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# General stability for a von Kármán plate system with memory boundary conditions

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**Abstract**

In this paper we consider a von Kármán plate system with memory condition at the boundary. We prove the asymptotic behavior of the corresponding solutions. We establish an explicit and general decay rate result using some properties of the convex functions. Our result is obtained without imposing any restrictive assumptions on the behavior of the relaxation function at infinity. These general decay estimates extend and improve on some earlier results-exponential or polynomial decay rates.

**Keywords:** von Kármán plate; general decay; memory term; relaxation function; convexity

## 1 Introduction

This paper is concerned with the general decay of the solutions to a von Kármán plate system with memory condition at the boundary:

$$u_{tt} + \Delta^2 u = [u, v] \quad \text{in } \Omega \times (0, \infty), \tag{1.1}$$

$$\Delta^2 v = -[u, u] \quad \text{in } \Omega \times (0, \infty), \tag{1.2}$$

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma \times (0, \infty), \tag{1.3}$$

$$\frac{\partial u}{\partial \nu} + \int_0^t g_1(t-s)\mathcal{B}_1 u(s)\,ds = 0 \quad \text{on } \Gamma \times (0, \infty), \tag{1.4}$$

$$u - \int_0^t g_2(t-s)\mathcal{B}_2 u(s)\,ds = 0 \quad \text{on } \Gamma \times (0, \infty), \tag{1.5}$$

$$u(x, y, 0) = u_0(x, y), \qquad u_t(x, y, 0) = u_1(x, y) \quad \text{in } \Omega, \tag{1.6}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^2$ with a smooth boundary $\Gamma$. Let us denote by $\nu = (\nu_1, \nu_2)$ the external unit normal vector on $\Gamma$ and by $\tau = (-\nu_2, \nu_1)$ the corresponding unit tangent vector. The relaxation functions $g_1, g_2 \in C^1(0, \infty)$ are positive and nondecreasing. The von Kármán bracket is given by

$$[u, v] = u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx}.$$

Here, $\mathcal{B}_1$, $\mathcal{B}_2$ denote the differential operators

$$\mathcal{B}_1 u = \Delta u + (1-\mu)B_1 u, \qquad \mathcal{B}_2 u = \frac{\partial \Delta u}{\partial \nu} + (1-\mu)\frac{\partial B_2 u}{\partial \tau}$$

and

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx},$$
$$B_2 u = \left(\nu_1^2 - \nu_2^2\right)u_{xy} + \nu_1\nu_2(u_{yy} - u_{xx})$$

and the constant $\mu \in (0, \frac{1}{2})$ represents Poisson's ratio.

From the physical point of view, this system describes the transversal displacement $u$ and the Airy-stress function $v$ of a vibrating plate. We know that the memory effect described in integral equations (1.4) and (1.5) can be caused by the interaction with another viscoelastic element. Problems related to equations (1.1)-(1.6) are interesting not only from the point of view of PDE general theory, but also due to its applications in mechanics.

The problem of stability of the solutions to a von Kármán system with dissipative effects has been studied by several authors [1–4]. Rivera and Menzala [5] considered the dynamical von Kármán equations for viscoelastic plates under the presence of a long range memory:

$$\begin{cases} u_{tt} - h\Delta u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)\,d\tau = [u,v] \quad \text{in } \Omega \times (0,\infty), \\ \Delta^2 v = -[u,u] \quad \text{in } \Omega \times (0,\infty), \\ v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma \times (0,\infty), \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \\ \mathcal{B}_1 u - \mathcal{B}_1\{\int_0^t g(t-\tau)u(\tau)\,d\tau\} = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \\ \mathcal{B}_2 u - h\frac{\partial u_{tt}}{\partial \nu} - \mathcal{B}_2\{\int_0^t g(t-\tau)u(\tau)\,d\tau\} = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \\ u(x,y,0) = u_0(x,y), \qquad u_t(x,y,0) = u_1(x,y) \quad \text{in } \Omega. \end{cases} \tag{1.7}$$

The equations describe small vibration of a thin homogeneous, isotropic plate of uniform thickness $h$. They studied that the energy decays uniformly exponentially or algebraically with the same rate of decay as the relaxation function. Later, Raposo and Santos [6] proved the general decay of the solutions to von Kármán plate model (1.7) under condition on $g$ such as

$$g'(t) \le -\xi(t)g(t), \qquad \xi(t) > 0, \qquad \xi'(t) < 0, \quad t \ge 0, \tag{1.8}$$

where $\xi$ is a differential function. This result generalized on the earlier ones in the literature. Kang [7] established an explicit and general decay rate result for von Kármán system (1.7) with nonlinear boundary damping $h(u_t)$. Kang improved the results of [6] without imposing any restrictive growth assumption on the damping function $h$ and strongly weakening the usual assumptions on the relaxation function $g$. Kang [8] showed the exponential decay result of solutions for von Kármán equations (1.7) without the assumption (1.8). She studied that solutions decay exponentially to zero as time goes to infinity in case

$$g'(t) + \gamma g(t) \ge 0, \qquad \int_0^\infty \bigl(g'(t) + \gamma g(t)\bigr)e^{\alpha t}\,dt < +\infty, \quad \forall t \ge 0 \tag{1.9}$$

for some $\gamma, \alpha > 0$. It is clear then that we are allowing $g'(t)$ to take negative values. The kernel $g(t)$ may oscillate. This result improved on the earlier ones concerning the exponential decay. Recently, Kang [9] considered the exponential decay for von Kármán equations (1.7) with acoustic boundary conditions when relaxation function satisfies (1.9). The construction of the Lyapunov function is based on the multiplier method.

On the other hand, Rivera *et al.* [10] studied the stability of the solutions to a von Kármán system for viscoelastic plates with memory effect in the boundary. They proved that the solution of system (1.1)-(1.6) decays uniformly exponentially or polynomially with the same rate of decay as the relaxation function. Later, Santos and Soufyane [11] improved the decay result of [10]. They assumed that the resolvent kernels satisfy

$$\begin{aligned} &k_i(0) > 0, \qquad k_i(t) \geq 0, \\ &k_i'(t) \leq 0, \qquad k_i''(t) \geq \gamma_i(t)\bigl(-k_i'(t)\bigr), \quad \forall t \geq 0 \ (i = 1,2), \end{aligned} \tag{1.10}$$

where $\gamma_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying the following conditions:

$$\gamma_i(t) > 0, \qquad \gamma_i'(t) \leq 0, \qquad \int_0^{+\infty} \gamma_i(t)\,dt = +\infty.$$

They studied that the energy decays with a rate of decay similar to the relaxation functions, which are not necessarily decaying like polynomial or exponential functions. Motivated by their results, we prove the general decay of the solution for a von Kármán plate system with memory boundary conditions (1.1)-(1.6) for resolvent kernels $k_i$ satisfying

$$k_i''(t) \geq H\bigl(-k_i'(t)\bigr), \quad \forall t \geq 0 \ (i = 1,2), \tag{1.11}$$

where $H$ is a positive function, with $H(0) = H'(0) = 0$, and $H$ is linear or strictly increasing and strictly convex on $(0, r]$ for some $0 < r < 1$. The proof is based on the multiplier method and makes use of some properties of convex functions including the use of general Young's inequality and Jensen's inequality. We establish an explicit decay rate result that allows a wider class of relaxation functions and generalizes previous decay results of [10, 11].

Moreover, there exists a large body of literature regarding viscoelastic problems with the memory term acting at the boundary. Cavalcanti *et al.* [12] considered the existence and uniform decay rates of solutions to a degenerate system with a memory condition at the boundary. Santos [13] and Santos *et al.* [14] proved the decay rates for solutions of a Timoshenko system and a nonlinear wave equation of Kirchhoff type with a memory condition at the boundary, respectively. Park and Kang [15] investigated the asymptotic behavior of the solutions of a multi-valued hyperbolic differential inclusion with a boundary condition of memory type. Precisely, denoting by $k$ the resolvent kernel of $-g'/g(0)$, they proved that the energy of the solution decays exponentially (polynomially) to zero provided $k$ decays exponentially (polynomially) to zero. Messaoudi and Soufyane [16], Mustafa and Messaoudi [17] and Kang [18, 19] obtained the general stability for wave equation, the Timoshenko system and Kirchhoff plates with viscoelastic boundary conditions under condition $k$ satisfying (1.10), respectively. Mustafa and Abusharkh [20] proved the general decay for plate equations with viscoelastic boundary damping when resolvent kernels $k_i$ satisfy (1.11).

Besides, Mustafa and Messaoudi [21, 22] investigated the general stability result of viscoelastic equation for relaxation function $g$ satisfying (1.11). These conditions on $H$ are weaker than those imposed in [23].

The paper is organized as follows. In Section 2 we present some notations and material needed for our work. In Section 3 we prove the general decay of the solutions to the von Kármán plate system with memory condition at the boundary.

## 2 Preliminaries

In this section, we present some material needed in the proof of our main result. Throughout this paper we define

$$(u,v) = \int_\Omega u(x)v(x)\,dx, \qquad (u,v)_\Gamma = \int_\Gamma u(x)v(x)\,d\Gamma.$$

For a Banach space $X$, $\|\cdot\|_X$ denotes the norm of $X$. For simplicity, we denote $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Gamma)}$ by $\|\cdot\|$ and $\|\cdot\|_\Gamma$, respectively. A simple calculation, based on the integration by parts formula, yields

$$\left(\Delta^2 u, v\right) = a(u,v) + (\mathcal{B}_2 u, v)_\Gamma - \left(\mathcal{B}_1 u, \frac{\partial v}{\partial \nu}\right)_\Gamma, \tag{2.1}$$

where the bilinear symmetric form $a(u,v)$ is given by

$$a(u,v) = \int_\Omega \left\{u_{xx}v_{xx} + u_{yy}v_{yy} + \mu(u_{xx}v_{yy} + u_{yy}v_{xx}) + 2(1-\mu)u_{xy}v_{xy}\right\}dx\,dy.$$

We know that $\sqrt{a(u,u)}$ is equivalent to $H^2(\Omega)$, that is,

$$c_0\|u\|^2_{H^2(\Omega)} \le a(u,u) \le \tilde{c}_0\|u\|^2_{H^2(\Omega)},$$

where $c_0$ and $\tilde{c}_0$ are generic positive constants. This and the Sobolev imbedding theorem imply that for some positive constants $C_p$ and $C_s$,

$$\|u\|^2 \le C_p a(u,u) \quad \text{and} \quad \|\nabla u\|^2 \le C_s a(u,u), \quad \forall u \in H^2(\Omega). \tag{2.2}$$

We assume that there exists $x_0 \in \mathbb{R}^2$ such that

$$\Gamma = \left\{x \in \Gamma : \nu(x)\cdot(x - x_0) > 0\right\}. \tag{2.3}$$

We define $m(x) = x - x_0$ and $R = \max_{x\in\Omega}|m(x)|$.

The following identity will be used later.

**Lemma 2.1** ([24]) *For every $u \in H^4(\Omega)$, we have*

$$\begin{aligned}\left(m\cdot\nabla u, \Delta^2 u\right) &= a(u,u) + \int_\Gamma \left[(\mathcal{B}_2 u)(m\cdot\nabla u) - (\mathcal{B}_1 u)\frac{\partial}{\partial\nu}(m\cdot\nabla u)\right]d\Gamma \\ &\quad + \frac{1}{2}\int_\Gamma m\cdot\nu\left[u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx}u_{yy} + 2(1-\mu)u_{xy}^2\right]d\Gamma.\end{aligned}$$

Now, we introduce the relative results of the Airy stress function and von Karman bracket $[\cdot,\cdot]$.

**Lemma 2.2** ([25]) *Let $u$, $w$ be functions in $H^2(\Omega)$ and $v$ in $H_0^2(\Omega)$, where $\Omega$ is an open bounded and connected set of $\mathbb{R}^2$ with regular boundary. Then*

$$\int_\Omega w[v,u]\,dx = \int_\Omega v[w,u]\,dx.$$

Now, we use the boundary conditions (1.4) and (1.5) to estimate the terms $\mathcal{B}_1 u$ and $\mathcal{B}_2 u$. Denoting by

$$(g*v)(t) = \int_0^t g(t-s)v(s)\,ds$$

the convolution product operator and differentiating equations (1.4) and (1.5), we arrive at the following Volterra equation:

$$\mathcal{B}_1 u + \frac{1}{g_1(0)} g_1' * \mathcal{B}_1 u = -\frac{1}{g(0)}\frac{\partial u_t}{\partial \nu}, \qquad \mathcal{B}_2 u + \frac{1}{g_2(0)} g_2' * \mathcal{B}_2 u = \frac{1}{g_2(0)} u_t.$$

Applying Volterra's inverse operator, we have

$$\mathcal{B}_1 u = -\frac{1}{g_1(0)}\left\{\frac{\partial u_t}{\partial \nu} + k_1 * \frac{\partial u_t}{\partial \nu}\right\}, \qquad \mathcal{B}_2 u = \frac{1}{g_2(0)}\{u_t + k_2 * u_t\},$$

where the resolvent kernels $k_i$ $(i=1,2)$ satisfy

$$k_i + \frac{1}{g_i(0)} g_i' * k_i = -\frac{1}{g_i(0)} g_i'.$$

Assuming throughout the paper that $u_0 \equiv 0$ on $\Gamma \times (0,\infty)$ and defining $\tau_1 = \frac{1}{g_1(0)}$ and $\tau_2 = \frac{1}{g_2(0)}$, we can rewrite $\mathcal{B}_1 u$ and $\mathcal{B}_2 u$ as

$$\mathcal{B}_1 u = -\tau_1\left\{\frac{\partial u_t}{\partial \nu} + k_1(0)\frac{\partial u}{\partial \nu} + k_1' * \frac{\partial u}{\partial \nu}\right\}, \tag{2.4}$$

$$\mathcal{B}_2 u = \tau_2\left\{u_t + k_2(0)u + k_2' * u\right\}. \tag{2.5}$$

Hence, we use conditions (2.4) and (2.5) instead of the boundary conditions (1.4) and (1.5).

Let us define

$$(g\,\square\,v)(t) := \int_0^t g(t-s)\left|v(t)-v(s)\right|^2\,ds.$$

By differentiating the term $g\,\square\,v$, we obtain the following lemma for the important property of the convolution product operator.

**Lemma 2.3** *For $g, v \in C^1([0,\infty):\mathbb{R})$, we have*

$$(g*v)v_t = -\frac{1}{2}g(t)\left|v(t)\right|^2 + \frac{1}{2}g'\,\square\,v - \frac{1}{2}\frac{d}{dt}\left[g\,\square\,v - \left(\int_0^t g(s)\,ds\right)|v|^2\right].$$

We consider the following assumptions on $k_1$ and $k_2$.

(A1) The resolvent kernels $k_i : \mathbb{R}_+ \to \mathbb{R}_+$ ($i = 1, 2$) are twice differentiable functions such that

$$k_i(0) > 0, \qquad \lim_{t\to\infty} k_i(t) = 0, \qquad k_i'(t) \le 0,$$

and there exists a positive function $H \in C^1(\mathbb{R}_+)$, and $H$ is a linear or strictly increasing and strictly convex $C^2$ function on $(0, r]$, $r < 1$, with $H(0) = H'(0) = 0$, such that

$$k_i''(t) \ge H\bigl(-k_i'(t)\bigr), \quad \forall t > 0. \tag{2.6}$$

The energy of system (1.1)-(1.6) is given by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\|u_t\|^2 + \frac{1}{2}a(u,u) + \frac{1}{4}\|\Delta v\|^2 + \frac{\tau_2}{2}k_2(t)\|u\|_\Gamma^2 - \frac{\tau_2}{2}\int_\Gamma k_2' \Box u \, d\Gamma \\ & + \frac{\tau_1}{2}k_1(t)\left\|\frac{\partial u}{\partial \nu}\right\|_\Gamma^2 - \frac{\tau_1}{2}\int_\Gamma k_1' \Box \frac{\partial u}{\partial \nu}\, d\Gamma. \end{aligned}$$

The well-posedness of von Kármán system plates with boundary conditions of memory type is given by the following theorem.

**Theorem 2.1** ([10]) *Let $k_i \in C^2(\mathbb{R}_+)$ be such that $k_i, -k_i', k_i'' \ge 0$ for $i = 1, 2$. If $(u_0, u_1) \in H^2(\Omega) \cap L^2(\Omega)$, then there exists a unique weak solution for system* (1.1)-(1.6). *Moreover, if $(u_0, u_1) \in (H^4(\Omega) \cap H^2(\Omega)) \times H^2(\Omega)$, then the solution of* (1.1)-(1.6) *has the following regularity*:

$$u \in C^1\bigl([0,T] : H^2(\Omega)\bigr) \cap C^0\bigl([0,T] : H^4(\Omega)\bigr), \qquad v \in C^0\bigl([0,T] : H^4(\Omega) \cap H_0^2(\Omega)\bigr).$$

We are now ready to state our main result.

**Theorem 2.2** *Assume that* (A1) *holds. Suppose that $D$ is a positive $C^1$ function, with $D(0) = 0$, for which $H_0$ is a strictly increasing and strictly convex $C^2$ function on* $(0, r]$ *and*

$$\int_0^{+\infty} \frac{-k_i'(s)}{H_0^{-1}(k_i''(s))}\, ds < +\infty \quad \textit{for } i = 1, 2. \tag{2.7}$$

*Then there exist positive constants $c_1$, $c_2$, $c_3$ and $\epsilon_0$ such that the solution of* (1.1)-(1.6) *satisfies*

$$E(t) \le c_3 H_1^{-1}(c_1 t + c_2), \quad \forall t \ge 0, \tag{2.8}$$

*where*

$$H_1(t) = \int_t^1 \frac{1}{sH_0'(\epsilon_0 s)}\, ds \quad \textit{and} \quad H_0(t) = H\bigl(D(t)\bigr).$$

*Moreover, for some choice of $D$, if $\int_0^1 H_1(t)\,dt < +\infty$, then we obtain*

$$E(t) \le c_3 G^{-1}(c_1 t + c_2), \tag{2.9}$$

*where*

$$G(t) = \int_t^1 \frac{1}{sH'(\epsilon_0 s)}\,ds. \tag{2.10}$$

*In particular,* (2.9) *is valid for the special case* $H(t) = ct^p$ *for* $1 \le p < \frac{3}{2}$.

**Remark 2.1** (1) From (A1), we conclude that $\lim_{t\to+\infty}(-k_i'(t)) = 0$ for $i = 1,2$. This implies that $\lim_{t\to+\infty} k_i''(t)$ cannot be equal to a positive number, and so it is natural to assume that $\lim_{t\to+\infty} k_i''(t) = 0$. Therefore, there is $t_0 > 0$ large enough such that $k_i'(t_0) < 0$ and

$$\max\{k_i(t), -k_i'(t), k_i''(t)\} < \min\{r, H(r), H_0(r)\}, \quad \forall t \ge t_0. \tag{2.11}$$

From $H$ is a positive continuous function, we obtain

$$d_1 \le H(-k_i'(t)) \le d_2, \quad \forall t \in [0, t_0] \tag{2.12}$$

for some positive constants $d_1$ and $d_2$. Since $k_i'$ is nondecreasing, $k_i'(0) < 0$ and $k_i'(t_0) < 0$, we have

$$0 < -k_i'(t_0) \le -k_i'(t) \le -k_i'(0), \quad \forall t \in [0, t_0]. \tag{2.13}$$

Hence, by (2.6), (2.12) and (2.13),

$$k_i''(t) \ge H(-k_i'(t)) \ge \frac{d_1}{k_i'(0)} k_i'(0) \ge \frac{d_1}{k_i'(0)} k_i'(t),$$

which gives

$$k_i''(t) \ge -d_3 k_i'(t), \quad \forall t \in [0, t_0], i = 1, 2, \tag{2.14}$$

for some positive constant $d_3$.

(2) By using the properties of $H$, we can prove that the function $H_1$ is strictly decreasing and convex on $(0,1]$, with $\lim_{t\to 0} H_1(t) = +\infty$. Thus, Theorem 2.2 ensures

$$\lim_{t\to+\infty} E(t) = 0.$$

**Remark 2.2** The well-known Jensen's inequality will be of essential use in establishing our main result. If $F_0$ is a convex function on $[a,b]$, $f : \Omega \to [a,b]$ and $h$ are integrable functions on $\Omega$, $h(x) \ge 0$, and $\int_\Omega h(x)\,dx = h_0 > 0$, then Jensen's inequality states that

$$F_0\left(\frac{1}{h_0}\int_\Omega f(x)h(x)\,dx\right) \le \frac{1}{h_0}\int_\Omega F_0(f(x))h(x)\,dx. \tag{2.15}$$

## 3 General decay

In this section, we study the asymptotic behavior of the solutions for system (1.1)-(1.6). To prove the decay property, we first obtain the dissipative property of system (1.1)-(1.6).

Multiplying equation (1.1) by $u_t$ and integrating by parts over $\Omega$, we get

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2+\frac{1}{2}a(u,u)+\frac{1}{4}\|\Delta v\|^2\right]=\left(\mathcal{B}_1 u,\frac{\partial u_t}{\partial \nu}\right)_\Gamma-(\mathcal{B}_2 u,u_t)_\Gamma.$$

From the boundary conditions (2.4) and (2.5) and Lemma 2.3, we have

$$\begin{aligned}E'(t)&\le-\tau_2\|u_t\|_\Gamma^2+\frac{\tau_2}{2}k_2'(t)\|u\|_\Gamma^2\\&\quad-\frac{\tau_2}{2}\int_\Gamma k_2''\Box u\,d\Gamma-\tau_1\left\|\frac{\partial u_t}{\partial \nu}\right\|_\Gamma^2+\frac{\tau_1}{2}k_1'(t)\left\|\frac{\partial u}{\partial \nu}\right\|_\Gamma^2-\frac{\tau_1}{2}\int_\Gamma k_1''\Box\frac{\partial u}{\partial \nu}\,d\Gamma.\end{aligned}\tag{3.1}$$

Let us introduce the following functional:

$$\Phi(t):=\int_\Omega\left\{m\cdot\nabla u+\frac{1}{2}u\right\}u_t\,dx.$$

The following lemma plays an important role in the construction of the Lyapunov functional.

**Lemma 3.1** *There exists $C>0$ such that*

$$\begin{aligned}\Phi'(t)&\le-\frac{1}{2}\|u_t\|^2-\frac{1}{2}a(u,u)-\|\Delta v\|^2+\frac{1}{2}\int_\Gamma(m\cdot\nu)|u_t|^2\,d\Gamma\\&\quad+C\int_\Gamma\left(|u_t|^2+\left|k_2(t)u\right|^2+\left|k_2'\circ u\right|^2\right)d\Gamma\\&\quad+C\int_\Gamma\left(\left|\frac{\partial u_t}{\partial \nu}\right|^2+\left|k_1(t)\frac{\partial u}{\partial \nu}\right|^2+\left|k_1'\circ\frac{\partial u}{\partial \nu}\right|^2\right)d\Gamma.\end{aligned}\tag{3.2}$$

*Proof* Differentiating $\Phi(t)$ and using (1.1), (2.1) and Lemma 2.1, we obtain

$$\begin{aligned}\Phi'(t)&=\int_\Omega u_t(m\cdot\nabla u_t)\,dx+\frac{1}{2}\|u_t\|^2+\int_\Omega\left[(m\cdot\nabla u)+\frac{1}{2}u\right]\left(-\Delta^2 u+[u,v]\right)dx\\&=\frac{1}{2}\int_\Gamma(m\cdot\nu)|u_t|^2\,d\Gamma-\frac{1}{2}\|u_t\|^2-\frac{3}{2}a(u,u)\\&\quad-\frac{1}{2}\|\Delta v\|^2-\int_\Gamma(\mathcal{B}_2 u)\left(m\cdot\nabla u+\frac{1}{2}u\right)d\Gamma\\&\quad+\int_\Gamma(\mathcal{B}_1 u)\frac{\partial}{\partial \nu}\left(m\cdot\nabla u+\frac{1}{2}u\right)d\Gamma\\&\quad-\frac{1}{2}\int_\Gamma m\cdot\nu\left[u_{xx}^2+u_{yy}^2+2\mu u_{xx}u_{yy}+2(1-\mu)u_{xy}^2\right]d\Gamma\\&\quad+\int_\Omega(m\cdot\nabla u)[u,v]\,dx.\end{aligned}\tag{3.3}$$

According to Lemma 2.2, we have

$$\int_\Omega(m\cdot\nabla u)[u,v]\,dx=\int_\Omega[m\cdot\nabla u,u]v\,dx=-\frac{1}{2}\|\Delta v\|^2-\frac{1}{2}\int_\Gamma(m\cdot\nu)|\Delta v|^2\,d\Gamma.\tag{3.4}$$

Using the trace theorem, (2.2), (2.3) and Young's inequality, we get, for $\epsilon_1 > 0$,

$$\begin{aligned}
&\left| -\int_\Gamma (\mathcal{B}_2 u)\left(m\cdot\nabla u + \frac{1}{2}u\right) d\Gamma\right| + \left|\int_\Gamma (\mathcal{B}_1 u)\frac{\partial}{\partial\nu}\left(m\cdot\nabla u + \frac{1}{2}u\right) d\Gamma\right| \\
&\quad\le \epsilon_1 \int_\Gamma \left(m\cdot\nabla u + \frac{1}{2}u\right)^2 d\Gamma + C_{\epsilon_1}\int_\Gamma |\mathcal{B}_2 u|^2\, d\Gamma \\
&\qquad + \epsilon_1 \int_\Gamma \left|\frac{\partial}{\partial\nu}\left(m\cdot\nabla u + \frac{1}{2}u\right)\right|^2 d\Gamma + C_{\epsilon_1}\int_\Gamma |\mathcal{B}_1 u|^2\, d\Gamma \\
&\quad\le \epsilon_1 c(C_p + C_s)a(u,u) + \epsilon_1 c\int_\Gamma (m\cdot\nu)\big[u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx}u_{yy} + 2(1-\mu)u_{xy}^2\big]\, d\Gamma \\
&\qquad + C_{\epsilon_1}\int_\Gamma |\mathcal{B}_2 u|^2\, d\Gamma + C_{\epsilon_1}\int_\Gamma |\mathcal{B}_1 u|^2\, d\Gamma,
\end{aligned} \tag{3.5}$$

where $\epsilon_1$ is a positive constant. From (3.3)-(3.5), we obtain

$$\begin{aligned}
\Phi'(t) &\le \frac{1}{2}\int_\Gamma (m\cdot\nu)|u_t|^2\, d\Gamma - \frac{1}{2}\|u_t\|^2 - \left(\frac{3}{2} - \epsilon_1 c(C_p + C_s)\right)a(u,u) - \|\Delta v\|^2 \\
&\quad - \left(\frac{1}{2} - \epsilon_1 c\right)\int_\Gamma (m\cdot\nu)\big[u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx}u_{yy} + 2(1-\mu)u_{xy}^2\big]\, d\Gamma \\
&\quad - \frac{1}{2}\int_\Gamma (m\cdot\nu)|\Delta v|^2\, d\Gamma + C_{\epsilon_1}\int_\Gamma |\mathcal{B}_2 u|^2\, d\Gamma + C_{\epsilon_1}\int_\Gamma |\mathcal{B}_1 u|^2\, d\Gamma.
\end{aligned} \tag{3.6}$$

Note that

$$k' * u(t) = -\big(k'\circ u\big)(t) + u(t)\big[k(t) - k(0)\big].$$

From the above inequality, the boundary conditions (2.4) and (2.5) can be written as

$$\mathcal{B}_1 u = -\tau_1\left\{\frac{\partial u_t}{\partial\nu} + k_1(t)\frac{\partial u}{\partial\nu} - k_1'\circ\frac{\partial u}{\partial\nu}\right\}, \tag{3.7}$$

$$\mathcal{B}_2 u = \tau_2\big\{u_t + k_2(t)u - k_2'\circ u\big\}. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6) and choosing $\epsilon_1$ small enough, we have estimate (3.2). □

Let us consider the following binary operator:

$$(k\circ v)(t) := \int_0^t k(t-s)\big(v(t) - v(s)\big)\, ds.$$

Then, applying Hölder's inequality for $0 \le \alpha \le 1$, we get

$$\big|(k\circ v)(t)\big|^2 \le \left[\int_0^t \big|k(s)\big|^{2(1-\alpha)}\, ds\right]\big(|k|^{2\alpha}\,\square\, v\big)(t). \tag{3.9}$$

*Proof of Theorem* 2.2 Let us introduce the Lyapunov functional

$$L(t) := NE(t) + \Phi(t)$$

with $N > 0$.

Using (3.1), (3.2) and (3.9) with $\alpha = \frac{1}{2}$, we obtain

$$\begin{aligned} L'(t) \le {} & -\frac{1}{2}\|u_t\|^2 - \frac{1}{2}a(u,u) - \|\Delta v\|^2 - \left(\tau_2 N - \frac{R}{2} - C\right)\|u_t\|_\Gamma^2 - (\tau_1 N - C)\left\|\frac{\partial u_t}{\partial \nu}\right\|_\Gamma^2 \\ & + \left(\frac{\tau_2 N}{2}k_2'(t) + Ck_2^2(t)\right)\|u\|_\Gamma^2 + \left(\frac{\tau_1 N}{2}k_1'(t) + Ck_1^2(t)\right)\left\|\frac{\partial u}{\partial \nu}\right\|_\Gamma^2 \\ & - \frac{\tau_2 N}{2}\int_\Gamma k_2'' \Box u \, d\Gamma - C\int_\Gamma k_2' \Box u \, d\Gamma - \frac{\tau_1 N}{2}\int_\Gamma k_1'' \Box \frac{\partial u}{\partial \nu}\, d\Gamma - C\int_\Gamma k_1' \Box \frac{\partial u}{\partial \nu}\, d\Gamma. \end{aligned}$$

Taking $N$ large, for some positive constant $\beta$,

$$L'(t) \le -\beta E(t) + Ck_2^2(t)\|u\|_\Gamma^2 - C\int_\Gamma k_2' \Box u \, d\Gamma + Ck_1^2(t)\left\|\frac{\partial u}{\partial \nu}\right\|_\Gamma^2 - C\int_\Gamma k_1' \Box \frac{\partial u}{\partial \nu}\, d\Gamma,$$

which, using trace theory and the fact that $\lim_{t\to\infty} k_i(t) = 0$ for $i = 1, 2$, yields, for large $t_0$,

$$L'(t) \le -\beta E(t) - C\int_\Gamma k_2' \Box u \, d\Gamma - C\int_\Gamma k_1' \Box \frac{\partial u}{\partial \nu}\, d\Gamma, \quad \forall t \ge t_0. \tag{3.10}$$

On the other hand, it is not difficult to see that $L(t)$ satisfies

$$q_0 E(t) \le L(t) \le q_1 E(t) \tag{3.11}$$

for some positive constants $q_0$, $q_1$. Using (2.14), (3.1) and (3.10), we have

$$\begin{aligned} L'(t) \le {} & -\beta E(t) + \frac{C}{d_3}\int_0^{t_0} k_2''(s)\int_\Gamma \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds \\ & - C\int_{t_0}^t k_2'(s)\int_\Gamma \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds \\ & + \frac{C}{d_3}\int_0^{t_0} k_1''(s)\int_\Gamma \left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 d\Gamma\, ds \\ & - C\int_{t_0}^t k_1'(s)\int_\Gamma \left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 d\Gamma\, ds \\ \le {} & -\beta E(t) - \frac{2C}{d_3\tau}E'(t) - C\int_{t_0}^t k_2'(s)\int_\Gamma \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds \\ & - C\int_{t_0}^t k_1'(s)\int_\Gamma \left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 d\Gamma\, ds, \quad \forall t \ge t_0, \end{aligned} \tag{3.12}$$

where $\tau = \min\{\tau_1, \tau_2\}$. We take $\mathcal{L}(t) = L(t) + \frac{2C}{d_3\tau}E(t)$, which is equivalent to $E(t)$, and use (3.12) to get

$$\begin{aligned} \mathcal{L}'(t) \le {} & -\beta E(t) - C\int_{t_0}^t k_2'(s)\int_\Gamma \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds \\ & - C\int_{t_0}^t k_1'(s)\int_\Gamma \left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 d\Gamma\, ds. \end{aligned} \tag{3.13}$$

(A) The general case: This case is obtained on account of the ideas presented in [20, 22] as follows. Let $H_0^*$ be the convex conjugate of $H_0$ in the sense of Young (see [26]); then

$$H_0^*(s) = s\big(H_0'\big)^{-1}(s) - H_0\big[\big(H_0'\big)^{-1}(s)\big], \quad \text{if } s \in \big(0, H_0'(r)\big] \tag{3.14}$$

and $H_0^*$ satisfies the following Young's inequality:

$$AB \le H_0^*(A) + H_0(B), \quad \text{if } A \in \big(0, H_0'(r)\big], B \in (0, r]. \tag{3.15}$$

We define $\eta(t)$ and $\xi(t)$ by

$$\begin{aligned}
\eta(t) &:= \int_{t_0}^t \frac{-k_2'(s)}{H_0^{-1}(k_2''(s))} \int_\Gamma \big|u(t) - u(t-s)\big|^2 \, d\Gamma \, ds, \\
\xi(t) &:= \int_{t_0}^t \frac{-k_1'(s)}{H_0^{-1}(k_1''(s))} \int_\Gamma \left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 d\Gamma \, ds,
\end{aligned}$$

where $H_0$ is such that (2.7) is satisfied. From (2.7), (3.1) and trace theory, and choosing $t_0$ even larger if needed, we find that $\eta(t)$ and $\xi(t)$ satisfy

$$\eta(t) < 1, \qquad \xi(t) < 1, \quad \forall t \ge t_0. \tag{3.16}$$

Besides, we define $\kappa(t)$ and $\chi(t)$ by

$$\begin{aligned}
\kappa(t) &:= \int_{t_0}^t k_2''(s) \frac{-k_2'(s)}{H_0^{-1}(k_2''(s))} \int_\Gamma \big|u(t) - u(t-s)\big|^2 \, d\Gamma \, ds, \\
\chi(t) &:= \int_{t_0}^t k_1''(s) \frac{-k_1'(s)}{H_0^{-1}(k_1''(s))} \int_\Gamma \left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 d\Gamma \, ds.
\end{aligned}$$

By (2.6) and the properties of $H_0$ and $D$, we obtain

$$\frac{-k_i'(s)}{H_0^{-1}(k_i''(s))} \le \frac{-k_i'(s)}{H_0^{-1}(H(-k_i'(s)))} = \frac{-k_i'(s)}{D^{-1}(-k_i'(s))} \le \alpha_0, \quad i = 1, 2, \tag{3.17}$$

for some positive constant $\alpha_0$. Then, using (2.11), (3.1) and (3.17) and choosing $t_0$ even larger, we can see that $\kappa(t)$ satisfies, for all $t \ge t_0$,

$$\begin{aligned}
\kappa(t) &\le \alpha_0 \int_{t_0}^t k_2''(s) \int_\Gamma \big|u(t) - u(t-s)\big|^2 \, d\Gamma \, ds \le cE(0) \int_{t_0}^t k_2''(s) \, ds \le -ck_2'(t_0)E(0) \\
&\le \min\big\{r, H(r), H_0(r)\big\}.
\end{aligned} \tag{3.18}$$

Similarly, we deduce that $\chi(t) \le \min\{r, H(r), H_0(r)\}$. Since $H_0$ is strictly convex on $(0, r]$ and $H_0(0) = 0$, then

$$H_0(\lambda x) \le \lambda H_0(x) \tag{3.19}$$

provided $0 \le \lambda \le 1$ and $x \in (0, r]$. Using (3.16), (3.19) and Jensen's inequality (2.15), we have

$$\begin{aligned}\kappa(t) &= \frac{1}{\eta(t)}\int_{t_0}^{t}\eta(t)H_0\bigl[H_0^{-1}\bigl(k_2''(s)\bigr)\bigr]\frac{-k_2'(s)}{H_0^{-1}(k_2''(s))}\int_\Gamma\bigl|u(t)-u(t-s)\bigr|^2\,d\Gamma\,ds\\ &\geq \frac{1}{\eta(t)}\int_{t_0}^{t}H_0\bigl[\eta(t)H_0^{-1}\bigl(k_2''(s)\bigr)\bigr]\frac{-k_2'(s)}{H_0^{-1}(k_2''(s))}\int_\Gamma\bigl|u(t)-u(t-s)\bigr|^2\,d\Gamma\,ds\\ &\geq H_0\biggl(\int_{t_0}^{t}-k_2'(s)\int_\Gamma\bigl|u(t)-u(t-s)\bigr|^2\,d\Gamma\,ds\biggr).\end{aligned}$$

This implies that

$$\int_{t_0}^{t}-k_2'(s)\int_\Gamma\bigl|u(t)-u(t-s)\bigr|^2\,d\Gamma\,ds\leq H_0^{-1}\bigl(\kappa(t)\bigr). \tag{3.20}$$

Similarly, we get

$$\int_{t_0}^{t}-k_1'(s)\int_\Gamma\left|\frac{\partial u(t)}{\partial\nu}-\frac{\partial u(t-s)}{\partial\nu}\right|^2 d\Gamma\,ds\leq H_0^{-1}\bigl(\chi(t)\bigr). \tag{3.21}$$

From (3.13), (3.20) and (3.21) we deduce that

$$\mathcal{L}'(t)\leq-\beta E(t)+CH_0^{-1}\bigl(\kappa(t)\bigr)+CH_0^{-1}\bigl(\chi(t)\bigr),\quad\forall t\geq t_0. \tag{3.22}$$

For $\epsilon_0<r$ and $d_0>0$, we define the functional

$$F(t):=\mathcal{L}(t)H_0'\biggl(\epsilon_0\frac{E(t)}{E(0)}\biggr)+d_0E(t),$$

which satisfies

$$\alpha_1F(t)\leq E(t)\leq\alpha_2F(t) \tag{3.23}$$

for some $\alpha_1,\alpha_2>0$. Also, by $\epsilon_0<r$, $E'\leq0$, we get $\epsilon_0\frac{E(t)}{E(0)}<r$. Using (3.1), (3.14), (3.15), (3.18) and (3.22) and the fact that $E'\leq0$, $H_0>0$, $H_0'>0$ and $H_0''>0$ on $(0,r]$, we obtain

$$\begin{aligned}F'(t) &= \mathcal{L}'(t)H_0'\biggl(\epsilon_0\frac{E(t)}{E(0)}\biggr)+\epsilon_0\mathcal{L}(t)\frac{E'(t)}{E(0)}H_0''\biggl(\epsilon_0\frac{E(t)}{E(0)}\biggr)+d_0E'(t)\\ &\leq-\beta E(t)H_0'\biggl(\epsilon_0\frac{E(t)}{E(0)}\biggr)+CH_0'\biggl(\epsilon_0\frac{E(t)}{E(0)}\biggr)H_0^{-1}\bigl(\kappa(t)\bigr)\\ &\quad+CH_0'\biggl(\epsilon_0\frac{E(t)}{E(0)}\biggr)H_0^{-1}\bigl(\chi(t)\bigr)+d_0E'(t)\\ &\leq-\beta E(t)H_0'\biggl(\epsilon_0\frac{E(t)}{E(0)}\biggr)+CH_0^*\biggl(H_0'\biggl(\epsilon_0\frac{E(t)}{E(0)}\biggr)\biggr)+C\kappa(t)+C\chi(t)+d_0E'(t)\\ &\leq-\beta E(t)H_0'\biggl(\epsilon_0\frac{E(t)}{E(0)}\biggr)+\epsilon_0C\frac{E(t)}{E(0)}H_0'\biggl(\epsilon_0\frac{E(t)}{E(0)}\biggr)\\ &\quad-CH_0\biggl(\epsilon_0\frac{E(t)}{E(0)}\biggr)+C\bigl(\kappa(t)+\chi(t)\bigr)+d_0E'(t)\\ &\leq-\bigl(\beta E(0)-\epsilon_0C\bigr)\frac{E(t)}{E(0)}H_0'\biggl(\epsilon_0\frac{E(t)}{E(0)}\biggr)-\frac{2C\alpha_0}{\tau}E'(t)+d_0E'(t).\end{aligned}$$

Therefore, with a suitable choice of $\epsilon_0$ and $d_0$, we have, for all $t \geq t_0$,

$$F'(t) \leq -\alpha_3 \left(\frac{E(t)}{E(0)}\right) H_0'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) = -\alpha_3 H_2\left(\frac{E(t)}{E(0)}\right), \tag{3.24}$$

where $\alpha_3 > 0$ and $H_2(t) = tH_0'(\epsilon_0 t)$. By the strict convexity of $H_0$ on $(0, r]$, we see that $H_2(t) > 0$, and $H_2'(t) = H_0'(\epsilon_0 t) + \epsilon_0 t H_0''(\epsilon_0 t) > 0$ on $(0, 1]$. We take

$$K(t) = \frac{\alpha_1 F(t)}{E(0)},$$

which is clearly equivalent to $E(t)$. Using (3.23), (3.24) and $H_2' > 0$, we get

$$K'(t) = \frac{\alpha_1 F'(t)}{E(0)} \leq -\frac{\alpha_1\alpha_3}{E(0)} H_2\left(\frac{E(t)}{E(0)}\right) \leq -k_0 H_2\bigl(K(t)\bigr), \quad \forall t \geq t_0,$$

where $k_0 = \frac{\alpha_1\alpha_3}{E(0)} > 0$. Then, using the properties of $H_2$, the fact that $H_1$ is a strictly decreasing function on $(0, 1]$ and $\lim_{t\to 0} H_1(t) = +\infty$, we obtain, for some $k_1, k_2 > 0$,

$$K(t) \leq H_1^{-1}(k_1 t + k_2), \quad \forall t \geq t_0, \tag{3.25}$$

where $H_1(t) = \int_t^1 \frac{1}{H_2(s)}\, ds$. Using (3.23) and (3.25), we have (2.8). Moreover, if $\int_0^t H_1(t)\, dt < +\infty$, then $\int_0^{+\infty} H_1^{-1}(t)\, dt < +\infty$. From (2.8) we get $\int_0^{+\infty} E(t)\, dt < \infty$ and

$$\begin{aligned}
&\int_0^t \int_\Gamma \bigl|u(t) - u(t-s)\bigr|^2\, d\Gamma\, ds \leq c \int_0^t E(s)\, ds < +\infty, \\
&\int_0^t \int_\Gamma \left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 d\Gamma\, ds \leq c \int_0^t E(s)\, ds < +\infty.
\end{aligned}$$

Similarly, we define, for large $t_0$,

$$\begin{aligned}
&\eta_1(t) := \int_{t_0}^t \int_\Gamma \bigl|u(t) - u(t-s)\bigr|^2\, d\Gamma\, ds < 1, \quad \forall t \geq t_0, \\
&\xi_1(t) := \int_{t_0}^t \int_\Gamma \left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 d\Gamma\, ds < 1, \quad \forall t \geq t_0
\end{aligned}$$

and

$$\begin{aligned}
&\kappa_1(t) := \int_{t_0}^t k_2''(s) \int_\Gamma \bigl|u(t) - u(t-s)\bigr|^2\, d\Gamma\, ds, \\
&\chi_1(t) := \int_{t_0}^t k_1''(s) \int_\Gamma \left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 d\Gamma\, ds.
\end{aligned}$$

From (2.6), the strict convexity of $H$ and Jensen's inequality (2.15), we obtain

$$\begin{aligned}\kappa_1(t) &\geq \frac{1}{\eta_1(t)}\int_{t_0}^{t}\eta_1(t)H\bigl(-k_2'(s)\bigr)\int_\Gamma\bigl|u(t)-u(t-s)\bigr|^2\,d\Gamma\,ds\\&\geq \frac{1}{\eta_1(t)}\int_{t_0}^{t}H\bigl(-\eta_1(t)k_2'(s)\bigr)\int_\Gamma\bigl|u(t)-u(t-s)\bigr|^2\,d\Gamma\,ds\\&\geq H\biggl(\int_{t_0}^{t}-k_2'(s)\int_\Gamma\bigl|u(t)-u(t-s)\bigr|^2\,d\Gamma\,ds\biggr).\end{aligned}$$

Therefore, we deduce that

$$\int_{t_0}^{t}\bigl(-k_2'(s)\bigr)\int_\Gamma\bigl|u(t)-u(t-s)\bigr|^2\,d\Gamma\,ds\leq H^{-1}\bigl(\kappa_1(t)\bigr).$$

Similarly, we have

$$\int_{t_0}^{t}\bigl(-k_1'(s)\bigr)\int_\Gamma\left|\frac{\partial u(t)}{\partial\nu}-\frac{\partial u(t-s)}{\partial\nu}\right|^2\,d\Gamma\,ds\leq H^{-1}\bigl(\chi_1(t)\bigr).$$

Thus, (3.13) becomes

$$\mathcal{L}'(t)\leq-\beta E(t)+CH^{-1}\bigl(\kappa_1(t)\bigr)+CH^{-1}\bigl(\chi_1(t)\bigr),\quad\forall t\geq t_0.$$

Hence, repeating the same procedures, we see that for some $c_1$, $c_2$ and $c_3>0$,

$$E(t)\leq c_3G^{-1}(c_1t+c_2),$$

where $G(t)=\int_t^1\frac{1}{sH'(\epsilon_0 s)}\,ds$.

(B) The special case $H(t)=ct^p$ and $1\leq p<\frac{3}{2}$:

*Case* 1. $p=1$: Using (2.6) and (3.1), estimate (3.13) yields, for all $t\geq t_0$,

$$\begin{aligned}\mathcal{L}'(t)&\leq-\beta E(t)+\frac{C}{c}\int_{t_0}^{t}k_2''(s)\int_\Gamma\bigl|u(t)-u(t-s)\bigr|^2\,d\Gamma\,ds\\&\quad+\frac{C}{c}\int_{t_0}^{t}k_1''(s)\int_\Gamma\left|\frac{\partial u(t)}{\partial\nu}-\frac{\partial u(t-s)}{\partial\nu}\right|^2\,d\Gamma\,ds\\&\leq-\beta E(t)-\frac{2C}{c\tau}E'(t),\end{aligned}$$

which gives

$$\left(\mathcal{L}+\frac{2C}{c\tau}E\right)'(t)\leq-\beta E(t),\quad\forall t\geq t_0.$$

From (3.11) we know that $\mathcal{L}+\frac{2C}{c\tau}E\sim E$. Hence, we easily get

$$E(t)\leq c'e^{-ct}=c'G^{-1}(t),$$

where

$$G(t)=\int_t^1\frac{1}{sH'(\epsilon_0 s)}\,ds=\int_t^1\frac{1}{sc}\,ds=-\frac{\ln t}{c}.$$

*Case* 2. $1 < p < \frac{3}{2}$: We can see that

$$\int_0^\infty \left(-k_i'(s)\right)^{1-\delta_0} ds < \infty, \quad i = 1, 2, \tag{3.26}$$

for any $\delta_0 < 2 - p$. Using (3.1) and (3.26) and choosing $t_0$ even larger if needed, we conclude that for all $t \geq t_0$,

$$\begin{aligned} I(t) &:= \int_{t_0}^t \left(-k_2'(s)\right)^{1-\delta_0} \int_\Gamma \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds \\ &\leq 2 \int_{t_0}^t \left(-k_2'(s)\right)^{1-\delta_0} \int_\Gamma \left[\left|u(t)\right|^2 + \left|u(t-s)\right|^2\right] d\Gamma\, ds \\ &\leq cE(0) \int_{t_0}^t \left(-k_2'(s)\right)^{1-\delta_0} ds < 1, \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} J(t) &:= \int_{t_0}^t \left(-k_1'(s)\right)^{1-\delta_0} \int_\Gamma \left|\frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu}\right|^2 d\Gamma\, ds \\ &\leq 2 \int_{t_0}^t \left(-k_1'(s)\right)^{1-\delta_0} \int_\Gamma \left[\left|\frac{\partial u(t)}{\partial \nu}\right|^2 + \left|\frac{\partial u(t-s)}{\partial \nu}\right|^2\right] d\Gamma\, ds \\ &\leq cE(0) \int_{t_0}^t \left(-k_1'(s)\right)^{1-\delta_0} ds < 1. \end{aligned}$$

From Hölder's inequality, Jensen's inequality, (2.6), (3.1) and (3.27), we get

$$\begin{aligned} &\int_{t_0}^t \left(-k_2'(s)\right) \int_\Gamma \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds \\ &\quad= \int_{t_0}^t \left(-k_2'(s)\right)^{(p-1+\delta_0)(\frac{\delta_0}{p-1+\delta_0})} \left(-k_2'(s)\right)^{1-\delta_0} \int_\Gamma \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds \\ &\quad\leq \left(\int_{t_0}^t \left(-k_2'(s)\right)^{p-1+\delta_0} \left(-k_2'(s)\right)^{1-\delta_0} \int_\Gamma \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds\right)^{\frac{\delta_0}{p-1+\delta_0}} \\ &\qquad\times \left(\int_{t_0}^t \left(-k_2'(s)\right)^{1-\delta_0} \int_\Gamma \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds\right)^{\frac{p-1}{p-1+\delta_0}} \\ &\quad= I(t) \left(\frac{1}{I(t)} \int_{t_0}^t \left(-k_2'(s)\right)^{p-1+\delta_0} \left(-k_2'(s)\right)^{1-\delta_0} \int_\Gamma \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds\right)^{\frac{\delta_0}{p-1+\delta_0}} \\ &\quad\leq \left(\int_{t_0}^t \left(-k_2'(s)\right)^p \int_\Gamma \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds\right)^{\frac{\delta_0}{p-1+\delta_0}} \\ &\quad\leq \left(\frac{1}{c}\right)^{\frac{\delta_0}{p-1+\delta_0}} \left(\int_{t_0}^t k_2''(s) \int_\Gamma \left|u(t) - u(t-s)\right|^2 d\Gamma\, ds\right)^{\frac{\delta_0}{p-1+\delta_0}} \\ &\quad\leq \left(\frac{1}{c}\right)^{\frac{\delta_0}{p-1+\delta_0}} \left(-E'(t)\right)^{\frac{\delta_0}{p-1+\delta_0}}. \end{aligned} \tag{3.28}$$

Similarly, we obtain

$$\int_{t_0}^{t} \bigl(-k_1'(s)\bigr) \int_{\Gamma} \left| \frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu} \right|^2 d\Gamma\, ds \le \left(\frac{1}{c}\right)^{\frac{\delta_0}{p-1+\delta_0}} \bigl(-E'(t)\bigr)^{\frac{\delta_0}{p-1+\delta_0}}. \tag{3.29}$$

Therefore, using (3.28) and (3.29), we see that (3.13) yields, for $\delta_0 = \frac{1}{2}$,

$$\mathcal{L}'(t) \le -\beta E(t) + \frac{2C}{c^{\frac{1}{2p-1}}} \bigl(-E'(t)\bigr)^{\frac{1}{2p-1}}. \tag{3.30}$$

Multiplying (3.30) by $E^{\gamma}(t)$, with $\gamma = 2p-2$, and using (3.1) and Young's inequality, with $\frac{1}{\gamma+1} + \frac{\gamma}{\gamma+1} = 1$, we have

$$\begin{aligned} \bigl(\mathcal{L}E^{\gamma}\bigr)'(t) &= \mathcal{L}'(t)E^{\gamma}(t) + \gamma \mathcal{L}(t)E^{\gamma-1}(t)E'(t) \le -\beta E^{\gamma+1}(t) + \frac{2C}{c^{\frac{1}{\gamma+1}}} E^{\gamma}(t)\bigl(-E'(t)\bigr)^{\frac{1}{\gamma+1}} \\ &\le -\beta E^{\gamma+1}(t) + \varepsilon E^{\gamma+1}(t) + C_{\varepsilon}\bigl(-E'(t)\bigr). \end{aligned}$$

Choosing $\varepsilon < \beta$, we get, for some $C_1 > 0$,

$$L_0'(t) \le -C_1 L_0^{\gamma+1}(t),$$

where $L_0 = \mathcal{L}E^{\gamma} + C_{\varepsilon}E \sim E$. Then we conclude that

$$E(t) \le \frac{c}{(c' + c''t)^{\frac{1}{\gamma}}}. \tag{3.31}$$

Since $p < \frac{3}{2}$ and using (3.31), we see that

$$\int_0^{\infty} E(t)\, dt \le \int_0^{\infty} \frac{c}{(c' + c''t)^{\frac{1}{2p-2}}}\, dt < +\infty.$$

Using this fact, we have

$$\int_0^t \int_{\Gamma} \bigl|u(t) - u(t-s)\bigr|^2 d\Gamma\, ds \le c \int_0^t E(s)\, ds < +\infty, \tag{3.32}$$

$$\int_0^t \int_{\Gamma} \left| \frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu} \right|^2 d\Gamma\, ds \le c \int_0^t E(s)\, ds < +\infty. \tag{3.33}$$

Then, by (2.6), (3.1), (3.32), (3.33) and Hölder's inequality, estimate (3.13) gives

$$\begin{aligned} \mathcal{L}'(t) &\le -\beta E(t) + C\left(\int_0^t \int_{\Gamma} \bigl|u(t) - u(t-s)\bigr|^2 d\Gamma\, ds\right)^{\frac{p-1}{p}} \left(\int_{\Gamma} \bigl(-k_2'\bigr)^p \Box u\, d\Gamma\right)^{\frac{1}{p}} \\ &\quad + C\left(\int_0^t \int_{\Gamma} \left| \frac{\partial u(t)}{\partial \nu} - \frac{\partial u(t-s)}{\partial \nu} \right|^2 d\Gamma\, ds\right)^{\frac{p-1}{p}} \left(\int_{\Gamma} \bigl(-k_1'\bigr)^p \Box \frac{\partial u}{\partial \nu}\, d\Gamma\right)^{\frac{1}{p}} \\ &\le -\beta E(t) + c\left(\int_{\Gamma} k_2'' \Box u\, d\Gamma\right)^{\frac{1}{p}} + c\left(\int_{\Gamma} k_1'' \Box \frac{\partial u}{\partial \nu}\, d\Gamma\right)^{\frac{1}{p}} \\ &\le -\beta E(t) + 2c\bigl(-E'(t)\bigr)^{\frac{1}{p}}. \end{aligned} \tag{3.34}$$

Now, we multiply (3.34) by $E^\gamma(t)$ with $\gamma = p-1$. Hence, repeating the above steps, we find that

$$E(t) \le \frac{c}{(c' + c''t)^{\frac{1}{\gamma}}} = cG^{-1}\bigl(a' + a''t\bigr),$$

where

$$G(t) = \frac{1}{cp\epsilon_0^{p-1}} \int_t^1 \frac{1}{s^p}\,ds = \frac{1}{cp(p-1)\epsilon_0^{p-1}}\left(\frac{1}{t^{p-1}} - 1\right). \qquad \square$$

**Example** We give an example to explain the energy decay rates given by Theorem 2.2.

(1) If

$$k_i'(t) = -\frac{1}{a + t^q}, \quad \forall i = 1, 2,$$

for $q > 3$ and $a > 1$, then $k_i''(t) = H(-k_i'(t))$, where

$$H(t) = qt^2\left(\frac{1}{t} - a\right)^{1-\frac{1}{q}}.$$

Because

$$H'(t) = \frac{q(1 + \frac{1}{q} - 2at)}{(\frac{1}{t} - a)^{\frac{1}{q}}}, \qquad H''(t) = \frac{\frac{2a^2q}{t^2}(t - \frac{1+q-\sqrt{q^2-1}}{2aq})(t - \frac{1+q+\sqrt{q^2-1}}{2aq})}{(\frac{1}{t} - a)^{1+\frac{1}{q}}},$$

then the function $H$ satisfies hypothesis (A1) on the interval $(0, r]$ for any $0 < r < \frac{1+q-\sqrt{q^2-1}}{2aq}$. By taking $D(t) = t^\alpha$, (2.7) is satisfied for any $\alpha > \frac{q}{q-1}$. Hence, an explicit rate of decay can be obtained by Theorem 2.2. The function $H_0(t) = H(t^\alpha)$ has derivative

$$H_0'(t) = \frac{q\alpha t^{\alpha-1}[1 + \frac{1}{q} - 2at^\alpha]}{(\frac{1}{t^\alpha} - a)^{\frac{1}{q}}}.$$

Thus,

$$H_1(t) = \int_t^1 \frac{[\frac{1}{(\epsilon_0 s)^\alpha} - a]^{\frac{1}{q}}}{q\alpha s(\epsilon_0 s)^{\alpha-1}[1 + \frac{1}{q} - 2a(\epsilon_0 s)^\alpha]}\,ds.$$

Let $\frac{1}{(\epsilon_0 s)^\alpha} = u$, then we have

$$H_1(t) = \int_{\frac{1}{\epsilon_0^\alpha}}^{\frac{1}{(\epsilon_0 t)^\alpha}} \frac{(u-a)^{\frac{1}{q}} u^{-\frac{1}{\alpha}}}{q\alpha^2[1 + \frac{1}{q} - \frac{2a}{u}]}\,du \le \frac{1}{q\alpha^2[1 + \frac{1}{q} - 2a\epsilon_0^\alpha]} \int_{\frac{1}{\epsilon_0^\alpha}}^{\frac{1}{(\epsilon_0 t)^\alpha}} (u-a)^{\frac{1}{q}} u^{-\frac{1}{\alpha}}\,du.$$

Using the fact that the function $f(u) = (u-a)^{\frac{1}{q}}$ is increasing on $(a, +\infty)$ and $(u-a)^{\frac{1}{q}} < u^{\frac{1}{q}}$ and taking $\epsilon_0 < a^{-\frac{1}{\alpha}}$, we get

$$H_1(t) \le \frac{1}{q\alpha^2[1 + \frac{1}{q} - 2a\epsilon_0^\alpha]} \int_{\frac{1}{\epsilon_0^\alpha}}^{\frac{1}{(\epsilon_0 t)^\alpha}} u^{\frac{1}{q} - \frac{1}{\alpha}}\,du = \frac{\epsilon_0^{\frac{q-\alpha-\alpha q}{q}}}{\alpha(\alpha - q + \alpha q)[1 + \frac{1}{q} - 2a\epsilon_0^\alpha]}\bigl[t^{\frac{q-\alpha-\alpha q}{q}} - 1\bigr].$$

Next, we find that if $\alpha < \frac{2q}{1+q}$,

$$\begin{aligned}\int_0^1 H_1(t)\,dt &\le \frac{\epsilon_0^{\frac{q-\alpha-\alpha q}{q}}}{\alpha(\alpha-q+\alpha q)[1+\frac{1}{q}-2a\epsilon_0^{\alpha}]}\int_0^1\bigl[t^{\frac{q-\alpha-\alpha q}{q}}-1\bigr]\,dt \\ &= \frac{\epsilon_0^{\frac{q-\alpha-\alpha q}{q}}}{\alpha(2q-\alpha-\alpha q)[1+\frac{1}{q}-2a\epsilon_0^{\alpha}]} < +\infty.\end{aligned}$$

Taking $\frac{1}{\epsilon_0 s} = \nu$ and $\epsilon_0 < a^{-1}$, we obtain

$$\begin{aligned}G(t) &= \int_t^1 \frac{1}{sH'(\epsilon_0 s)}\,ds = \int_t^1 \frac{(\frac{1}{\epsilon_0 s}-a)^{\frac{1}{q}}}{sq(1+\frac{1}{q}-2a\epsilon_0 s)}\,ds = \int_{\frac{1}{\epsilon_0}}^{\frac{1}{\epsilon_0 t}} \frac{(\nu-a)^{\frac{1}{q}}\nu^{-1}}{q(1+\frac{1}{q}-\frac{2a}{\nu})}\,d\nu \\ &\le \frac{1}{q(1+\frac{1}{q}-2a\epsilon_0)}\int_{\frac{1}{\epsilon_0}}^{\frac{1}{\epsilon_0 t}} \nu^{\frac{1}{q}-1}\,d\nu = \frac{1}{1+\frac{1}{q}-2a\epsilon_0}\left[\left(\frac{1}{\epsilon_0 t}\right)^{\frac{1}{q}} - \left(\frac{1}{\epsilon_0}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Therefore,

$$G^{-1}(t) \le \frac{1}{\epsilon_0[(\frac{1}{\epsilon_0})^{\frac{1}{q}} + (1+\frac{1}{q}-2a\epsilon_0)t]^q}.$$

Hence we can use (2.9) to deduce that the energy decays

$$E(t) \le \frac{\tilde{c}_1}{\tilde{c}_2 + \tilde{c}_3 t^q},$$

where $\tilde{c}_i$ $(i = 1, 2, 3)$ are constants.

(2) As in [20], let $0 < q < 1$

$$k_i'(t) = -\exp\bigl(-t^q\bigr), \quad \forall i = 1, 2,$$

then $k_i''(t) = H(-k_i'(t))$, where, for $t \in (0, r]$, $r < 1$, $H(t) = \frac{qt}{[\ln(1/t)]^{\frac{1}{q}-1}}$. Hence

$$E(t) \le c\exp\bigl(-\omega t^q\bigr).$$

**Competing interests**

The author declares that they have no competing interests.

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**References**

1. Favini, A, Horn, M, Lasiecka, I, Tataru, D: Global existence, uniqueness and regularity of solutions to a von Kármán system with nonlinear boundary dissipation. Differ. Integral Equ. **9**(2), 267-294 (1996)
2. Horn, MA, Lasiecka, I: Uniform decay of weak solutions to a von Kármán plate with nonlinear boundary dissipation. Differ. Integral Equ. **7**, 885-908 (1994)
3. Horn, MA, Lasiecka, I: Global stabilization of a dynamic von Kármán plate with nonlinear boundary feedback. Appl. Math. Optim. **31**, 57-84 (1995)
4. Puel, J, Tucsnak, M: Boundary stabilization for the von Kármán equations. SIAM J. Control **33**, 255-273 (1996)

5. Rivera, JEM, Menzala, GP: Decay rates of solutions to a von Kármán system for viscoelastic plates with memory. Q. Appl. Math. **LVII**, 181-200 (1999)
6. Raposo, CA, Santos, ML: General decay to a von Kármán system with memory. Nonlinear Anal. **74**, 937-945 (2011)
7. Kang, JR: Energy decay rates for von Kármán system with memory and boundary feedback. Appl. Math. Comput. **218**, 9085-9094 (2012)
8. Kang, JR: Exponential decay for a von Kármán equations with memory. J. Math. Phys. **54**, 033501 (2013)
9. Kang, JR: Exponential decay for a von Kármán equations of memory type with acoustic boundary conditions. Math. Methods Appl. Sci. **38**, 598-608 (2015)
10. Rivera, JEM, Oquendo, HP, Santos, ML: Asymptotic behavior to a von Kármán plate with boundary memory conditions. Nonlinear Anal. **62**, 1183-1205 (2005)
11. Santos, ML, Soufyane, A: General decay to a von Kármán plate system with memory boundary conditions. Differ. Integral Equ. **24**(1-2), 69-81 (2011)
12. Cavalcanti, MM, Domingos Cavalcanti, VN, Santos, ML: Existence and uniform decay rates of solutions to a degenerate system with memory conditions at the boundary. Appl. Math. Comput. **150**(2), 439-465 (2004)
13. Santos, ML: Decay rates for solutions of a Timoshenko system with a memory condition at the boundary. Abstr. Appl. Anal. **7**(10), 531-546 (2002)
14. Santos, ML, Ferreira, J, Pereira, DC, Raposo, CA: Global existence and stability for wave equation of Kirchhoff type with memory condition at the boundary. Nonlinear Anal. **54**, 959-976 (2003)
15. Park, JY, Kang, JR: Uniform decay for hyperbolic differential inclusion with memory condition at the boundary. Numer. Funct. Anal. Optim. **27**(7-8), 875-888 (2006)
16. Messaoudi, SA, Soufyane, A: General decay of solutions of a wave equation with a boundary control of memory type. Nonlinear Anal., Real World Appl. **11**, 2896-2904 (2010)
17. Mustafa, MI, Messaoudi, SA: Energy decay rates for a Timoshenko system with viscoelastic boundary conditions. Appl. Math. Comput. **218**, 9125-9131 (2012)
18. Kang, JR: General decay for Kirchhoff plates with a boundary condition of memory type. Bound. Value Probl. **2012**, 129 (2012)
19. Kang, JR: General decay for a differential inclusion of Kirchhoff type with a memory condition at the boundary. Acta Math. Sci. **34**(3), 729-738 (2014)
20. Mustafa, MI, Abusharkh, GA: Plate equations with viscoelastic boundary damping. Indag. Math. **26**, 307-323 (2015)
21. Messaoudi, SA, Mustafa, MI: A general stability result for a quasilinear wave equation with memory. Nonlinear Anal., Real World Appl. **14**, 1854-1864 (2013)
22. Mustafa, MI, Messaoudi, SA: General stability result for viscoelastic wave equations. J. Math. Phys. **53**, 053702 (2012)
23. Alabau-Boussouira, F, Cannarsa, P: A general method for proving sharp energy decay rates for memory dissipative evolution equations. C. R. Math. Acad. Sci. Paris **347**, 867-872 (2009)
24. Lagnese, JE: Boundary Stabilization of Thin Plates. SIAM, Philadelphia (1999)
25. Chueshov, I, Lasiecka, I: Global attractors for von Karman evolutions with a nonlinear boundary dissipation. J. Differ. Equ. **198**, 196-231 (2004)
26. Arnold, VI: Mathematical Methods of Classical Mechanics. Springer, New York (1989)