx}(z) + u(z)u_y(z) - u_t(z) = f, \quad \text{in } \mathbb{R} \times \mathbb{R} \times (0, T), \tag{1.1}$$

where $z = (x, y, t)$ denotes the point in $\mathbb{R}^3$. This equation arises in mathematical finance [1] and in the physical phenomena such as diffusion and convection of matter. One of the main features of equation (1.1) is the strong degeneracy due to the lack of diffusion in the y-direction. We restrict our consideration to two cases: the initial boundary value problems of (1.1) and the initial value problems of (1.1).

Regarding the theoretical analysis of (1.1), most scholars have been devoted to the study of well-posedness and regularity of solutions [2–5]. Antonelli and Pascucci [2] proved that there exists a unique viscosity solution to the initial value problem for (1.1) in a small time. The existence and uniqueness of a global solution in an unbounded domain was studied by Vol'pert and Hudjaev [5]. On the regularity of solutions, Citti *et al.* [3] proved that the viscosity solution of (1.1) is a classical solution in the sense that u_{xx}, $uu_y - u_t$ are continuous and the equation is pointwise satisfied. Furthermore, they obtained the smooth solution of (1.1) when $f(z) \in C^\infty(\Omega)$ and $\partial_x u \neq 0$, in an open set $\Omega \subset \mathbb{R}^3$ in [4].

Blow-up and nonexistence of solutions for (1.1) are as important aspects of properties of partial differential equations. In [6], Fujita described the initial problem of a semi-linear parabolic equation, which takes place blowing up even when the initial data is very nice. Ever since then, results about blow-up and nonexistence have been generalized to deal with some more general semilinear, quasilinear and fully nonlinear parabolic equations and systems. Without being exhaustive with the amount of references concerned with this topic, let us mention the works [7–11]. For a more extensive list of references, we refer to the book by Quittner and Souplet [12].

For degenerate parabolic equations, blow-up results have been obtained by many authors, see [13–16]. It is mentioned that the initial value problem

$$\begin{cases} u_{xx} + xu_y - u_t = -u^{1+\alpha}, \alpha > 0 & \text{in } \mathbb{R}^2 \times (0,T), \\ u|_{t=0} = g, & \text{in } \mathbb{R}^2, \end{cases}$$

has no nontrivial nonnegative solutions in [14]. There is an interesting thing that replacing the right term $u^{1+\alpha}$ by $u|u|^\alpha$ in the first equation of the above problem, Haraux and Weissler [17] obtained global solutions.

In this paper, we will mainly deal with the following problems:

$$\begin{cases} u_{xx} + uu_y - u_t = -u|u|^\alpha, & \text{in } \mathbb{R}^+ \times \mathbb{R} \times (0,T), \\ u|_{t=0} = g, & \text{in } \mathbb{R}^+ \times \mathbb{R}, \\ \lim_{x\to\infty} u = u|_{x=0} = 0, & \text{in } \mathbb{R} \times (0,T), \end{cases} \tag{1.2}$$

and

$$\begin{cases} u_{xx} + uu_y - u_t = -t^k |x|^{-\gamma} |u|^{\alpha+1}, & \text{in } \mathbb{R} \times \mathbb{R} \times (0,T), \\ u|_{t=0} = g, & \text{in } \mathbb{R} \times \mathbb{R}. \end{cases} \tag{1.3}$$

It is known that the local solutions are obtained for (1.2) and (1.3) in [2]. Our interest is the blow-up of spatial derivatives of solutions in finite time to the initial boundary value problem (1.2) and the nonexistence of the weak solutions to the initial value problem (1.3).

Our main results are the following theorems.

Firstly, we define energy functionals

$$F(a) = \int_0^\infty a^2\,dx \quad \text{and} \quad E(a) = \int_0^\infty \left(\frac{1}{2}a_x^2 - \frac{1}{3}a^3\right)dx. \tag{1.4}$$

Theorem 1.1 *Let $a_0(x)$ have compact support such that $E(a_0) < 0$. Assume that the initial value $g(x,y)$ takes the form $g = yb_0(x,y)$, $b_0(x,0) = a_0(x)$. Then spatial derivatives of smooth solutions of* (1.2) *blow up in finite time. More precisely, there exists $T = \frac{F(a_0)}{6(1-\beta)E(a_0)}$, $\beta \in (1, \frac{3}{2})$, such that either*

$$\lim_{t\to T} \max_{x\in\mathbb{R}^+} |u_y(x,0,t)| = +\infty \quad \text{or} \quad \lim_{t\to T} u_{xy}(0,0,t) = +\infty.$$

This is our first result. A smooth solution u of (1.2) means $u \in C^1([0,T_0), C^2(\mathbb{R}^+ \times \mathbb{R}))$ for $T_0 > 0$. It is remarkable that Theorem 1.1 remains valid if we replace $-u|u|^\alpha$ by $u|u|^\alpha$ or 0.

Next, we consider the more general case

$$\begin{cases} u_{xx} + uu_y - u_t = f(u), & \text{in } \mathbb{R}^+ \times \mathbb{R} \times (0,T), \\ u|_{t=0} = g, & \text{in } \mathbb{R}^+ \times \mathbb{R}, \\ \lim_{x\to\infty} u = u|_{x=0} = 0, & \text{in } \mathbb{R} \times (0,T). \end{cases} \tag{1.5}$$

Continuing with the description of our results, let us introduce the precise assumptions on our f:

(H) $f(0) = 0$ and there exists an increasing continuous function ϕ on $[0,+\infty)$ such that

$$\bigl|f(r_1) - f(r_2)\bigr| \le \phi\bigl(|r_1 - r_2|\bigr),$$

and $\frac{1}{\phi(r)}$ is not integrable near $r = +0$, that is,

$$\int_0^\delta \frac{dr}{\phi(r)} = +\infty,$$

where δ is a positive constant.

Then Theorem 1.1 can be extended to the following theorem.

Theorem 1.2 *Suppose that f satisfies* (H), $f'(0) \le 0$, *and $g(x,y)$ satisfies the conditions of Theorem* 1.1. *Then there exist blow-up solutions of* (1.5) *in finite time.*

For initial value problems, we derive two theorems.

The following theorem considers blow-up of solutions to the initial value problem

$$\begin{cases} u_{xx} + uu_y - u_t = f(u), & \text{in } Q_T = \mathbb{R}^2 \times (0,T), \\ u|_{t=0} = g, & \text{in } \mathbb{R}^2. \end{cases} \tag{1.6}$$

Theorem 1.3 *Assume that u is the bounded classical solution of* (1.6) *in* $\overline{Q}_{T_\varepsilon}$, $Q_{T_\varepsilon} = \mathbb{R}^2 \times (0, T-\varepsilon)$, *for any given* $\varepsilon \in (0,T)$. *If* $f \le 0$ *and* $g(x,c_0) \ge \frac{c_0}{T}$, $c_0 > 0$, *then u blows up in time T at* $y = c_0$.

This improves the result of Example 1.1 in [2].

Finally, we consider nonexistence of weak solutions to the initial problem (1.3). Here, a weak solution of (1.3) is defined as follows.

Definition 1 A function $u \in L^2_{\mathrm{loc}}(Q)$ is called a weak solution of (1.3) with the initial data $g(x,y) \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ in $Q = \mathbb{R}^2 \times (0,\infty)$ if $t^k|x|^{-\gamma}|u|^{\alpha+1} \in L^1_{\mathrm{loc}}(Q)$ and

$$\int_Q -t^k|x|^{-\gamma}|u|^{\alpha+1}\phi\,dx\,dy\,dt = \int_Q \left(u\phi_{xx} + \frac{1}{2}u^2\phi_y - u\phi_t\right)dx\,dy\,dt - \int_{\mathbb{R}^2} g\phi(x,y,0)\,dx\,dy$$

hold for any nonnegative $\phi \in C^2_0(\mathbb{R}^2 \times [0,\infty))$.

Now, we address our result.

Theorem 1.4 *Let* $\alpha > 1$, $k - \frac{\gamma}{2} > 0$. *Assume that* $\int_{\mathbb{R}^2} g(x,y)\,dx\,dy \ge 0$. *If* $\alpha \le k - \frac{\gamma}{2} + 1$, *then there exists no nontrivial weak solution of* (1.3).

The rest of the paper is organized as follows. Section 2 is devoted to initial boundary value problems (1.2) and (1.5) through energy methods. In Section 3, we investigate initial value problems (1.3) and (1.6) by a comparison principle and choosing a special test function. Finally, we describe a numerical result about the blow-up of solutions in Theorem 1.1 in Section 4.

2 Initial boundary value problems

In this section, we obtain the blow-up results of initial boundary value problems (1.2) and (1.5).

2.1 Proof of Theorem 1.1

Suppose that a smooth solution u of (1.2) exists locally and the initial value $g(x,y)$ satisfies the form $g = yb_0(x,y)$. If we restrict (1.2) to the half line $l = \{x > 0, y = 0\}$ and let $v(x,t) = u(x,0,t)$, v obviously satisfies an equation of the form

$$v_{xx} + w(x,t)v - v_t = -v|v|^{\alpha},$$

where $w(x,t) = u_y(x,0,t)$ is smooth, with the initial data $v(x,0) = 0$ and the boundary data $v(0,t) = 0$. By the maximum principle, we conclude that $u(x,0,t) = v(x,t) = 0$ as long as u stays smooth. Any smooth function that vanishes at $y = 0$ can be written in this form

$$u(x,y,t) = yb(x,y,t). \tag{2.1}$$

Substituting (2.1) into the first equation of (1.2), we obtain

$$b_{xx} + b(b + yb_y) - b_t = -b|yb|^{\alpha}.$$

Let $a(x,t) = b(x,0,t)$ and $a_0(x) = b(x,0,0)$. Then a satisfies

$$a_t = a_{xx} + a^2, \tag{2.2}$$

with the initial boundary value conditions

$$a(0,t) = 0, \qquad \lim_{x\to+\infty} a(x,t) = 0, \qquad a(x,0) = a_0(x). \tag{2.3}$$

The proof of Theorem 1.1 is based on the following lemma.

Lemma 2.1 *If $a_0(x)$ has compact support such that $E(a_0) < 0$ (E is defined as in* (1.4)), *then there exists a finite time T such that either*

$$\lim_{t\to T}\max_{x\in\mathbb{R}^+}\big|a(x,t)\big| = +\infty \quad \text{or} \quad \lim_{t\to T} a_x(0,t) = +\infty.$$

Proof Assume that $\max_{x\in\mathbb{R}^+} a$ stays bounded. Since a satisfies equation (2.2), the standard result shows that a decays exponentially fast at infinity as long as its maximum norm stays bounded.

Next, we will show that $F(a)$ (F is defined as in (1.4)) blows up in finite time assuming that $a_x(0,t)$ stays finite. We will use the following integral identities that are valid for the smooth solutions of (2.2)-(2.3):

$$\frac{d}{dt}F(a) = -2\int_0^\infty a_x^2\,dx + 2\int_0^\infty a^3\,dx, \tag{2.4}$$

$$\frac{d}{dt}\int_0^\infty \frac{1}{2}a_x^2\,dx = -\int_0^\infty a_{xx}^2\,dx - \int_0^\infty a^2 a_{xx}\,dx, \tag{2.5}$$

$$\frac{d}{dt}\int_0^\infty \frac{1}{3}a^3\,dx = \int_0^\infty a^2 a_{xx}\,dx + \int_0^\infty a^4\,dx. \tag{2.6}$$

Employing (2.4), (2.5) and (2.6), we find

$$\begin{aligned} \frac{d}{dt}E(a) &= -\int_0^\infty \left(a_{xx}+a^2\right)^2 dx, \\ \frac{dF(a)}{dt} &= \int_0^\infty a_x^2\,dx - 6E(a) \ge -6E(a). \end{aligned}$$

Thus, we have $E(a) < 0$ for $t > 0$ under the condition $E(a_0) < 0$.

At last, we compute the time derivative of $H(a) = -\frac{E(a)}{F(a)^\beta}$. Firstly, we have

$$-F(a)\frac{dE(a)}{dt} = \int_0^\infty a^2\,dx \int_0^\infty \left(a_{xx}+a^2\right)^2 dx \ge \left(\int_0^\infty a\left(a_{xx}+a^2\right)dx\right)^2 = \frac{1}{4}\left(\frac{dF(a)}{dt}\right)^2.$$

Furthermore,

$$-F(a)\frac{dE(a)}{dt} \ge -\frac{3}{2}E(a)\frac{dF(a)}{dt}.$$

If we choose $\beta \in (1, \frac{3}{2})$, then

$$\begin{aligned} \frac{d}{dt}H(a) &= F(a)^{-\beta-1}\left(-F(a)\frac{dE(a)}{dt} + \beta E(a)\frac{dF(a)}{dt}\right) \\ &\ge F(a)^{-\beta-1}\left(\beta - \frac{3}{2}\right)E(a)\frac{dF(a)}{dt} \\ &\ge 0. \end{aligned}$$

By the definition of $H(a)$, we get $-E(a) \ge H(a_0)F(a)^\beta$, where $H(a)|_{t=0} = H(a_0)$.

Since

$$\frac{dF(a)}{dt} \ge -6E(a) \ge 6H(a_0)F(a)^\beta,$$

we deduce

$$F(a) \ge \frac{1}{(F(a_0)^{-\beta+1} - 6(\beta-1)H(a_0)t)^{\frac{1}{\beta-1}}}.$$

Hence there exists a finite time $T = \frac{F(a_0)}{6(1-\beta)E(a_0)}$, $\beta \in (1, \frac{3}{2})$ such that

$$\lim_{t\to T} F(a) = +\infty.$$

Due to the condition

$$\lim_{x\to\infty} a = 0,$$

we get

$$\lim_{t\to T}\max_{x\in\mathbb{R}^+}|a| = +\infty.$$

This completes the proof of Lemma 2.1. □

Proof of Theorem 1.1 We note that the smooth solution of (2.2)-(2.3) is unique. From Lemma 2.1, we get either

$$\lim_{t\to T}\max_{x\in\mathbb{R}^+}\bigl|b(x,0,t)\bigr| = +\infty \quad \text{or} \quad \lim_{t\to T} b_x(0,0,t) = +\infty.$$

This implies that either

$$\lim_{t\to T}\max_{x\in\mathbb{R}^+}\bigl|u_y(x,0,t)\bigr| = +\infty \quad \text{or} \quad \lim_{t\to T} u_{xy}(0,0,t) = +\infty.$$ □

2.2 Proof of Theorem 1.2

As the proof of Lemma 2.1, the smooth solution of (1.5) can be written in this form

$$u(x,y,t) = yI(x,y,t)$$

if $g = yb_0(x,y)$ and f satisfies hypothesis (H). Substituting $u(x,y,t) = yI(x,y,t)$ into the first equation of (1.5), we get

$$yI_t = yI_{xx} + yI(I + yI_y) - f(yI). \tag{2.7}$$

Let $s(x,t) = I(x,0,t)$ and $s_0(x) = I(x,0,0)$. By $f(0) = 0$ and $f'(0) \le 0$, multiply (2.7) by $\frac{1}{y}$ and take limit as $y \to 0$ to get

$$s_t = s_{xx} + s^2 - f'(0)s,$$

with the initial boundary value conditions

$$s(0,t) = 0, \qquad \lim_{x\to+\infty} s(x,t) = 0, \qquad s(x,0) = s_0(x).$$

Setting $\psi = \exp(f'(0)t)s$, ψ satisfies

$$\psi_t = \psi_{xx} + \exp\bigl(-f'(0)t\bigr)\psi^2. \tag{2.8}$$

Lemma 2.2 *Define*

$$E_1(\psi) = \int_0^\infty \left(\frac{1}{2}\psi_x^2 - \frac{1}{3}\exp\bigl(-f'(0)t\bigr)\psi^3\right) dx.$$

If the initial value $\psi_0 = \psi(x,0)$ *has compact support such that* $E_1(\psi_0) < 0$ *and* $f'(0) \le 0$, *then there exists a finite time* T *such that either*

$$\lim_{t\to T}\max_{x\in\mathbb{R}^+}|\psi| = +\infty \quad \textit{or} \quad \lim_{t\to T}\psi_x(0,t) = +\infty.$$

Proof It is proceeded by a contradiction to the proof of Lemma 2.1, that is, we assume that $\psi_x(0,t)$ stays finite and we will get $F(\psi)$ (F is defined as in (1.4)) blows up in finite time. The following integral identities are valid for smooth solutions of (2.8):

$$\begin{aligned}
&\frac{d}{dt}F(\psi) = -2\int_0^\infty \psi_x^2\,dx + 2\int_0^\infty \exp\bigl(-f'(0)t\bigr)\psi^3\,dx,\\
&\frac{d}{dt}\int_0^\infty \frac{1}{2}\psi_x^2\,dx = -\int_0^\infty \psi_{xx}^2\,dx - \int_0^\infty \exp\bigl(-f'(0)t\bigr)\psi^2\psi_{xx}\,dx,\\
&\frac{d}{dt}\int_0^\infty \frac{1}{3}\exp\bigl(-f'(0)t\bigr)\psi^3\,dx = \int_0^\infty \exp\bigl(-f'(0)t\bigr)\psi^2\psi_{xx}\,dx + \int_0^\infty \exp\bigl(-2f'(0)t\bigr)\psi^4\,dx\\
&\qquad - \int_0^\infty \frac{1}{3}f'(0)\exp\bigl(-f'(0)t\bigr)\psi^3\,dx.
\end{aligned}$$

Due to $f'(0) \le 0$, we get

$$\begin{aligned}
&\frac{d}{dt}E_1(\psi) = -\int_0^\infty \biggl[\bigl(\psi_{xx} + \exp\bigl(-f'(0)t\bigr)\psi^2\bigr)^2 - \frac{1}{3}f'(0)\exp\bigl(-f'(0)t\bigr)\psi^3\biggr]\,dx \le 0,\\
&-F(\psi)\frac{dE_1(\psi)}{dt}\\
&\quad = \int_0^\infty \psi^2\,dx\int_0^\infty \biggl[\bigl(\psi_{xx} + \exp\bigl(-f'(0)t\bigr)\psi^2\bigr)^2 - \frac{1}{3}f'(0)\exp\bigl(-f'(0)t\bigr)\psi^3\biggr]\,dx\\
&\quad \ge \biggl(\int_0^\infty \psi\bigl(\psi_{xx} + \exp\bigl(-f'(0)t\bigr)\psi^2\bigr)\,dx\biggr)^2 = \frac{1}{4}\biggl(\frac{dF}{dt}\biggr)^2.
\end{aligned}$$

Since

$$\frac{dF}{dt} = \int_0^\infty \psi_x^2\,dx - 6E_1 \ge -6E_1 > 0,$$

we have

$$-F\frac{dE_1}{dt} \ge -\frac{3}{2}E_1\frac{dF}{dt}.$$

If we define $H(a) = -\frac{E_1(a)}{F(a)^\beta}$ and choose $\beta \in (1, \frac{3}{2})$, then

$$\begin{aligned}
\frac{d}{dt}H(\psi) &= F^{-\beta-1}\biggl(-F\frac{dE_1}{dt} + \beta E_1\frac{dF}{dt}\biggr)\\
&\ge F^{-\beta-1}\biggl(\beta - \frac{3}{2}\biggr)E_1\frac{dF}{dt}\\
&\ge 0.
\end{aligned}$$

We have $-E_1(\psi) \geq H(\psi_0)F^\beta$ and $\frac{dF}{dt} \geq -6E_1 \geq 6H(\psi_0)F^\beta$, where $H(\psi)|_{t=0} = H(\psi_0)$. Hence there exists a finite time $T = \frac{F(\psi_0)}{6(1-\beta)E_1(\psi_0)}$, $\beta \in (1, \frac{3}{2})$ such that

$$\lim_{t\to T} \max_{x\in\mathbb{R}^+} |\psi| = +\infty.$$

This completes the proof of Lemma 2.2. □

By Lemma 2.2, the solution of (1.5) has either

$$\lim_{t\to T} \max_{x\in\mathbb{R}^+} \left|u_y(x,0,t)\right| = +\infty \quad \text{or} \quad \lim_{t\to T} u_{xy}(0,0,t) = +\infty.$$

Then Theorem 1.2 is obtained.

Remark 1 Replacing the semilinear term uu_y of (1.5) by $h(u)u_y$, if $h(u)$ satisfies hypothesis (H) and $f'(0)h'(0) \leq 0$, then the smooth solutions of (1.5) have the same result as Theorem 1.2.

Remark 2 Theorem 1.1 and Theorem 1.2 describe the lower dimensional problems. The higher dimensional cases are parallel to the lower dimensional cases. For example, the high dimensional problem is as follows:

$$\begin{cases} \Delta_x u + h(u)u_y - u_t = f(u), & \text{in } \mathbb{R}_+^N \times \mathbb{R} \times (0,t), \\ u(x,y,0) = g, & \text{in } \mathbb{R}_+^N \times \mathbb{R}, \\ u(x,y,t) = 0, & \text{in } \partial\mathbb{R}_+^N \times \mathbb{R} \times (0,t), \end{cases}$$

where Δ_x is the Laplace operator acting in the variable $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}_+^N$.

3 Initial value problems

The section describes initial value problems (1.3) and (1.6) for deriving the proofs of Theorem 1.3 and Theorem 1.4.

For the convenience of description, we set

$$Lu = u_{xx} + uu_y - u_t.$$

Next, we get a comparison principle about the initial value problem (1.6).

Lemma 3.1 *Assume that there are two solutions u_i of* (1.6) *satisfying $u_i \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and $u_i, (u_2)_y \in L^\infty(Q_T)$, $i = 1, 2$. Let $f(u_1) \leq f(u_2)$, $g(u_1) \geq g(u_2)$, then $u_1 \geq u_2$.*

Proof Set $w = u_1 - u_2$,

$$\begin{aligned} Lu_1 - Lu_2 &= (u_1 - u_2)_{xx} - (u_1 - u_2)_t + u_1(u_1 - u_2)_y + (u_2)_y(u_1 - u_2) \\ &= w_{xx} + u_1 w_y + (u_2)_y w - w_t \leq 0. \end{aligned}$$

We suppose that $r_0 > 0$, $\alpha > 0$, $N > 0$, and

$$m \geq \max_{Q_T} \bigl(|u_1| + |u_2|\bigr),$$

and set

$$v = \frac{m}{r_0^2}\bigl(r^2 + Nt\bigr)\exp(\alpha t) + w$$

for $r^2 = x^2 + y^2$.

Defining $\overline{L}$ by

$$\overline{L}w = w_{xx} + u_1 w_y + (u_2)_y w - w_t,$$

we have

$$\overline{L}v = \overline{L}w + \frac{m}{r_0^2}\exp(\alpha t)\bigl[2 + 2yu_1 + \bigl((u_2)_y - \alpha\bigr)\bigl(r^2 + Nt\bigr) - N\bigr].$$

Choosing

$$\alpha \geq \max_{Q_T}\bigl(|u_1| + \bigl|(u_2)_y\bigr|\bigr) \quad \text{and} \quad N > 2,$$

we get $\overline{L}v \leq 0$.

In $\Omega_{r_0} = \{(x,y,t)|x^2 + y^2 \leq r_0^2, 0 \leq t \leq T\}$, due to $v|_{t=0} \geq 0$, $v|_{r=r_0} \geq 0$, by the maximum principle, we obtain $v \geq 0$.

For any $p \in Q_T$, if we choose r_0 sufficiently large such that $p \in \Omega_{r_0}$, then $v|_p \geq 0$.

Set $r_0 \to \infty$, we get $w|_p = (u_1 - u_2)|_p \geq 0$. □

Using Lemma 3.1, we get the following proof of Theorem 1.3.

Proof of Theorem 1.3 Taking $u_1 = \frac{y}{T-t}$, it shows that

$$Lu_1 = (u_1)_{xx} + u_1(u_1)_y - (u_1)_t = 0.$$

Fixing $y = c_0 > 0$, we have $Lu_1(x, c_0, t) \geq Lu(x, c_0, t)$. When $g(x, c_0) \geq \frac{c_0}{T}$, we get $u \geq \frac{c_0}{T-t}$ by Lemma 3.1.

At $y = c_0$,

$$u(x, y, T - \varepsilon) \geq \frac{c_0}{\varepsilon},$$

$$\lim_{t \to T} u(x, y, t) = \lim_{\varepsilon \to 0} u(x, y, T - \varepsilon) = \infty. \qquad \square$$

Finally, we give the proof of Theorem 1.4.

Proof of Theorem 1.4 Let u be such a weak solution of (1.3) and $\phi \in C_0^2(\mathbb{R}^2 \times [0, \infty))$ be a nonnegative test function. Applying the first equation of (1.3) and Young's inequality, we obtain

$$\begin{aligned}
&\int_Q t^k |x|^{-\gamma} |u|^{\alpha+1}\phi \, dx\, dy\, dt \\
&\quad = \int_Q \left[-u\phi_{xx} + \frac{1}{2}u^2\phi_y - u\phi_t\right] dx\, dy\, dt - \int_{\mathbb{R}^2} g\phi(x, y, 0)\, dx\, dy
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2}\int_Q t^k|x|^{-\gamma}|u|^{\alpha+1}\phi\,dx\,dy\,dt + c\biggl[\int_Q t^{-\frac{k}{\alpha}}|x|^{\frac{\gamma}{\alpha}}\phi^{-\frac{1}{\alpha}}(\phi_{xx})^{\frac{\alpha+1}{\alpha}}\,dx\,dy\,dt \\
&\quad + \int_Q t^{-\frac{2k}{\alpha-1}}|x|^{\frac{2\gamma}{\alpha-1}}\phi^{-\frac{2}{\alpha-1}}(\phi_y)^{\frac{\alpha+1}{\alpha-1}}\,dx\,dy\,dt + \int_Q t^{-\frac{k}{\alpha}}|x|^{\frac{\gamma}{\alpha}}\phi^{-\frac{1}{\alpha}}(\phi_t)^{\frac{\alpha+1}{\alpha}}\,dx\,dy\,dt\biggr] \\
&\quad - \int_{\mathbb{R}^2} g\phi(x,y,0)\,dx\,dy,
\end{aligned}$$

where $\alpha > 1$.

We define

$$\phi(x,y,t) = \psi\left(\frac{t+x^2+y^2}{r^2}\right),$$

where $\psi \in C_0^\infty(\mathbb{R}^+)$ satisfies $0 \le \psi \le 1$ and

$$\psi(s) = \begin{cases} 1, & 0 \le s \le 1, \\ 0, & s \ge 2. \end{cases}$$

Then

$$\begin{aligned}
&\int_Q t^k|x|^{-\gamma}|u|^{\alpha+1}\psi\,dx\,dy\,dt \\
&\quad\le c\biggl[\int_Q t^{-\frac{k}{\alpha}}|x|^{\frac{\gamma}{\alpha}}\psi^{-\frac{1}{\alpha}}(\psi_{xx})^{\frac{\alpha+1}{\alpha}}\,dx\,dy\,dt + \int_Q t^{-\frac{2k}{\alpha-1}}|x|^{\frac{2\gamma}{\alpha-1}}\psi^{-\frac{2}{\alpha-1}}(\psi_y)^{\frac{\alpha+1}{\alpha-1}}\,dx\,dy\,dt \\
&\quad\quad + \int_Q t^{-\frac{k}{\alpha}}|x|^{\frac{\gamma}{\alpha}}\psi^{-\frac{1}{\alpha}}(\psi_t)^{\frac{\alpha+1}{\alpha}}\,dx\,dy\,dt\biggr] \le c\bigl[r^{\frac{\gamma}{\alpha}-\frac{2k}{\alpha}}r^{\frac{-2(\alpha+1)}{\alpha}+4} + r^{\frac{2\gamma}{\alpha-1}-\frac{4k}{\alpha-1}}r^{\frac{-(\alpha+1)}{\alpha-1}+4}\bigr].
\end{aligned} \tag{3.1}$$

In the case where $\gamma + 2\alpha - 2k - 2 < 0$, the exponents of the right terms in (3.1) are negative. Taking the limit as $r \to \infty$ in (3.1), we deduce that

$$\int_{t+x^2+y^2\le r^2} t^k|x|^{-\gamma}|u|^{\alpha+1}\,dx\,dy\,dt \to 0.$$

This implies that $u \equiv 0$ in Q.

In the case where $\gamma + 2\alpha - 2k - 2 = 0$, we get from (3.1) that

$$\int_Q t^k|x|^{-\gamma}|u|^{\alpha+1}\,dx\,dy\,dt < \infty.$$

Set $\Omega_r = \{(x,y,t) \in \mathbb{R}^2 \times (0,\infty) : r^2 \le t + x^2 + y^2 \le 2r^2\}$. Since $\psi(s)$ is constant for $s \in [0,1] \cup [2,\infty)$, we have

$$\begin{aligned}
&\left|\int_Q \left[-u\psi_{xx} + \frac{1}{2}u^2\psi_y - u\psi_t\right]dx\,dy\,dt\right| \\
&\quad = \left|\int_{\Omega_r}\left[-u\psi_{xx} + \frac{1}{2}u^2\psi_y - u\psi_t\right]dx\,dy\,dt\right| \\
&\quad \le c\biggl\{\biggl[\int_{\Omega_r} t^{-\frac{k}{\alpha}}|x|^{\frac{\gamma}{\alpha}}\psi^{-\frac{1}{\alpha}}(\psi_{xx})^{\frac{\alpha+1}{\alpha}}\,dx\,dy\,dt\biggr]^{\frac{\alpha}{\alpha+1}}
\end{aligned}$$

$$+\left[\int_{\Omega_r} t^{-\frac{2k}{\alpha-2}}|x|^{\frac{2\gamma}{\alpha-1}}\psi^{-\frac{2}{\alpha-1}}(\psi_y)^{\frac{\alpha+1}{\alpha-1}}\,dx\,dy\,dt\right]^{\frac{\alpha-1}{\alpha+1}}$$

$$+\left[\int_{\Omega_r} t^{-\frac{k}{\alpha}}|x|^{\frac{\gamma}{\alpha}}\psi^{-\frac{1}{\alpha}}(\psi_t)^{\frac{\alpha+1}{\alpha}}\,dx\,dy\,dt\right]^{\frac{\alpha}{\alpha+1}}\Bigg\}\left\{\int_{\Omega_r} t^k|x|^{-\gamma}|u|^{\alpha+1}\psi\,dx\,dy\,dt\right\}^{\frac{2}{\alpha+1}}$$

$$\le c\left\{\int_{\Omega_r} t^k|x|^{-\gamma}|u|^{\alpha+1}\,dx\,dy\,dt\right\}^{\frac{2}{\alpha+1}}. \tag{3.2}$$

It follows from the integrability of $t^k|x|^{-\gamma}|u|^{\alpha+1}$ in Q that

$$\lim_{r\to\infty}\int_{\Omega_r} t^k|x|^{-\gamma}|u|^{\alpha+1}\,dx\,dy\,dt = 0.$$

From (3.1) and (3.2), we know that

$$\int_Q t^k|x|^{-\gamma}|u|^{\alpha+1}\,dx\,dy\,dt = 0.$$

This implies that $u \equiv 0$. □

4 A numerical experiment

Next, we present a numerical experiment. Our goal is to show that the result presented in Theorem 1.1 can be observed when one performs numerical computations. For a numerical experiment, we choose an adaptive bounded space to problem (1.2).

At $y = 0$, (1.2) in a bounded domain can be written to the following problem:

$$\begin{cases} a_t = a_{xx} + a^2, & 0 < x < 3, t > 0, \\ a|_{t=0} = -x(x-3), & 0 < x < 3, \\ a|_{x=0} = 0, & t > 0, \\ a|_{x=3} = 0, & t > 0. \end{cases} \tag{4.1}$$

Figure 1 shows the evolution of the numerical solution of (4.1) with a space step size 0.01, whose blow-up time turns out to be $T = 0.56$. In $(0, T)$, for any $T \geq 0.56$, we fail to show the figure in Matlab since the function value increases rapidly. In Figure 2, we display the profile of Figure 1 at $t = 0.55$.

Figure 1 Evolution of the numerical solution in $t \in [0, 0.55]$.

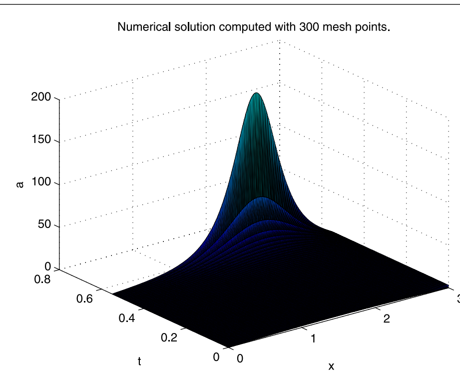
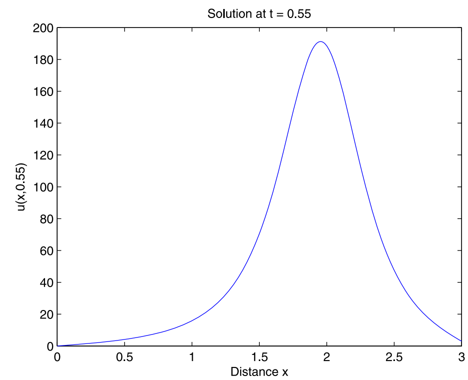


Figure 2 The profile at $t = 0.55$.



Competing interests
The author declares that they have no competing interests.

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