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Positive periodic solutions for second order differential equations with impulsive effects

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Abstract

The existence of positive periodic solutions for a class of second order impulsive differential equations is studied. By using a fixed point theorem in cone, we obtain two existence results, which extend some known results.

MSC: 34B18; 34B37

Keywords: impulsive differential equation; periodic solution; fixed point theorem

1 Introduction

In this paper, we discuss the existence of positive periodic solutions for the following second order impulsive differential equation:

$$\begin{cases} u''(t) + (a-b)u'(t) + f(t,u(t)) = 0, & t \neq t_k, \\ \Delta u(t_k) = I_k(u(t_k)), \qquad \Delta u'(t_k) = J_k(u(t_k)), & k \in \mathbb{Z}, \end{cases} \tag{1.1}$$

where a, b are positive constants, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ represent the right limit and left limit of $u(t)$ at t_k, respectively, $u'(t_k^-) = \lim_{h\to 0^-} h^{-1}[u(t_k+h) - u(t_k)]$, $u'(t_k^+) = \lim_{h\to 0^+} h^{-1}[u(t_k+h) - u(t_k)]$.

Throughout this paper, we suppose that the following conditions are fulfilled:

(P) $f(t+T,u) = f(t,u)$, $T > 0$ is a constant, $\lim_{k\to\pm\infty} t_k = \pm\infty$, $t_k < t_{k+1}$. There exists $p \in \mathbb{N}$ such that $I_{k+p}(u) = I_k(u)$, $J_{k+p}(u) = J_k(u)$, $t_{k+p} = t_k + T$.

To define the solution of (1.1), we introduce the space $PC^r(\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R} | u^{(j)}(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and each $u^{(j)}(t_k^+)$ exists for $k \in \mathbb{Z}$, where $j = 0, 1, \ldots, r\}$.

By a solution of (1.1) we mean a function $x \in PC^2(\mathbb{R})$ satisfying (1.1).

The theory of impulsive differential equations has been a significant development in the last two decades. Periodic solutions and periodic boundary value problems of impulsive differential equations have received considerable attention and much literature has been published; for instance, see [1–5] and the references therein. It should be noted that compared to first order impulsive differential equations, there exist very few existence results of positive periodic solutions for second order impulsive equations, especially for second order impulsive equations with derivative term, see [6–11]. In [12], authors considered the special case $a = b$ of (1.1), in which the functions $f(\cdot, u)$, $I_k(u)$, $J_k(u)$ satisfy the given growth conditions at the origin and infinity. Hence, one cannot obtain the multiplicity of periodic solutions under their conditions. If $0 < t_1 < t_2 < \cdots < t_p < T$, our problem is equivalent to the periodic boundary value problem

$$\begin{cases} u''(t) + (a-b)u'(t) + f(t,u(t)) = 0, & t \neq t_k, t \in [0,T], \\ \Delta u(t_k) = I_k(u(t_k)), \qquad \Delta u'(t_k) = J_k(u(t_k)), & k = 1,2,\dots,p, \\ u(0) = u(T), \qquad u'(0) = u'(T). \end{cases} \tag{1.2}$$

In recent paper [13], authors discussed the special case of (1.2)

$$\begin{cases} -u''(t) + \rho^2 u(t) = g(t,u(t)), & t \neq t_k, t \in [0,2\pi], \\ \Delta u(t_k) = I_k(u(t_k)), \qquad \Delta u'(t_k) = J_k(u(t_k)), & k = 1,2,\dots,p, \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi) \end{cases} \tag{1.3}$$

and obtained multiplicity of positive solutions of (1.3), where they made the best of the properties about the Green's function of

$$\begin{cases} -u''(t) + \rho^2 u(t) = 0, \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi). \end{cases}$$

However, their method is invalid for (1.1).

In this paper, by using a fixed point theorem in cone, we obtain two existence theorems of a single positive periodic solution for (1.1) under suitable behavior of functions f, I_k, J_k on some closed set. In addition, some information on the location of periodic solution is obtained, which can lead to the results on multiple periodic solutions.

2 Main results

Put $u = x$, $u' + au = y$ or $u = \phi$, $u' - bu = -\varphi$, then (1.1) can be written as

$$\begin{cases} x'(t) = -ax(t) + y(t), & t \neq t_k, \\ y'(t) = by(t) - F(t,x(t)), & t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), \\ \Delta y(t_k) = H_k(x(t_k)), & k \in \mathbb{Z}, \end{cases} \tag{2.1}$$

$$\begin{cases} \phi'(t) = b\phi(t) - \varphi(t), & t \neq t_k, \\ \varphi'(t) = -a\varphi(t) + F(t,\phi(t)), & t \neq t_k, \\ \Delta \phi(t_k) = I_k(\phi(t_k)), \\ \Delta \varphi(t_k) = D_k(\phi(t_k)), & k \in \mathbb{Z}, \end{cases} \tag{2.2}$$

where $F(t,u) = f(t,u) + abu$, $H_k(u) = J_k(u) + aI_k(u)$ and $D_k(u) = bI_k(u) - J_k(u)$.

If $u(t)$ is a T-periodic solution of (1.1), then $x, y, \phi, \varphi \in PC^1(\mathbb{R})$ are T-periodic functions, and (x,y), (ϕ,φ) satisfy (2.1), (2.2), respectively.

A function pair $(x,y) \in \{(u_1,u_2) \in PC^1(\mathbb{R},\mathbb{R}) \times PC^1(\mathbb{R},\mathbb{R})\}$ and x, y satisfy (2.1) (or (2.2)), we call $z = (x,y)$ a solution of (2.1) (or (2.2)). If $z = (x,y)$ is a solution of (2.1) (or (2.2)) and x, y are T-periodic, then x is a T-periodic solution of (1.1). If (x,y) and (ϕ,φ) are T-periodic solutions of (2.1) and (2.2), respectively, similar to Lemma 2.2 in [12], one can obtain that

x, y and ϕ, φ satisfy the integral equation system

$$\begin{aligned}
x(t) &= \int_t^{t+T} G_a(t,s)y(s)\,ds + \sum_{t\le t_k<t+T} G_a(t,t_k)I_k\bigl(x(t_k)\bigr),\\
y(t) &= \int_t^{t+T} G_b^*(t,s)F\bigl(s,x(s)\bigr)\,ds - \sum_{t\le t_k<t+T} G_b^*(t,t_k)H_k\bigl(x(t_k)\bigr),\\
\phi(t) &= \int_t^{t+T} G_b^*(t,s)\varphi(s)\,ds - \sum_{t\le t_k<t+T} G_b^*(t,t_k)I_k\bigl(\phi(t_k)\bigr),\\
\varphi(t) &= \int_t^{t+T} G_a(t,s)F\bigl(s,\phi(s)\bigr)\,ds + \sum_{t\le t_k<t+T} G_a(t,t_k)D_k\bigl(\phi(t_k)\bigr),
\end{aligned}$$

where

$$G_\lambda(t,s) = \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, \qquad G_\lambda^*(t,s) = \frac{e^{\lambda(t+T-s)}}{e^{\lambda T}-1}.$$

Lemma 2.1 [14] *Let X be a Banach space and P be a cone in X. Suppose that Ω_1 and Ω_2 are open subsets of X such that $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$ and suppose that*

$$\Phi : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$$

is a completely continuous operator such that

(i) $\inf \|\Phi u\| > 0$, $u \neq \mu\Phi u$ *for* $u \in P \cap \partial\Omega_1$ *and* $\mu \ge 1$, *and* $u \neq \mu\Phi u$ *for* $u \in P \cap \partial\Omega_2$ *and* $0 < \mu \le 1$, *or*

(ii) $\inf \|\Phi u\| > 0$, $u \neq \mu\Phi u$ *for* $u \in P \cap \partial\Omega_2$ *and* $\mu \ge 1$, *and* $u \neq \mu\Phi u$ *for* $u \in P \cap \partial\Omega_1$ *and* $0 < \mu \le 1$.

Then Φ has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Set $\delta = e^{-aT}$, $\sigma = e^{-bT}$ and

$$\begin{aligned}
S(v_1,v_2,\dots,v_p) &= -\sum_{k=1}^{p}\bigl[J_k(v_k) + (a-b)I_k(v_k)\bigr], \quad v_1,\dots,v_p \in \mathbb{R}, \gamma = \delta \text{ or } \sigma,\\
\varphi_\gamma(s) &= \int_0^T \sup_{u\in[\gamma s,s]} f(t,u)\,dt + \sup\bigl\{S(v_1,v_2,\dots,v_p) : v_k \in [\gamma s,s], 1\le k\le p\bigr\},\\
\psi_\gamma(s) &= \int_0^T \inf_{u\in[\gamma s,s]} f(t,u)\,dt + \inf\bigl\{S(u,v_1,v_2,\dots,v_p) : v_k \in [\gamma s,s], 1\le k\le p\bigr\}.
\end{aligned}$$

The following theorems are the main results of this paper.

Theorem 2.1 *Assume that* (P) *holds and there exist two positive constants $r < R$ such that*

$$\begin{aligned}
&F \in C\bigl(J \times [\delta r,R],[0,+\infty)\bigr),\\
&I_k \in C\bigl([\delta r,R],[0,+\infty)\bigr) \quad (1\le k\le p),\\
&H_k \in C\bigl([\delta r,R],(-\infty,0]\bigr) \quad (1\le k\le p),
\end{aligned}$$

where $F(t,u) = f(t,u) + abu$ and $H_k = J_k + aI_k$. Further suppose that $\varphi_\delta(r) < 0 < \psi_\delta(R)$ or $\varphi_\delta(R) < 0 < \psi_\delta(r)$, then (1.1) *has at least one positive T-periodic solution x with $r \le \|x\| \le R$ and $x(t) \ge \delta r$ for all $t \in \mathbb{R}$.*

Remark 2.1 The condition $\varphi_\delta(r) < 0 < \psi_\delta(R)$ (or $\varphi_\delta(R) < 0 < \psi_\delta(r)$) in Theorem 2.1 can be replaced by the condition easily verified

$$\Upsilon(r) < 0 < \gamma(R) \quad \bigl(\text{or } \Upsilon(R) < 0 < \gamma(r)\bigr),$$

where

$$\begin{aligned}\Upsilon(s) &= T \sup_{(t,u)\in[0,T]\times[\delta s,s]} f(t,u) + \sup\bigl\{S(v_1,v_2,\ldots,v_p) : v_k \in [\delta s,s], 1 \le k \le p\bigr\},\\ \gamma(s) &= T \inf_{(t,u)\in[0,T]\times[\delta s,s]} f(t,u) + \inf\bigl\{S(v_1,v_2,\ldots,v_p) : v_k \in [\delta s,s], 1 \le k \le p\bigr\}.\end{aligned}$$

Proof Here we only prove the case $\varphi_\delta(r) < 0 < \psi_\delta(R)$. Let

$$\tilde{f}(t,u) = \begin{cases} f(t,\delta r) & \text{if } u < \delta r,\\ f(t,u) & \text{if } \delta r \le u \le R,\\ f(t,R) & \text{if } u > R,\end{cases}$$

$$\tilde{I}_k(u) = \begin{cases} I_k(\delta r) & \text{if } u < \delta r,\\ I_k(u) & \text{if } \delta r \le u \le R,\\ I_k(R) & \text{if } u > R,\end{cases} \qquad \tilde{J}_k(u) = \begin{cases} J_k(\delta r) & \text{if } u < \delta r,\\ J_k(u) & \text{if } \delta r \le u \le R,\\ J_k(R) & \text{if } u > R,\end{cases}$$

$$\tilde{F}(t,u) = \tilde{f}(t,u) + abu, \qquad \tilde{H}_k(u) = \tilde{J}_k(u) + a\tilde{I}_k(u),$$

$$\tilde{S}(v_1,\ldots,v_p) = \sum_{k=1}^{p} \tilde{S}_k(v_k), \qquad \tilde{S}_k(v) = -\tilde{J}_k(v) - (a-b)\tilde{I}_k(v).$$

At first, we show that

$$\int_0^T \sup_{\delta s \le u \le s} \tilde{f}(t,u)\,dt + \sup\bigl\{\tilde{S}(v_1,\ldots,v_p) : \delta s \le v_k \le s\bigr\} < 0, \quad \forall s \in (0,r]. \tag{2.3}$$

We claim that

$$\sup_{\delta s \le u \le s} \tilde{f}(t,u) \le \sup_{\delta r \le u \le r} \tilde{f}(t,u) = \sup_{\delta r \le u \le r} f(t,u), \quad s \in (0,r]. \tag{2.4}$$

In fact, if $s \le \delta r$, $\sup_{\delta s \le u \le s} \tilde{f}(t,u) = \sup_{\delta s \le u \le s} f(t,\delta r) = f(t,\delta r) \le \sup_{\delta r \le u \le r} f(t,u)$. If $\delta r \le s \le r$, there exists $u_t \in [\delta s, s]$ such that $\tilde{f}(t,u_t) = \sup_{\delta s \le u \le s} \tilde{f}(t,u)$. We consider two subcases.

Subcase 1: $\delta r \le u_t \le s$. $\tilde{f}(t,u_t) \le \sup_{\delta r \le u \le r} \tilde{f}(t,u)$.

Subcase 2: $\delta s \le u_t \le \delta r$. $\tilde{f}(t,u_t) = f(t,\delta r) \le \sup_{\delta r \le u \le r} f(t,u)$.

Hence, (2.4) holds and

$$\int_0^T \sup_{\delta s \le u \le s} \tilde{f}(t,u)\,dt \le \int_0^T \sup_{\delta r \le u \le r} f(t,u)\,dt, \quad s \in (0,r].$$

Similar to (2.4), we obtain that

$$\sup_{\delta s\le v\le s}\tilde{S}_k(v)\le \sup_{\delta r\le v\le r}\tilde{S}_k(v),\quad \forall s\in(0,r],$$

$$\begin{aligned}\sup_{\delta s\le v_k\le s}\tilde{S}(v_1,\dots,v_p)&=\sup_{\delta s\le v_k\le s}\sum_{k=1}^{p}\tilde{S}_k(v_k)=\sum_{k=1}^{p}\sup_{\delta s\le v\le s}\tilde{S}_k(v)\\&\le\sum_{k=1}^{p}\sup_{\delta r\le v\le r}\tilde{S}_k(v)=\sup_{\delta r\le v_k\le r}\sum_{k=1}^{p}\tilde{S}_k(v_k)\\&=\sup_{\delta r\le v_k\le r}\tilde{S}(v_1,\dots,v_p)=\sup_{\delta r\le v_k\le r}S(v_1,\dots,v_p),\quad \forall s\in(0,r].\end{aligned}$$

Thus

$$\int_0^T\sup_{\delta s\le u\le s}\tilde{f}(t,u)\,dt+\sup\bigl\{\tilde{S}(v_1,\dots,v_p):\delta s\le v_k\le s\bigr\}\le\varphi_\delta(r)<0,\quad\forall s\in(0,r].$$

Next, we define operator and cone. Let $E=\{z=(x,y)\in PC(\mathbb{R})\times PC(\mathbb{R}),x(t+T)=x(t),y(t+T)=y(t)\}$, then E is a Banach space with the norm $\|z\|_E=\|x\|+\|y\|$, where $\|x\|=\max_{t\in[0,T]}|x(t)|$ and $\|y\|=\max_{t\in[0,T]}|y(t)|$.

Define the mapping $A:E\to E$ and the cone P in E by

$$A(x,y)=(X,Y),$$

where

$$\begin{aligned}X(t)&=\int_t^{t+T}G_a(t,s)y(s)\,ds+\sum_{t\le t_k<t+T}G_a(t,t_k)\tilde{I}_k\bigl(x(t_k)\bigr),\\Y(t)&=\int_t^{t+T}G_b^*(t,s)\tilde{F}\bigl(s,x(s)\bigr)\,ds-\sum_{t\le t_k<t+T}G_b^*(t,t_k)\tilde{H}_k\bigl(x(t_k)\bigr),\\P&=\bigl\{z=(x,y)\in E:x(t)\ge\delta\|x\|,y(t)\ge\sigma\|y\|,t\in\mathbb{R}\bigr\}.\end{aligned}$$

Put

$$\begin{aligned}\Omega_1&=\bigl\{z=(x,y)\in E:\|x\|<r,\|y\|<\varepsilon+a\sigma^{-1}r\bigr\},\\\Omega_2&=\bigl\{z\in E:\|x\|<R,\|y\|<\varepsilon+a\sigma^{-1}R\bigr\},\end{aligned}$$

where $0<\varepsilon<a\sigma^{-1}(R-r)$. At first, we show that $A:P\cap(\bar{\Omega}_2\setminus\Omega_1)\to P$. For any $(x,y)\in P$, it is easy to verify that $A(x,y)(t+T)=A(x,y)(t)$, that is, $X(t+T)=X(t)$ and $Y(t+T)=Y(t)$. We need to show that $A(x,y)(t)=(X(t),Y(t))\in P$ for $(x,y)\in P$, that is, $X(t)\ge\delta\|X\|$, $Y(t)\ge\sigma\|Y\|$ for any $t\in[0,T]$. Noting that

$$\begin{aligned}X(t)&\le\frac{e^{aT}}{e^{aT}-1}\int_t^{t+T}y(s)\,ds+\frac{e^{aT}}{e^{aT}-1}\sum_{t\le t_k<t+T}\tilde{I}_k\bigl(x(t_k)\bigr)\\&=\frac{e^{aT}}{e^{aT}-1}\left(\int_0^T y(s)\,ds+\sum_{k=1}^{p}\tilde{I}_k\bigl(x(t_k)\bigr)\right),\end{aligned}$$

$$X(t) \ge \frac{1}{e^{aT}-1}\left(\int_0^T y(s)\,ds + \sum_{k=1}^{p} \tilde{I}_k\bigl(x(t_k)\bigr)\right),$$

$$\begin{aligned} Y(t) &\le \frac{e^{bT}}{e^{bT}-1}\int_t^{t+T} \tilde{F}\bigl(s,x(s)\bigr)\,ds - \frac{e^{bT}}{e^{bT}-1}\sum_{t\le t_k<t+T} \tilde{H}_k\bigl(x(t_k)\bigr) \\ &= \frac{e^{bT}}{e^{bT}-1}\left(\int_0^T \tilde{F}\bigl(s,x(s)\bigr)\,ds - \sum_{k=1}^{p} \tilde{H}_k\bigl(x(t_k)\bigr)\right), \end{aligned}$$

$$Y(t) \ge \frac{1}{e^{bT}-1}\left(\int_0^T \tilde{F}\bigl(s,x(s)\bigr)\,ds - \sum_{k=1}^{p} \tilde{H}_k\bigl(x(t_k)\bigr)\right),$$

we obtain that $X(t) \ge \delta\|X\|$ and $Y(t) \ge \sigma\|Y\|$, and hence $A(P) \subset P$. In addition, one can easily check that $A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ is completely continuous.

Finally, we show that the condition (ii) of Lemma 2.1 is satisfied. We firstly show

$$z \ne \mu Az, \quad \forall z \in P \cap \partial\Omega_1, 0 < \mu \le 1.$$

If not, there exist $z_0 = (x_0, y_0) \in P \cap \partial\Omega_1$ and $0 < \mu_0 \le 1$ such that $z_0 = \mu_0 A z_0$. Then

$$\begin{cases} x_0'(t) = -ax_0(t) + \mu_0 y_0(t), & t \ne t_k, \\ y_0'(t) = by_0(t) - \mu_0\tilde{F}(t, x_0(t)), & t \ne t_k, \\ \Delta x_0(t_k) = \mu_0\tilde{I}_k(x_0(t_k)), & \\ \Delta y_0(t_k) = \mu_0\tilde{H}_k(x_0(t_k)), & k \in \mathbb{Z}. \end{cases} \tag{2.5}$$

By integration, we obtain that

$$a\int_0^T x_0(s)\,ds = \mu_0\int_0^T y_0(s)\,ds + \mu_0\sum_{k=1}^{p} \tilde{I}_k\bigl(x_0(t_k)\bigr), \tag{2.6}$$

$$b\int_0^T y_0(s)\,ds + \mu_0\sum_{k=1}^{p} \tilde{H}_k\bigl(x_0(t_k)\bigr) = \mu_0\int_0^T \tilde{F}\bigl(s,x_0(s)\bigr)\,ds. \tag{2.7}$$

From (2.6) and (2.7), we have

$$\begin{aligned} &\int_0^T \tilde{f}\bigl(s,x_0(s)\bigr)\,ds - \sum_{k=1}^{p}\bigl[\tilde{J}_k\bigl(x_0(t_k)\bigr) + (a-b)\tilde{I}_k\bigl(x_0(t_k)\bigr)\bigr] \\ &\quad = ab\bigl(\mu_0^{-1}-1\bigr)\int_0^T x_0(s)\,ds + b\bigl(\mu_0^{-1}-1\bigr)\int_0^T y_0(s)\,ds \ge 0. \end{aligned}$$

Since $z_0 = (x_0, y_0) \in P \cap \partial\Omega_1$, $\|x_0\| = r$ or $\|y_0\| = \varepsilon + a\sigma^{-1}r$. If $\|x_0\| = 0$, one easily gets that $y_0 = 0$, $t \ne t_k$. On the other hand, the fact that $\|y_0\| = \varepsilon + a\sigma^{-1}r$ and y_0 is left continuous implies that $y_0 \not\equiv 0$ for $t \ne t_k$, a contradiction. Thus $0 < \|x_0\| \le r$. Noting that $\int_0^T \tilde{f}(s, x_0(s))\,ds \le \int_0^T \sup_{\delta\|x_0\|\le u\le\|x_0\|}\tilde{f}(s,u)\,ds$, we obtain that

$$\int_0^T \sup_{\delta\|x_0\|\le u\le\|x_0\|} \tilde{f}(s,u)\,ds - \sum_{k=1}^{p}\bigl[\tilde{J}_k\bigl(x_0(t_k)\bigr) + (a-b)\tilde{I}_k\bigl(x_0(t_k)\bigr)\bigr] \ge 0,$$

which is in contradiction with (2.3). Hence,

$$z \neq \mu Az, \quad z \in P \cap \partial\Omega_1,\ 0 < \mu \leq 1.$$

We show that for $z \in P \cap \partial\Omega_2$ and $\mu \geq 1$,

$$\inf \|Az\|_E > 0 \quad \text{and} \quad z \neq \mu Az.$$

Assume that $\inf_{z\in P\cap\partial\Omega_2} \|Az\|_E = 0$. There exists the sequence $z_n = (x_n, y_n) \in P \cap \partial\Omega_2$ such that $\|A(x_n, y_n) = (X_n, Y_n)\|_E \to 0$ as $n \to \infty$. Hence, $x_n \geq \delta\|x_n\|$, $y_n \geq \sigma\|y_n\|$, $\|x_n\| = R$ or $\|y_n\| = \varepsilon + a\sigma^{-1}R$ and

$$\begin{aligned}
&\frac{1}{e^{aT}-1}\left(\int_0^T y_n(s)\,ds + \sum_{k=1}^{p} \tilde{I}_k\bigl(x_n(t_k)\bigr)\right) \leq \bigl\|X_n(t)\bigr\| \to 0, \\
&\frac{1}{e^{bT}-1}\left(\int_0^T \tilde{F}\bigl(s, x_n(s)\bigr)\,ds - \sum_{k=1}^{p} \tilde{H}_k\bigl(x_n(t_k)\bigr)\right) \leq \bigl\|Y_n(t)\bigr\| \to 0
\end{aligned}$$

as $n \to \infty$, which imply that

$$\begin{aligned}
&0 \leq \sigma T\|y_n\| \leq \int_0^T y_n(t)\,dt \to 0, \qquad \tilde{I}_k\bigl(x_n(t_k)\bigr) \to 0 \quad (1 \leq k \leq p), \\
&\int_0^T \tilde{F}\bigl(s, x_n(s)\bigr)\,ds - \sum_{k=1}^{p} \tilde{H}_k\bigl(x_n(t_k)\bigr) \to 0
\end{aligned}$$

as $n \to \infty$. Hence $\|x_n\| = R$ and $x_n(t) \geq \delta\|x_n\| = \delta R$. We have

$$\begin{aligned}
0 < \psi_\delta(R) &\leq \int_0^T \tilde{f}\bigl(t, x_n(t)\bigr)\,dt - \sum_{k=1}^{p}\bigl[\tilde{J}_k\bigl(x_n(t_k)\bigr) + (a-b)\tilde{I}_k\bigl(x_n(t_k)\bigr)\bigr] \\
&\leq \int_0^T \tilde{F}\bigl(s, x_n(s)\bigr)\,ds - \sum_{k=1}^{p} \tilde{H}_k\bigl(x_n(t_k)\bigr) + b\sum_{k=1}^{p} \tilde{I}_k\bigl(x_n(t_k)\bigr) \to 0, \quad n \to \infty,
\end{aligned}$$

which is a contradiction.

Suppose that there exist $z_1 = (x_1, y_1) \in P \cap \partial\Omega_2$ and $\mu_1 \geq 1$ such that $z_1 = \mu_1 Az_1$. Then

$$\begin{cases}
x_1'(t) = -ax_1(t) + \mu_1 y_1(t), \quad t \neq t_k, \\
y_1'(t) = ay_1(t) - \mu_1\tilde{F}(t, x_1(t)), \quad t \neq t_k, \\
\Delta x_1(t_k) = \mu_1\tilde{I}_k(x_1(t_k)), \\
\Delta y_1(t_k) = \mu_1\tilde{H}_k(x_1(t_k)), \quad k \in \mathbb{Z}.
\end{cases} \tag{2.8}$$

Similar to (2.6) and (2.7), we have

$$a\int_0^T x_1(s)\,ds = \mu_1\int_0^T y_1(s)\,ds + \mu_1\sum_{k=1}^{p} \tilde{I}_k\bigl(x_1(t_k)\bigr), \tag{2.9}$$

$$b\int_0^T y_1(s)\,ds + \mu_1\sum_{k=1}^{p} \tilde{H}_k\bigl(x_1(t_k)\bigr) = \mu_1\int_0^T \tilde{F}\bigl(s, x_1(s)\bigr)\,ds, \tag{2.10}$$

$$\begin{aligned} &ab(1-\mu_1)\int_0^T x_1(s)\,ds + b(1-\mu_1)\int_0^T y_1(s)\,ds \\ &\quad = \mu_1\left(\int_0^T \tilde{f}\big(s,x_1(s)\big)\,ds - \sum_{k=1}^{p}\big[\tilde{J}_k\big(x_1(t_k)\big) + (a-b)\tilde{I}_k\big(x_1(t_k)\big)\big]\right). \end{aligned} \tag{2.11}$$

We consider two cases.

Case 1 If $\|x_1\| = R$, then $x_1 \geq \delta\|x_1\| \geq \delta R$. Noting that $x_1 \geq 0$, $y_1 \geq 0$ for all $t \in J$ and $\mu_1 \geq 1$, we obtain that

$$\int_0^T \tilde{f}\big(s,x_1(s)\big)\,ds - \sum_{k=1}^{p}\big[\tilde{J}_k\big(x_1(t_k)\big) + (a-b)\tilde{I}_k\big(x_1(t_k)\big)\big] \leq 0,$$

which is in contradiction with the fact $\psi_\delta(R) > 0$.

Case 2 If $\|x_1\| < R$, then $\|y_1\| = \varepsilon + a\sigma^{-1}R$. From (2.9) and $\tilde{I}_k \geq 0$, we have

$$aTR \geq aT\|x_1\| \geq a\int_0^T x_1(s)\,ds \geq \mu_1\int_0^T y_1(s)\,ds \geq T\sigma\|y_1\|,$$

which implies that $\|y_1\| \leq a\sigma^{-1}R$, a contradiction.

The condition (ii) of Lemma 2.1 is fulfilled and it follows that A has at least one fixed point $z = (x,y) \in P \cap (\bar{\Omega}_2\backslash\Omega_1)$. Clearly, x, y satisfy

$$\begin{cases} x'(t) = -ax(t) + y(t), & t \neq t_k, \\ y'(t) = ay(t) - \tilde{F}(t,x(t)), & t \neq t_k, \\ \Delta x(t_k) = \tilde{I}_k(x(t_k)), \\ \Delta y(t_k) = \tilde{H}_k(x(t_k)), & k \in \mathbb{Z}. \end{cases} \tag{2.12}$$

Suppose that $\|x\| < r$. By integrating the first equation of (2.12), we obtain that

$$aTr \geq aT\|x\| \geq a\int_0^T x(s)\,ds \geq \int_0^T y(s)\,ds \geq T\sigma\|y\|,$$

which implies that $\|y\| \leq a\sigma^{-1}r$, a contradiction to $(x,y) \in P\cap(\bar{\Omega}_2\backslash\Omega_1)$. Hence, $r \leq \|x\| \leq R$, $\tilde{f}(t,x(t)) = f(t,x(t))$, $\tilde{I}_k(x(t_k)) = I_k(x(t_k))$, $\tilde{J}_k(x(t_k)) = J_k(x(t_k))$, and

$$\begin{cases} x'(t) = -ax(t) + y(t), & t \neq t_k, \\ y'(t) = ay(t) - F(t,x(t)), & t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), \\ \Delta y(t_k) = H_k(x(t_k)), & k \in \mathbb{Z}. \end{cases} \tag{2.13}$$

It is easy to check that x is one positive T-periodic solution of (1.1). The proof is complete. □

We introduce the following assumptions:

$$\lim_{x\to 0^+}\frac{J_k(x) + (a-b)I_k(x)}{x^m} = \alpha_k(m), \qquad \lim_{x\to +\infty}\frac{J_k(x) + (a-b)I_k(x)}{x^m} = \beta_k(m),$$

$$\liminf_{x\to 0^+}\min_{t\in[0,T]}\frac{f(t,x)}{x^m} = h(m), \qquad \limsup_{x\to 0^+}\max_{t\in[0,T]}\frac{f(t,x)}{x^m} = \bar{h}(m),$$

$$\liminf_{x\to+\infty}\min_{t\in[0,T]}\frac{f(t,x)}{x^m}=g(m),\qquad \limsup_{x\to+\infty}\max_{t\in[0,T]}\frac{f(t,x)}{x^m}=\bar{g}(m).$$

(C_1) There exist $n, m_k \in [0,\infty)$ $(1\le k\le p)$ such that $-\infty<\alpha_k(m_k)\le 0$, $-\infty<\bar{h}(n)<0$, $n<\min\{m_1,\dots,m_p\}$.
(C_2) There exist $0\le k_0\le p$, $l_1,l_2: 0\le l_1<l_2$ such that $-\infty<\beta_{k_0}(l_2)<0$, $g(l_1)>-\infty$.
(C_3) There exists a constant $m\ge 0$ such that $g(m)>0$.
(C_4) There exist $n, m_k\in[0,\infty)$ $(1\le k\le p)$ such that $-\infty<\beta_k(m_k)\le 0$, $-\infty<\bar{g}(n)<0$, $n>\max\{m_1,\dots,m_p\}$.
(C_5) There exist $0\le k_0\le p$, $l_1,l_2: 0\le l_1<l_2$ such that $-\infty<\alpha_{k_0}(l_1)<0$, $h(l_2)>-\infty$.
(C_6) There exists a constant $m\ge 0$ such that $h(m)>0$.

Corollary 2.1 *Assume that* $F\in C(J\times[0,+\infty),[0,+\infty))$, $I_k\in C([0,+\infty),[0,+\infty))$ $(1\le k\le p)$, $H_k\in C([0,+\infty),(-\infty,0])$ $(1\le k\le p)$. *Then* (1.1) *has at least one positive* T*-periodic solution if one of the following conditions is satisfied*:
(1) (C_1) *and* (C_2);
(2) (C_1) *and* (C_3);
(3) (C_4) *and* (C_5);
(4) (C_4) *and* (C_6).

Proof We only consider case (1). By (C_1) and (C_2), there exist $R_1>1>r_1>0$ such that

$$\begin{aligned}
&J_k(x)+(a-b)I_k(x)\ge\bigl(\alpha_k(m_k)+\bar{h}(n)\bigr)x^{m_k},\quad \forall 0<x\le r_1,\\
&f(t,x)\le\frac{1}{2}\bar{h}(n)x^n,\quad \forall t\in R, 0<x\le r_1,\\
&J_{k_0}(x)+(a-b)I_{k_0}(x)\le\frac{1}{2}\beta_{k_0}(l_2)x^{l_2},\quad x\ge R_1,\\
&f(t,x)\ge\frac{1}{2}g(l_1)x^{l_1},\quad \forall t\in R, x\ge R_1.
\end{aligned}$$

Choosing

$$\begin{aligned}
&r=\frac{1}{2}\min\biggl\{r_1,\biggl(\frac{-T\bar{h}(n)\delta^n}{2\sum_{k=1}^p|\alpha_k(m_k)+\bar{h}(n)|}\biggr)^{\frac{1}{\min\{m_1,m_2,\dots m_p\}-n}}\biggr\},\\
&R=2\max\biggl\{R_1/\delta,\biggl(\frac{|g(l_1)|T}{-\beta_{k_0}(l_2)\delta^{l_2}}\biggr)^{\frac{1}{l_2-l_1}}\biggr\},
\end{aligned}$$

we have

$$\begin{aligned}
&\int_0^T\sup_{\delta r\le u\le r}f(t,u)\,dt-\sum_{k=1}^p\bigl[J_k(v_k)+(a-b)I_k(v_k)\bigr]\\
&\quad\le\frac{1}{2}\bar{h}(n)T(\delta r)^n+\sum_{k=1}^p\bigl|\alpha_k(m_k)+\bar{h}(n)\bigr|r^{m_k}\\
&\quad\le\frac{1}{2}\bar{h}(n)T(\delta r)^n+r^m\sum_{k=1}^p\bigl|\alpha_k(m_k)+\bar{h}(n)\bigr|<0,
\end{aligned}$$

where $m = \min\{m_1, m_2, \ldots, m_p\}$ and $v_k \in [\delta r, r]$, and

$$\int_0^T \inf_{\delta R \le u \le R} f(t,u)\,dt - \sum_{k=1}^{p} \bigl[J_k(v_k) + (a-b)I_k(v_k)\bigr] \ge -\frac{1}{2}\bigl|g(l_1)\bigr|TR^{l_1} + \frac{1}{2}\bigl|\beta_{k_0}(l_2)\bigr|(\delta R)^{l_2} > 0$$

for $v_k \in [\delta R, R]$. By Theorem 2.1, (1.1) has at least one T-periodic solution x with $r \le \|x\| \le R$. □

Remark 2.2 From Corollary 2.1, we easily obtain Theorem 3.1 and Theorem 3.2 in [12]. Moreover, the conditions of cases (1) and (3) are weaker than those of Theorem 3.1 and Theorem 3.2 in [12].

Define the mapping $B : E \to E$ by

$$B(x,y) = (\hat{X}, \hat{Y}),$$

where

$$\hat{X}(t) = \int_t^{t+T} G_b^*(t,s)y(s)\,ds - \sum_{t \le t_k < t+T} G_b^*(t,t_k)\tilde{I}_k\bigl(x(t_k)\bigr),$$

$$\hat{Y}(t) = \int_t^{t+T} G_a(t,s)\tilde{F}\bigl(s,x(s)\bigr)\,ds + \sum_{t \le t_k < t+T} G_a(t,t_k)\bigl(b\tilde{I}_k\bigl(x(t_k)\bigr) - \tilde{J}_k\bigl(x(t_k)\bigr)\bigr).$$

Similar to Theorem 2.1, we have the following result.

Theorem 2.2 *Assume that* (P) *holds and there exist two positive constants $r < R$ such that*

$$F \in C\bigl(J \times [\sigma r, R], [0, +\infty)\bigr),$$
$$I_k \in C\bigl([\sigma r, R], (-\infty, 0]\bigr) \quad (1 \le k \le p),$$
$$D_k \in C\bigl([\sigma r, R], [0, +\infty)\bigr) \quad (1 \le k \le p),$$

where $F(t,u) = f(t,u) + abu$ and $D_k = bI_k - J_k$. Further suppose that $\varphi_\sigma(r) < 0 < \psi_\sigma(R)$ or $\varphi_\sigma(R) < 0 < \psi_\sigma(r)$ is satisfied, then (1.1) *has at least one positive T-periodic solution x with $r \le \|x\| \le R$ and $x(t) \ge \sigma r$ for all $t \in \mathbb{R}$.*

3 Application

In this section, some examples are provided to highlight our results obtained in previous section and the results in [12, 13] cannot been applied.

Example 3.1 Consider the differential equation

$$\begin{cases} u''(t) + Cu'(t) + f(u(t)) = 0, & t \ne t_k, \\ \Delta u'(t_k) = -J_k(u(t_k)), & k \in \mathbb{Z}, \end{cases} \tag{3.1}$$

where $C \in \mathbb{R}$, and there exists $p \in \mathbb{N}$ such that $J_{k+p}(u) = J_k(u)$, $t_{k+p} = t_k + T$. Moreover, the following condition holds:

(PP) there exist constants $\alpha > 0$, $0 < \xi < 1$, $0 < r < R$ such that

$$f \in C\bigl([\xi r, R], \mathbb{R}\bigr), \qquad J_k \in C\bigl([\xi r, R], [0, +\infty)\bigr) \quad (1 \le k \le p), \tag{3.2}$$

$$f(u) + \alpha u \ge 0 \quad \text{for } u \in [\xi r, R], \tag{3.3}$$

$$\left(f(r)T + \sum_{k=1}^{p} J_k(r)\right)\left(f(R)T + \sum_{k=1}^{p} J_k(R)\right) < 0. \tag{3.4}$$

Proposition 3.1 *Problem* (3.1) *has at least one positive T-periodic solution $x : r \le \|x\| \le R$ for $|C|$ sufficiently large.*

Proof Without loss of generality, we suppose that

$$Tf(r) + \sum_{k=1}^{p} J_k(r) < 0, \qquad Tf(R) + \sum_{k=1}^{p} J_k(R) > 0.$$

Since $f(u)$, $J_k(u)$ are continuous in $u \in [\xi r, R]$, there exists $\xi < \tau < 1$ such that

$$T \sup_{\tau r \le u \le r} f(u) + \sum_{k=1}^{p} J_k(v_k) < 0, \quad \forall \tau r \le v_1, \dots, v_p \le r,$$

$$T \inf_{\tau R \le u \le R} f(u) + \sum_{k=1}^{p} J_k(v_k) > 0, \quad \forall \tau R \le v_1, \dots, v_p \le R.$$

Choose $b = \alpha C^{-1}$, $a = b + C$ if $C > 0$, $a = \alpha |C|^{-1}$, $b = a - C$ if $C < 0$, then $a > 0$, $b > 0$ and

$$f(u) + abu \ge f(u) + \alpha u \ge 0 \quad \text{for } u \in [\tau r, R].$$

Note that

$$\tau < e^{-bT} < 1 \quad \text{for } C > \frac{\alpha T}{-\ln \tau}, \qquad \tau < e^{-aT} < 1 \quad \text{for } C < \frac{\alpha T}{\ln \tau},$$

$$\int_0^T \sup_{\nu r \le u \le r} f(u)\,dt + \sum_{k=1}^{p} J_k(v_k) = T \sup_{\nu r \le u \le r} f(u) + \sum_{k=1}^{p} J_k(v_k) < 0, \quad \forall \nu r \le v_1, \dots, v_p \le r,$$

$$\int_0^T \inf_{\nu R \le u \le R} f(t,u)\,dt + \sum_{k=1}^{p} J_k(v_k) = T \inf_{\nu r \le u \le r} f(u) + \sum_{k=1}^{p} J_k(v_k) > 0, \quad \forall \nu R \le v_1, \dots, v_p \le R,$$

where $\nu = e^{-aT}$ if $C < 0$, $\nu = e^{-bT}$ if $C > 0$. By Theorem 2.1 ($C < 0$) or Theorem 2.2 ($C > 0$), we obtain that (3.1) has at least one positive T-periodic solution $x : r \le \|x\| \le R$.

Consider the differential equation

$$\begin{cases} u''(t) + Cu'(t) + u(t) \sin u(t) = 0, & t \ne t_k, \\ \Delta u'(t_k) = -\sqrt[3]{u(t_k)}, & k \in \mathbb{Z}, \end{cases} \tag{3.5}$$

where $C \in \mathbb{R}$, $t_k < t_{k+1}$ and there exists $p \in \mathbb{N}$ such that $t_{k+p} = t_k + T$.

By Proposition 3.1, one can obtain that (3.5) has at least one positive T-periodic solution $x : 2m\pi - 0.5\pi \le \|x\| \le 2m\pi + 0.5\pi$ with sufficiently large $m \in \mathbb{N}$ if $|C|$ is sufficiently large. Thus

$$\lim_{C\to+\infty} \lambda^\sharp = +\infty,$$

where $\lambda^\sharp$ is the number of positive periodic solutions of (3.5).

Now, consider the special case of (3.5)

$$\begin{cases} u''(t) + 900u'(t) + u\sin u = 0, & t \ne k, \\ \Delta u'(k) = -\sqrt[3]{u(k)}, & k \in \mathbb{Z}. \end{cases} \tag{3.6}$$

Let $b = (-900 + \sqrt{900^2 + 4})/2$, $a = 900 + b$, $T = 1$, $\sigma = e^{-b}$, $f(u) = u\sin u$, $r_i^1 = (2i - 0.5)\pi$, $r_i^2 = (2i + 0.5)\pi$, $r_i^3 = (2i + 1.5)\pi$, $i = 1, 2, \dots, q = [\frac{1}{8b} - \frac{5}{8}]$, where $[d]$ denotes the integer part of d. It is easy to check that

$$f(u) + abu \ge 0, \quad \forall u \ge 0, \qquad (1-\sigma)r_i^j \le \pi/4, \quad j = 1,2,3, 1 \le i \le q,$$

$$f \text{ is nonincreasing in } \bigl[\sigma r_i^j, r_i^j\bigr], \quad j = 1,3, \text{ and nondecreasing in } \bigl[\sigma r_i^2, r_i^2\bigr].$$

Hence,

$$\begin{aligned} T\sup u\sin u + \sqrt[3]{v} &\le \sigma r_i^j \sin\bigl(\delta r_i^j\bigr) + \sqrt[3]{r_i^j} \le \sigma r_i^j \sin\left(r_i^j - \frac{\pi}{4}\right) + \sqrt[3]{r_i^j} \\ &\le -\frac{\sigma\sqrt{2}}{2} r_i^j + \sqrt[3]{r_i^j} < 0, \quad \sigma r_i^j \le u, v \le r_i^j, j = 1,3, \\ T\inf u\sin u + \sqrt[3]{v} &\ge \sigma r_i^j \sin\bigl(\sigma r_i^j\bigr) \ge \sigma r_i^j \sin\left(r_i^j - \frac{\pi}{4}\right) \ge \frac{\sigma\sqrt{2}}{2} r_i^j > 0, \quad \sigma r_i^2 \le u, v \le r_i^2. \end{aligned}$$

Using Theorem 2.2, we obtain that (3.6) has at least $2q$ positive 1-periodic solutions x_i, y_i, $1 \le i \le q$ with $r_i^1 \le \|x_i\| \le r_i^2 \le \|y_i\|$, $1 \le i \le q$. □

Example 3.2 Consider the differential equation

$$\begin{cases} u''(t) - u'(t) - u = 0, & t \ne t_k, \\ \Delta u(t_k) = u^2(t_k), \qquad \Delta u'(t_k) = -u^2(t_k), & k \in \mathbb{Z}, \end{cases} \tag{3.7}$$

where $t_k < t_{k+1}$, and there exists $p \in \mathbb{N}$ such that $t_{k+p} = t_k + T$.

Taking $m_1 = \cdots = m_p = 2$, $n = 1$, $k_0 = 1$, $l_1 = 1$, $l_2 = 2$ and $a = 0.7$, $b = 1.7$, one has

$$\alpha_k = -2, \qquad \bar{h}(1) = 1, \qquad \beta_{k_0} = -2, \qquad g(l_1) = -1,$$

where $\alpha_k, \bar{h}, \beta_k, g$ are defined in Corollary 2.1. The conditions (C_1) and (C_2) of Corollary 2.1 hold, and hence (3.7) has at least one positive T-periodic solution.

If $t_k = k$, (3.7) has a unique positive 1-periodic solution

$$u(t) = c_1 \exp\lambda_1(t-k) + c_2 \exp\lambda_2(t-k), \quad k-1 < t \le k,$$

where

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}, \qquad A = \frac{(1+\lambda_2)(e^{-\lambda_2}-1)}{(1+\lambda_1)(1-e^{-\lambda_1})},$$
$$c_2 = \frac{\lambda_1-\lambda_2}{(1+A)^2(1+\lambda_1)}\left(e^{-\lambda_2}-1\right), \qquad c_1 = Ac_2.$$

Example 3.3 Consider the differential equation

$$\begin{cases} u''(t) + \frac{1}{100}(1-u)u = 0, & t \neq t_k, \\ \Delta u(t_k) = \lambda u(t_k), \qquad \Delta u'(t_k) = -\lambda u(t_k), & k \in \mathbb{Z}, \end{cases} \tag{3.8}$$

where $\lambda > 0$ and $t_{k+4} = t_k + 1$.

We claim that (3.8) has one 1-periodic solution for $\lambda \in (0, 0.03)$. In fact, $f(u) = 0.01(1-u)u$, $I_k(u) = \lambda u$, $J_k(u) = -\lambda u$, $p = 4$. Taking $a = b = 1$, $r = 0.1$, $R = 100$, then

$$f(u) + u \geq 0, \quad u \in (0, 101],$$
$$f(u) - \sum_{k=1}^{4} J_k(v_k) > 0 \quad \text{for } \frac{r}{e} \leq u, v_1, \ldots, v_4 \leq r,$$
$$f(u) - \sum_{k=1}^{4} J_k(v_k) < \left(1-\frac{R}{e}\right)\frac{1}{e} + 400\lambda < \left(1-\frac{R}{e}\right)\frac{1}{e} + 12 < 0 \quad \text{for } \frac{R}{e} \leq u, v_1, \ldots, v_4 \leq R.$$

By Theorem 2.1, (3.8) has at least one positive 1-periodic solution.

Since

$$\lim_{u\to+\infty} \frac{f(u)}{u} = -\infty,$$

there is no constant $\rho > 0$ such that $f(u) + \rho^2 u \geq 0$ for all $u \geq 0$. Hence, the fundamental condition in [12] is not satisfied.

Example 3.4 Consider the differential equation

$$\begin{cases} u''(t) + Cu'(t) - u^{\alpha}(t) + \frac{1}{u^{\beta}(t)} = 0, & t \neq t_k, \\ \Delta u(t_k) = -\lambda u(t_k), \qquad \Delta u'(t_k) = -\lambda(u^2(t_k) + u(t_k)), \end{cases} \tag{3.9}$$

where $C > 1$, $0 < \alpha < 1$, $\beta > 0$, $t_k < t_{k+1}$, λ is a positive real parameter, and there exists $p \in \mathbb{N}$ such that $t_{k+p} = t_k + T$, $1 \leq T \leq C$.

We claim that (3.9) has at least two T-periodic solutions if $\lambda > 0$ is sufficiently small. In fact, $f(u) = u^{-\beta} - u^{\alpha}$, $I_k(u) = -\lambda u$, $J_k(u) = -\lambda(u^2 + u)$. Setting

$$b = \frac{1}{T}, \qquad a = b + C, \qquad r_1 = 10^{-4}, \qquad r_2 = 100^{\frac{1}{\alpha}}, \qquad r_3 = r_2 e/\lambda,$$

we have

$$f(u) + abu = u^{-\beta} - u^{\alpha} + abu \geq u^{-\beta} - u^{\alpha} + u > 0 \quad \text{for } u > 0,$$

$$D_k(u) = \lambda(1-b)u + \lambda u^2 \geq 0 \quad \text{for } u > 0.$$

Taking

$$0 < \lambda < \min\left\{\left(\frac{r_2^{2-\alpha}}{2Te^\alpha}\right)^{\frac{1}{1-\alpha}}, \frac{T}{2(C+1)pr_2^2}, \frac{1}{2}\right\},$$

we have

$$Tf(u) - \sum_{k=1}^{p}\bigl[J_k(v_k) + (a-b)I_k(v_k)\bigr] > 0 \quad \text{for } \frac{r_1}{e} \leq u, v_1, v_2, \ldots, v_p \leq r_1,$$

$$\begin{aligned} Tf(u) &- \sum_{k=1}^{p}\bigl[J_k(v_k) + (a-b)I_k(v_k)\bigr] \\ &\leq T\bigl(u^{-\beta} - 100/e^\alpha\bigr) + \lambda p\bigl[r_2^2 + (C+1)r_2\bigr] \\ &\leq T\bigl(1 - 100/e^\alpha\bigr) + \lambda p(C+1)\bigl(r_2^2 + r_2\bigr) \\ &\leq -T + 2(C+1)p\lambda r_2^2 < 0 \quad \text{for } \frac{r_2}{e} \leq u, v_1, v_2, \ldots, v_p \leq r_2, \end{aligned}$$

$$\begin{aligned} Tf(u) &- \sum_{k=1}^{p}\bigl[J_k(v_k) + (a-b)I_k(v_k)\bigr] \\ &> -T\bigl(1 + (r_3)^\alpha\bigr) + \lambda\frac{r_3^2}{e^2} \\ &\geq -2T(r_3)^\alpha + \frac{r_2^2}{\lambda} = -2T\frac{(r_2e)^\alpha}{\lambda^\alpha} + \frac{r_2^2}{\lambda} > 0 \quad \text{for } \frac{r_3}{e} \leq u, v_1, v_2, \ldots, v_p \leq r_3. \end{aligned}$$

Hence, (3.8) has at least two T-periodic solutions x_1, x_2 with $r_1 \leq \|x_1\| \leq r_2 \leq \|x_2\| \leq r_3$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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