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Inverse problem for non-stationary system of magnetohydrodynamics

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Abstract

We study the inverse problem for non-stationary system of magnetic hydrodynamics in which it is required to determine the velocity of the fluid $\vec{v}(x,t)$, the magnetic tension $\vec{H}(x,t)$, the pressure gradient $\nabla p(x,t)$, but also the external forces $\vec{f}(x)$ and the current $\operatorname{rot}\vec{j}(x)$. In this case, to the conditions constituting the direct problem are added additional conditions. The trace speed, the magnetic tension, and the pressure gradient in the final moment, time $t = T$, are taken as additional information. The strong generalized solvability of the inverse problem in the two-dimensional case is proved.

Keywords: magnetohydrodynamics; inverse problem; final overdetermination condition

1 Introduction

The mathematical description of the processes occurring in moving fluids leads to the solution of the Navier-Stokes equations. References [1–8] are devoted to the study of the questions of solvability and stability of solutions of initial-boundary value problems for the linearized and general nonlinear Navier-Stokes equations.

Magnetohydrodynamics (MHD) is a theory of macroscopic interaction of electrically conductive fluid and electromagnetic fields. It has important applications in astronomy and geophysics, as well as in engineering fields such as controlled thermonuclear fusion, nuclear reactor cooling liquid metals, electromagnetic casting of metals, MHD generators, and MHD ion engines.

In the 1960s and 1970s the efforts of mathematicians were directed to the study of a class of problems of magnetohydrodynamics. Fundamental work in this direction was done by Ladyzhenskaya and Solonnikov [9, 10].

In this paper we study the three initial-boundary value problems for non-stationary magnetohydrodynamic equations. Results on the solvability of these problems are similar to the corresponding results on the solvability of the initial-boundary value problems for non-stationary Navier-Stokes equations. Similar results were obtained by Mosconi and Solonnikov in [11] for stationary MHD equations. A variety of approaches to the mathematical study of MHD systems are reflected in the work of Ladyzhenskaya and Solonnikov [12], Sahaev and Solonnikov [13], Stupyalis [14, 15], Alekseev and Tereshko [16], Duvant and Lions [17], Sermange and Temam [18], Giga and Yoshida [19], and Dyer and Edmuns [20].

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2 Statement of the problem

We consider the inverse problem of magnetic hydrodynamic in the cylinder $Q_T = \Omega \times [0,T]$, $\Omega \subset R^2$. One can take the border of the area Ω from C^2, namely $\partial\Omega \subset C^2$, $\Gamma = \partial\Omega \times [0,T]$. We need to determine $\vec{v}(x,t)$, $\vec{H}(x,t)$, $\nabla p(x,t)$, $\vec{f}(x)$, and $\operatorname{rot}\vec{j}(x)$ that satisfy the following equations:

$$\frac{\partial \vec{v}}{\partial t} + \sum_{k=1}^{2} v_k \vec{v}_{x_k} - \frac{\mu}{\rho}\sum_{k=1}^{2} H_k \vec{H}_{x_k} - \nu\Delta\vec{v} = -\frac{1}{\rho}\operatorname{grad}\left(p + \frac{\mu\vec{H}^2}{2}\right) + g(x,t)\vec{f}(x), \tag{1}$$

$$\frac{\partial \vec{H}}{\partial t} + \frac{1}{\sigma\mu}\operatorname{rot}\operatorname{rot}\vec{H} - \operatorname{rot}[\vec{v}\times\vec{H}] = \frac{\xi(x,t)}{\sigma\mu}\operatorname{rot}\vec{j}(x), \tag{2}$$

$$\operatorname{div}\vec{v} = 0, \qquad \operatorname{div}(\mu\vec{H}) = 0. \tag{3}$$

We have the initial conditions

$$\vec{v}(x,0) = \vec{v}_0(x), \qquad \vec{H}(x,0) = \vec{H}_0(x), \tag{4}$$

the boundary conditions

$$\vec{v}|_\Gamma = 0, \qquad H_n|_\Gamma = 0, \qquad \left.\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2}\right|_\Gamma = j|_\Gamma = 0, \tag{5}$$

and the overdetermination conditions

$$\vec{v}(x,T) = \vec{U}(x), \qquad \vec{H}(x,T) = \vec{\Psi}(x), \qquad \nabla p(x,T) = \nabla\pi(x). \tag{6}$$

Here $\vec{v}(x,t)$ is the velocity of the fluid, $\vec{H}(x,t)$ the magnetic tension, p the pressure, $g(x,t)\vec{f}(x)$ the external hydrodynamic forces, $\xi(x,t)\operatorname{rot}\vec{j}(x)$ the current, μ the magnetic permeability, σ the conductivity, ρ the condensation, ν the kinematic viscosity coefficient of the fluid, $\vec{n}$ the outward pointing normal vector to the surface area S, and $H_n = \vec{H}\cdot\vec{n}$.

The first results of the well-posedness of the inverse problem with homogeneous boundary conditions for the Navier-Stokes equations appeared in the works of Prilepko and Vasin [21] and Abylkairov [22]. Reference [23] studied the solvability of the inverse spatial problem with unknown right part where we obtain a local existence theorem for the solutions. Different inverse problems for the Navier-Stokes equations and hydrodynamics were presented in [24–35]. In the work of Abylkairov [30] the inverse problem for the Navier-Stokes equations was studied with non-standard boundary conditions. The controllability of the systems of magnetic hydrodynamics has been studied in many papers (see, for example, [16, 31–35]).

We denote by $\overset{0}{J}_1(\Omega)$ the subspace $\overset{0}{W}{}^1_2(\Omega)$, consisting of solenoidal vectors and by $\hat{J}(\Omega)$ the subspace $W_2^2(\Omega)$, consisting of solenoidal vectors, satisfying the following $\Gamma = \partial\Omega\times[0,T]$ conditions: $u_n|_\Gamma = 0$, $y\operatorname{rot}\vec{u}|_\Gamma = 0$. Let $V_2(Q_T)$ be the Banach space of functions with the norm $\|\vec{u}\|_{V_2(Q_T)} = \operatorname{vrai}\max_{0\le t\le T}\|\vec{u}(x,t)\|_{2,\Omega} + \|\vec{u}_x\|_{2,Q_T}$ that is obtained as a result of the closure of set of smooth, solenoidal, and zero vectors near $\Gamma = \partial\Omega\times[0,T]$. $\overset{0}{J}_{l,n}(\Omega)$ is a subset of $W_2^l(\Omega)$ (l is an integer number), for which the closure in the norm of $W_2^l(\Omega)$ of the set of continuously differentiable solenoidal vector-functions with $u_n|_\Gamma = 0$. $G(\Omega)$

consists of grad φ, where φ is a single-valued function in Ω, a locally square integrable and differentiable function in $L_2(\Omega)$.

Lemma 1 ([4]) *The following inequalities hold*:

$$\|u\|^4_{4,\Omega} \leq 4\|u\|^2_{2,\Omega}\|u_{x_1}\|_{2,\Omega}\|u_{x_2}\|_{2,\Omega} \leq 2\|u\|^2_{2,\Omega}\|u_x\|^2_{2,\Omega}$$

for all $u(x) \in \overset{0}{W}{}^1_2(\Omega)$, $\Omega \subset R^2$,

$$\|u\|^4_{4,\Omega} \leq \varepsilon\|u_x\|^4_{2,\Omega} + \varepsilon^{-1}\|u\|^4_{2,\Omega},$$

where ε *is optional* ($\varepsilon > 0$),

$$\|u\|_{4,\Omega} \equiv \left(\int_\Omega |u|^4\,dx\right)^{\frac{1}{4}}, \qquad |u_x| \equiv \left(\sum_{k=1}^{2} u^2_{x_k}\right)^{\frac{1}{2}},$$

$$\|u_x\|^2_{2,\Omega} \equiv \int_\Omega |u_x|^2\,dx \equiv \int_\Omega u^2_x\,dx.$$

Theorem 1 ([15]) *If* $\partial\Omega \in C^2$, *then the operator* rot *sets a one-to-one correspondence between the spaces* $\overset{0}{W}{}^1_2(\Omega)$ *and* $\overset{0}{J}(\Omega)$, *moreover, the following inequality holds*:

$$\|\operatorname{rot}\vec{V}\| \leq \|\vec{V}\|^{(1)}_{2,\Omega} \leq \left(\sqrt{2} + \frac{1}{\sqrt{\mu_1}}\right)\|\operatorname{rot}\vec{V}\| \tag{7}$$

for all $\vec{V} \in \overset{0}{W}{}^1_2(\Omega)$.

Here the number μ_1 *is the smallest eigenvalue of the operator* Δ *in the field* Ω *at the zero boundary condition.*

Theorem 2 ([15]) *If* $\partial\Omega \in C^2$, *then the operator* rot *gives a one-to-one map of* $\overset{0}{J}_{1,n}(\Omega)$ *onto* $L_2(\Omega)$, *moreover, the following inequality holds*:

$$\|\operatorname{rot}\vec{\psi}\| \leq \|\vec{\psi}\|^{(1)}_{2,\Omega} \leq C_3\|\operatorname{rot}\vec{\psi}\| \tag{8}$$

for all $\vec{\psi} = (\psi_1, \psi_2) \in \overset{0}{J}_{1,n}(\Omega)$. *Here* $\vec{\psi}$ *does not depend on the constant* C_3.

3 Main results

Definition 1 The generalized solution of the inverse problem (1)-(6) is a set of function $\{\vec{v}(x,t), \nabla p(x,t), \vec{H}(x,t), \vec{f}(x), \operatorname{rot}\vec{j}(x)\}$, satisfying the correspondence (1)-(6) in the case if $\vec{v}(x,t) \in W^{2,1}_2(Q_T) \cap \overset{0}{J}_1(Q_T)$, $\vec{H}(x,t) \in W^{2,1}_2(Q_T) \cap \hat{J}(Q_T)$, $\vec{f}(x) \in L_2(\Omega)$, $\operatorname{rot}\vec{j}(x) \in L_2(\Omega)$, and the function $p(x,t) \in G(Q_T)$ ($\nabla p \in L_2(Q_T)$) at any t from $[0,T]$, and it continuously depends on t in the norm of this space for $[0,T]$.

Let us fix the functions $g(x,t)$ and $\xi(x,t)$ and define the nonlinear operators $T_g : L_2(\Omega) \to L_2(\Omega)$, $S_\xi : L_2(\Omega) \to L_2(\Omega)$ by the following expression:

$$(T_g\vec{f})(x) = \vec{v}_t(x,T), \qquad (S_\xi\vec{r})(x) = \vec{H}_t(x,T). \tag{9}$$

Here $r = \operatorname{rot}\vec{j}(x), \vec{f} = \vec{f}(x)$, but $\vec{v}(x,t)$ and $\vec{H}(x,t)$ are the solution of the direct problem (1)-(5) with $\vec{F} = g(x,t)\vec{f}(x)$, $\operatorname{rot}\vec{J}_0 = \xi(x,t)\operatorname{rot}\vec{j}(x)$.

The introduced operators T_g and S_ξ were well-posed as regards their definitions, since the necessary conditions for differentiability of $\vec{v}$, $\vec{H}$, and p are ensured by the theory in the work of Ladyzhenskaya and Solonnikov (see [9] Theorem 18, p.168, Theorem 19, p.169).

We suppose that $g(x,T) \neq 0$ and $\xi(x,T) \neq 0$ for all $x \in \Omega$; we introduce the nonlinear operators $A : L_2(\Omega) \to L_2(\Omega)$ and $yB : L_2(\Omega) \to L_2(\Omega)$, by the following expressions:

$$(A\vec{f})(x) = \frac{1}{g(x,T)}(T_g\vec{f})(x), \qquad (B\vec{r})(x) = \frac{\sigma\mu}{\xi(x,T)}(S_\xi\vec{r})(x). \tag{10}$$

Thus, if $g(x,t)\vec{f}(x) \in L_2(Q_T)$, $g_t(x,t)\vec{f}(x) \in L_{2,1}(Q_T)$ and $\xi(x,t)\operatorname{rot}\vec{j}(x) \in L_2(Q_T)$, $\xi_t(x,t) \times \operatorname{rot}\vec{j}(x) \in L_{2,1}(Q_T)$, additionally $g(x,t), g_t(x,t) \in C(\bar{Q}_T)$, $\xi(x,t), \xi_t(x,t) \in C(\bar{Q}_T)$, then, by (1) and (2) and in terms of these operators, (7), (8) given the inverse problem can be rewritten as

$$A\vec{f} + \vec{\aleph} = \vec{f}, \qquad B\vec{r} + \vec{\lambda} = \vec{r}, \tag{11}$$

where

$$\vec{\aleph} = \frac{1}{g(x,T)}\left[-\nu\Delta\vec{U} + U_k\vec{U}_{x_k} - \frac{\mu}{\rho}\Psi_k\vec{\Psi}_{x_k} + \frac{1}{\rho}\nabla\left(\pi + \frac{\mu\vec{\Psi}^2}{2}\right)\right],$$
$$\vec{\lambda} = \frac{\sigma\mu}{\xi(x,T)}\left[\frac{1}{\sigma\mu}\operatorname{rot}\operatorname{rot}\vec{h} - \operatorname{rot}(\vec{U}\times\vec{\Psi})\right].$$

Theorem 3 *Assume that* $\Omega \subset R^2$, $g, g_t \in C(\bar{Q}_T)$, $\xi, \xi_t \in C(\bar{Q}_T)$, $|g(x,t)| \geq g_T > 0$, $|\xi(x,t)| \geq \xi_T > 0$ *for* $x \in \Omega$, $\vec{U}(x) \in W_2^2(\Omega) \cap \overset{0}{J}_1(\Omega)$, $\vec{H}_0(x) \in \hat{J}(\Omega)$, $\vec{\Psi}(x) \in \hat{J}(\Omega)$, $\vec{v}_0(x) \in W_2^2(\Omega) \cap \overset{0}{J}_1(\Omega)$, $\nabla\pi(x) \in G(\Omega)$. *Then the operators A and B are completely continuous from* $L_2(\Omega)$ *to* $L_2(\Omega)$.

Proof Now we show that the operators T_g and S_ξ are completely continuous. Assume that $\vec{f}(x)$ and $\vec{r}(x)$ are arbitrary elements of $L_2(\Omega)$. We take arbitrary sequences $\{\vec{f}^N\}$ and $\{\vec{r}^N\}$ of $L_2(\Omega)$, such that

$$\|\vec{f}^N - \vec{f}\| \to 0, \qquad \|\vec{r}^N - \vec{r}\| \to 0, \quad \text{as } N \to \infty. \tag{12}$$

Let us show (10) implies that

$$\|T_g\vec{f}^N - T_g\vec{f}\| \to 0, \qquad \|S_\xi\vec{r}^N - S_\xi\vec{r}\| \to 0, \quad \text{as } N \to \infty. \tag{13}$$

We consider in Q_T the following problem:

$$\vec{W}_t - \nu\Delta\vec{W} + v_k^N\vec{W}_{x_k} - \frac{\mu}{\rho}H_k^N\vec{h}_{x_k} = -\frac{1}{\rho}\nabla\left(p + \frac{\mu(\vec{H}^N - \vec{H})^2}{2}\right) + \vec{F}, \tag{14}$$

$$\vec{h}_t + \frac{1}{\sigma\mu}\operatorname{rot}\operatorname{rot}\vec{h} - H_k^N\vec{W}_{x_k} + v_k^N\vec{h}_{x_k} = \vec{G}, \tag{15}$$

$$\operatorname{div}\vec{W} = 0, \qquad \operatorname{div}\vec{h} = 0, \tag{16}$$

$$h_n|_\Gamma = 0, \qquad \mathrm{rot}_\tau \vec{h}|_\Gamma = 0, \qquad \vec{W}|_\Gamma = 0, \tag{17}$$

$$\vec{W}|_{t=0} = 0, \qquad \vec{h}|_{t=0} = 0. \tag{18}$$

Here we introduce the notations $\vec{W} = \vec{v}^N - \vec{v}$, $\vec{h} = \vec{H}^N - \vec{H}$, $\vec{F} = (\vec{f}^N - \vec{f})g - W_k \vec{v}_{x_k} + \frac{\mu}{\rho} h_k \vec{H}_{x_k}$, $\vec{G} = \frac{\xi(x,t)}{\sigma\mu}(\vec{r}^N - \vec{r}) + h_k \vec{v}_{x_k} - W_k \vec{H}_{x_k}$. Here the functions $\vec{v}^N$, $\vec{H}^N$, and ∇p^N are the generalized solution of the direct problem (1)-(5), corresponding to the external forces $\vec{f}^N(x)g(x,t)$ and the currents $\xi(x,t)\vec{r}^N(x)$. We denote the solution of this problem, corresponding to the external forces $\vec{f}(x)g(x,t)$ and the currents $\xi(x,t)\vec{r}(x)$, by $\vec{v}$, $\vec{H}$, and ∇p.

We can consider the problem (14)-(18) with respect to the functions $\vec{W}$ and $\vec{h}$ as linear, since $(\vec{v}, \vec{H})$ and $(\vec{v}^N, \vec{H}^N)$ have the following [9, 13–15] differential conditions: $(\vec{v}_{tx}, \vec{H}_{tx}) \in L_2(Q_T) \times L_2(Q_T)$ and $(\vec{v}^N_{tx}, \vec{H}^N_{tx}) \in L_2(Q_T) \times L_2(Q_T)$; then $(\vec{v}, \vec{H})$ and $(\vec{v}^N, \vec{H}^N)$ are elements $L_q(\Omega) \times L_q(\Omega)$, for all $t \in [0, T]$ with any finite q, and they continuously depend on t in the norm $L_q(\Omega) \times L_q(\Omega)$. The following inequality holds:

$$\begin{aligned} &\|\vec{W}_t\|^2_{2,Q_T} + \|\vec{h}_t\|^2_{2,Q_T} + \nu\|\vec{W}_x\|^2_{2,\Omega} + \frac{1}{\sigma\mu}\|\vec{h}_x\|^2_{2,\Omega} \\ &\quad + \nu\|P\Delta\vec{W}\|^2_{2,Q_T} + \frac{1}{\sigma\mu}\|\mathrm{rot}\,\mathrm{rot}\,\vec{h}\|^2_{2,Q_T} \le c\big(\|\vec{F}\|^2_{2,Q_T} + \|\vec{G}\|^2_{2,Q_T}\big) \end{aligned} \tag{19}$$

for the solution of the problem [8, 9, 13].

Let us estimate $\|\vec{F}\|^2_{2,Q_T}$ and $\|\vec{G}\|^2_{2,Q_T}$. By applying Theorems 1 and 2, and Lemma 1, we obtain

$$\begin{aligned} \|\vec{F}\|^2_{2,Q_T} &\le 3\bigg[\big\|(\vec{f}^N - \vec{f})g\big\|^2_{2,Q_T} + \|W_k\vec{v}_{x_k}\|^2_{2,Q_T} + \frac{\mu}{\rho}\|h_k\vec{H}_{x_k}\|^2_{2,Q_t}\bigg], \\ \|W_k\vec{v}_{x_k}\|^2_{2,Q_T} &= \big\|(\vec{v}^N - \vec{v})_k \cdot \vec{v}_{x_k}\big\|^2_{2,Q_T} = \int_0^T \int_\Omega (\vec{v}^N - \vec{v})_k^2 \cdot \vec{v}^2_{x_k}\,dx\,dt \\ &\le \int_0^T \big\|\vec{v}^N - \vec{v}\big\|^2_{4,\Omega} \cdot \|\vec{v}_x\|^2_{4,\Omega}\,dt \\ &\le c(\Omega)\int_0^T \big\|(\vec{v}^N - \vec{v})_x\big\|^2_{2,\Omega} \cdot \|\vec{v}_{xx}\|^2_{2,\Omega}\,dt \\ &\le c(\Omega)\sup_{[0,T]}\big(\|\vec{v}\|^{(2)}_{2,\Omega}\big)^2 \cdot \big\|(\vec{v}^N - \vec{v})_x\big\|^2_{2,Q_T}, \\ \|h_k\vec{H}_{x_k}\|^2_{2,Q_T} &= \big\|(\vec{H}^N - \vec{H})_k \cdot \vec{H}_{x_k}\big\|^2_{2,Q_T} \\ &\le c(\Omega)\sup_{[0,T]}\big(\|\vec{H}\|^{(2)}_{2,\Omega}\big)^2 \big\|(\vec{H}^N - \vec{H})_x\big\|^2_{2,Q_T}. \end{aligned}$$

For the difference of two generalized solutions of problems [9] the following inequality holds:

$$\begin{aligned} \chi(t) &\le \int_0^t \bigg(\sqrt{\rho}\big\|(\vec{f}^N - \vec{f})g(x,\tau)\big\| + \frac{1}{\sqrt{\sigma}}\big\|(\vec{r}^N - \vec{r})\xi(x,\tau)\big\|\bigg)\exp\bigg\{c_0\int_\tau^t \Phi''^2(s)\,ds\bigg\}\,d\tau \\ &= \int_0^t \bigg[\sqrt{\rho}\bigg(\int_\Omega (\vec{f}^N - \vec{f})^2|g|^2\,dx\bigg)^{\frac{1}{2}} \\ &\quad + \frac{1}{\sqrt{\sigma}}\bigg(\int_\Omega (\vec{r}^N - \vec{r})^2|\xi|^2\,dx\bigg)^{\frac{1}{2}}\bigg]\,d\tau\,\exp\bigg\{c_0\int_\tau^t \Phi''^2(s)\,ds\bigg\}\,d\tau \end{aligned}$$

$$
\begin{aligned}
&\leq \exp\left\{c_0\int_0^T \Phi''^2(t)\,dt\right\}\int_0^t\left[\sqrt{\rho}\|\vec{f}^N-\vec{f}\|\sup_\Omega\left|g(x,\tau)\right|\right.\\
&\quad\left.+\frac{1}{\sqrt{\sigma}}\|\vec{r}^N-\vec{r}\|\sup_\Omega\left|\xi(x,\tau)\right|\right]d\tau\\
&\leq \exp\left\{c_0\int_0^T \Phi''^2(t)\,dt\right\}\left(\sqrt{\rho}\|\vec{f}^N-\vec{f}\|\int_0^T\sup_\Omega\left|g(x,\tau)\right|d\tau\right.\\
&\quad\left.+\frac{1}{\sqrt{\sigma}}\|\vec{r}^N-\vec{r}\|\int_0^T\sup_\Omega\left|\xi(x,\tau)\right|d\tau\right),
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\int_0^t \Phi^2(s)\,ds &\leq 2c_0\int_0^t \chi^2(s)\Phi''^2(s)\,ds\\
&\quad+\int_0^t \chi(s)\left(\sqrt{\rho}\|\vec{f}\|+\frac{1}{\sqrt{\sigma}}\|\operatorname{rot}\vec{j}\|\right)ds,
\end{aligned}
\tag{21}
$$

where

$$
\begin{aligned}
&\chi^2(t)=\rho\|\vec{W}\|^2+\mu\|\vec{h}\|^2,\qquad \Phi^2(t)=\rho\nu\|\vec{W}_x\|^2+\frac{1}{\sigma}\|\operatorname{rot}\vec{h}\|^2,\\
&\Phi''^2(t)=\rho\nu\|\vec{v}_x\|^2+\frac{1}{\sigma}\|\operatorname{rot}\vec{H}\|^2,\\
&\vec{f}=\left(\vec{f}^N(x)-\vec{f}(x)\right)g(x,t),\qquad \operatorname{rot}\vec{j}=\left(\vec{r}^N(x)-\vec{r}(x)\right)\xi(x,t),\qquad \chi(0)=0.
\end{aligned}
$$

By squaring and integrating both sides of inequality (20) on t from 0 to T, we have

$$
\begin{aligned}
\int_0^T \chi^2(t)\,dt &= \rho\|\vec{W}\|^2_{2,Q_T}+\mu\|\vec{h}\|^2_{2,Q_T}\\
&\leq 2T\exp\left\{2c_0\int_0^T \Phi''^2(t)\,dt\right\}\left(\rho\|\vec{f}^N-\vec{f}\|^2\left(\int_0^T\sup_\Omega\left|g(x,t)\right|dt\right)^2\right.\\
&\quad\left.+\frac{1}{\sigma}\|\vec{r}^N-\vec{r}\|^2\left(\int_0^T\sup_\Omega\left|\xi(x,t)\right|dt\right)^2\right),
\end{aligned}
\tag{22}
$$

then

$$
\begin{aligned}
\int_0^T \Phi^2(t)\,dt &= \rho\nu\left\|\left(\vec{v}^N-\vec{v}\right)_x\right\|^2_{2,Q_T}+\frac{1}{\sigma}\left\|\operatorname{rot}\vec{H}^N-\operatorname{rot}\vec{H}\right\|^2_{2,Q_T}\\
&\leq 2c_0\int_0^T \chi^2(t)\Phi''^2(t)\,dt\\
&\quad+\int_0^T \chi(t)\left(\sqrt{\rho}\left\|\left(\vec{f}^N-\vec{f}\right)g(x,t)\right\|+\frac{1}{\sqrt{\sigma}}\left\|\left(\vec{r}^N-\vec{r}\right)\xi(x,t)\right\|\right)dt\\
&\leq 4Tc_0\exp\left\{2c_0\int_0^T \Phi''^2(t)\,dt\right\}\int_0^T\left(\rho\|\vec{f}^N-\vec{f}\|^2\left(\int_0^t\sup_\Omega\left|g(x,\tau)\right|d\tau\right)^2\right.\\
&\quad\left.+\frac{1}{\sigma}\|\vec{r}^N-\vec{r}\|^2\left(\int_0^t\sup_\Omega\left|\xi(x,\tau)\right|d\tau\right)^2\right)\Phi''^2(t)\,dt\\
&\quad+2\sqrt{2T}\exp\left\{c_0\int_0^T \Phi''^2(t)\,dt\right\}\left(\rho\|\vec{f}^N-\vec{f}\|^2\left(\int_0^T\sup_\Omega\left|g(x,t)\right|dt\right)^2\right.\\
&\quad\left.+\frac{1}{\sigma}\|\vec{r}^N-\vec{r}\|^2\left(\int_0^T\sup_\Omega\left|\xi(x,t)\right|dt\right)^2\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\times \left(\int_0^T \left(\rho \|\vec{f}^N - \vec{f}\|^2 \left(\sup_\Omega |g(x,t)| \right)^2 \right.\right.$$
$$\left.\left. + \frac{1}{\sigma} \|\vec{r}^N - \vec{r}\|^2 \left(\sup_\Omega |\xi(x,t)| \right)^2 \right) dt \right)^{\frac{1}{2}}. \tag{23}$$

Furthermore, (12), (22), (23), and (19) imply that

$$\|(\vec{v}^N - \vec{v})_t\|^2_{2,Q_T} + \|(\vec{H}^N - \vec{H})_t\|^2_{2,Q_T} \to 0, \quad \text{as } N \to \infty. \tag{24}$$

Let us consider in Q_T the problem

$$\vec{W}_t - \nu \Delta \vec{W} = -\frac{1}{\rho} \nabla P + \vec{F}_1, \qquad \operatorname{div} \vec{W} = 0, \tag{25}$$
$$\vec{W}|_{t=0} = \vec{v}_2(x), \qquad \vec{W}|_\Gamma = 0, \tag{26}$$
$$\vec{h}_t + \frac{1}{\sigma\mu} \operatorname{rot}\operatorname{rot}\vec{h} = \vec{G}_1, \qquad \operatorname{div}\vec{h} = 0, \tag{27}$$
$$h_n|_S = 0, \qquad \operatorname{rot}\vec{h}|_\Gamma = 0, \qquad \vec{h}|_{t=0} = \vec{h}_2(x), \tag{28}$$

where

$$\vec{W} = (\vec{v}^N - \vec{v})_t, \qquad \vec{h} = (\vec{H}^N - \vec{H})_t,$$
$$\vec{F}_1 = \frac{\partial}{\partial t}\left[(\vec{f}^N - \vec{f})g - v_k^N \vec{v}^N_{x_k} + v_k \vec{v}_{x_k} + \frac{\mu}{\rho}\left(H_k^N \vec{H}^N_{x_k} - H_k \vec{H}_{x_k} \right) \right],$$
$$\vec{G}_1 = \frac{\partial}{\partial t}\left[\frac{\xi(x,t)}{\sigma\mu}(\vec{r}^N - \vec{r}) + \left(\operatorname{rot}[\vec{v}^N \times \vec{H}^N] - \operatorname{rot}[\vec{v} \times \vec{H}] \right) \right],$$
$$\vec{v}_2(x) = P(\vec{f}^N - \vec{f})g(x,0), \qquad \vec{h}_2(x) = P[\vec{r}^N - \vec{r}]\frac{\xi(x,0)}{\sigma\mu}.$$

The problem (25)-(26), as well as the problem (27)-(28) with the conditions in the space $V_2(Q_T)$ is uniquely solvable [4, 9, 13]. We now take the dot product of the first equation of (25) with the function $\vec{W}(x,t)$ and of (27) with $\vec{h}(x,t)$, respectively, in $L_2(Q_T)$, then we obtain

$$\frac{1}{2}\|\vec{W}(x,T)\|^2_{2,\Omega} + \nu \int_0^T \|\vec{W}_x(x,t)\|^2_{2,\Omega}\,dt$$
$$= \int_{Q_T} \vec{F}_1 \vec{W}\,dx\,dt + \frac{1}{2}\|\vec{W}(x,0)\|^2_{2,\Omega}, \tag{29}$$
$$\frac{1}{2}\|\vec{h}(x,T)\|^2_{2,\Omega} + \nu \int_0^T \|\operatorname{rot}\vec{h}(x,t)\|^2_{2,\Omega}\,dt$$
$$= \int_{Q_T} \vec{G}\vec{W}\,dx\,dt + \frac{1}{2}\|\vec{h}(x,0)\|^2_{2,\Omega}. \tag{30}$$

Using the method of [4] we prove that $(\vec{v}^N - \vec{v})_t$ and $(\vec{H}^N - \vec{H})_t$ are the solutions of the problems (23)-(24) and (25)-(26) from $V_2(Q_T)$. Estimating the right-hand sides of (29) and (30), and by (12) and (24), we have

$$\|(\vec{v}^N - \vec{v})_t(x,T)\|^2_{2,Q_T} + \|(\vec{H}^N - \vec{H})_t(x,T)\|^2_{2,Q_T} \to 0, \quad \text{as } N \to \infty. \tag{31}$$

Thus we have proved that the operators T_g and S_ξ are continuous.

Now we show the complete continuity of the operators T_g and S_ξ. Let us consider the following problem:

$$\vec{W}_t - \nu\Delta\vec{W} = -\frac{1}{\rho}\nabla P + \vec{F}_1, \qquad \operatorname{div}\vec{W} = 0, \tag{32}$$

$$\vec{h}_t + \frac{1}{\sigma\mu}\operatorname{rot}\operatorname{rot}\vec{h} = \vec{G}_1, \qquad \operatorname{div}\vec{h} = 0, \tag{33}$$

$$\begin{aligned} &h_n|_\Gamma = H_n|_\Gamma = 0, \qquad \operatorname{rot}\vec{h}|_\Gamma = 0, \qquad \vec{W}|_\Gamma = 0, \\ &\vec{W}|_{t=0} = \vec{v}_2(x), \qquad \vec{h}|_{t=0} = \vec{h}_2(x). \end{aligned} \tag{34}$$

Here

$$\begin{aligned} &\vec{W} = \vec{v}_t, \qquad \vec{h} = \vec{H}_t, \qquad \vec{F}_1 = \frac{\partial}{\partial t}\left[\vec{f}(x)g(x,t) - \nu_k\vec{v}_{x_k} + \frac{\mu}{\rho}H_k\vec{H}_{x_k}\right], \\ &\vec{G}_1 = \frac{\partial}{\partial t}\left[\frac{\xi(x,t)}{\sigma\mu}\vec{r}(x) + \operatorname{rot}[\vec{v}\times\vec{H}]\right], \\ &\vec{v}_2(x) = P\left(\nu\Delta\vec{v}_0 + \vec{f}(x)g(x,0) - \nu_{0k}\vec{v}_{0x_k} + \frac{\mu}{\rho}H_{0k}\vec{H}_{0x_k}\right), \\ &\vec{h}_2(x) = P\left[\frac{1}{\sigma\mu}\operatorname{rot}\operatorname{rot}\vec{H}_0 + \frac{\xi(x,0)}{\sigma\mu}\vec{r}(x) + \operatorname{rot}[\vec{v}_0\times\vec{H}_0]\right]. \end{aligned}$$

By the differential conditions of the problem (32)-(34) we fix an arbitrary number ε. Since the functions $\|\vec{W}_x(x,t)\|_{2,\Omega}$ and $\|\operatorname{rot}\vec{h}(x,t)\|_{2,\Omega}$ are continuous on $[\varepsilon,T]$, we could choose $t_* \in [\varepsilon,T]$ such that the following equalities hold:

$$\begin{aligned} &\int_\varepsilon^T \|\vec{W}_x(x,t)\|_{2,\Omega}^2\,dt = (T-\varepsilon)\|\vec{W}_x(x,t_*)\|_{2,\Omega}^2, \\ &\int_\varepsilon^T \|\operatorname{rot}\vec{h}(x,t)\|_{2,\Omega}^2\,dt = (T-\varepsilon)\|\operatorname{rot}\vec{h}(x,t_*)\|_{2,\Omega}^2. \end{aligned} \tag{35}$$

Equations (32) and (33) imply that

$$\begin{aligned} &\int_{t_*}^T\int_\Omega |\vec{W}_t - \nu P\Delta\vec{W}|^2\,dx\,dt = \int_{t_*}^T\int_\Omega |P\vec{F}_1|^2\,dx\,dt, \\ &\int_{t_*}^T\int_\Omega \left|\vec{h}_t + \frac{1}{\sigma\mu}P\operatorname{rot}\operatorname{rot}\vec{h}\right|^2\,dx\,dt = \int_{t_*}^T\int_\Omega |P\vec{G}_1|^2\,dx\,dt. \end{aligned} \tag{36}$$

Integrating by parts (36), we obtain

$$\begin{aligned} &\nu\|\vec{W}_x(x,T)\|_{2,\Omega}^2 + \int_{t_*}^T\int_\Omega \left(|\vec{W}_t|^2 + |\nu P\Delta\vec{W}|^2\right)dx\,dt \\ &\quad = \int_{t_*}^T\int_\Omega |P\vec{F}_1|^2\,dx\,dt + \nu\|\vec{W}_x(x,t_*)\|_{2,\Omega}^2, \\ &\frac{1}{\sigma\mu}\|\operatorname{rot}\vec{h}(x,T)\|_{2,\Omega}^2 + \int_{t_*}^T\int_\Omega \left(|\vec{h}_t|^2 + \left|\frac{1}{\sigma\mu}P\operatorname{rot}\operatorname{rot}\vec{h}\right|^2\right)dx\,dt \\ &\quad = \int_{t_*}^T\int_\Omega |P\vec{G}_1|^2\,dx\,dt + \frac{1}{\sigma\mu}\|\operatorname{rot}\vec{h}(x,t_*)\|_{2,\Omega}^2. \end{aligned} \tag{37}$$

By equality (35) and equality (37), we obtain

$$\begin{aligned}&\nu\big\|\vec{W}_x(x,T)\big\|_{2,\Omega}^2\leq\int_{t_*}^T\int_\Omega|P\vec{F}_1|^2\,dx\,dt+\frac{\nu}{T-\varepsilon}\big\|\vec{W}_x(x,t)\big\|_{2,Q_T}^2,\\&\frac{1}{\sigma\mu}\big\|\operatorname{rot}\vec{h}(x,T)\big\|_{2,\Omega}^2\leq\int_{t_*}^T\int_\Omega|P\vec{G}_1|^2\,dx\,dt+\frac{1}{\sigma\mu(T-\varepsilon)}\big\|\operatorname{rot}\vec{h}(x,t)\big\|_{2,Q_T}^2.\end{aligned}\tag{38}$$

By the well-known theory in [4, 13], the following inequalities hold:

$$\begin{aligned}&\nu\|\vec{W}_x\|_{2,Q_T}^2\leq\big\|\vec{v}_2(x)\big\|_{2,\Omega}^2+\frac{3}{2}\left(\int_0^T\big\|\vec{F}_1(x,t)\big\|_{2,\Omega}\,dt\right)^2,\\&\frac{1}{\sigma\mu}\|\operatorname{rot}\vec{h}\|_{2,Q_T}^2\leq\big\|\vec{h}_2(x)\big\|_{2,\Omega}^2+\frac{3}{2}\left(\int_0^T\big\|\vec{G}_1(x,t)\big\|_{2,\Omega}\,dt\right)^2\end{aligned}\tag{39}$$

for the solutions of problem (32)-(34).

Since $\Omega\subset R^2$, $g,g_t\in C(\bar{Q}_T)$, $\xi,\xi_t\in C(\bar{Q}_T)$, and $\vec{f}(x),\operatorname{rot}\vec{j}(x)\in L_2(\Omega)$, the following inequalities hold:

$$\begin{aligned}&\left\|\vec{f}(x)g(x,t)-\nu_k\vec{v}_{x_k}+\frac{\mu}{\rho}H_k\vec{H}_{x_k}\right\|_{2,\Omega}\\&\quad\leq\big\|\vec{f}(x)g(x,t)\big\|_{2,\Omega}+c(\Omega)\sup_{[0,T]}\big\|\vec{v}(x,t)\big\|_{2,\Omega}^{(2)}\cdot\big\|\vec{v}_x(x,t)\big\|_{2,\Omega}\\&\qquad+c(\Omega)\sup_{[0,T]}\big\|\vec{H}(x,t)\big\|_{2,\Omega}^{(2)}\cdot\big\|\operatorname{rot}\vec{H}(x,t)\big\|_{2,\Omega},\\&\left\|\frac{\xi(x,t)}{\sigma\mu}\vec{r}(x)+\operatorname{rot}[\vec{v}\times\vec{H}]\right\|_{2,\Omega}\\&\quad\leq\left\|\frac{\xi(x,t)}{\sigma\mu}\vec{r}(x)\right\|_{2,\Omega}+c(\Omega)\sup_{[0,T]}\left(\big\|\vec{v}(x,t)\big\|_{2,\Omega}^{(2)}\right)^2\cdot\big\|\operatorname{rot}\vec{H}(x,t)\big\|_{2,\Omega}^2\\&\qquad+c(\Omega)\sup_{[0,T]}\left(\big\|\vec{H}(x,t)\big\|_{2,\Omega}^{(2)}\right)^2\cdot\big\|\vec{v}_x(x,t)\big\|_{2,\Omega}^2,\end{aligned}\tag{40}$$

$$\begin{aligned}\big\|\vec{F}_1(x,t)\big\|_{2,\Omega}&\leq\big\|\vec{f}(x)g_t(x,t)\big\|_{2,\Omega}+c_1(\Omega)\sup_{[0,T]}\big\|\vec{v}(x,t)\big\|_{2,\Omega}^{(2)}\cdot\big\|\vec{v}_{tx}(x,t)\big\|_{2,\Omega}\\&\quad+c_1(\Omega)\sup_{[0,T]}\big\|\vec{H}(x,t)\big\|_{2,\Omega}^{(2)}\cdot\big\|\operatorname{rot}\vec{H}_t(x,t)\big\|_{2,\Omega},\\\big\|\vec{G}_1(x,t)\big\|_{2,\Omega}&\leq\left\|\frac{\xi_t(x,t)}{\sigma\mu}\vec{r}(x)\right\|_{2,\Omega}+c_1(\Omega)\sup_{[0,T]}\left(\big\|\vec{v}(x,t)\big\|_{2,\Omega}^{(2)}\right)^2\cdot\big\|\operatorname{rot}\vec{H}_t(x,t)\big\|_{2,\Omega}^2\\&\quad+c_1(\Omega)\sup_{[0,T]}\left(\big\|\vec{H}(x,t)\big\|_{2,\Omega}^{(2)}\right)^2\cdot\big\|\vec{v}_{tx}(x,t)\big\|_{2,\Omega}^2.\end{aligned}\tag{41}$$

By inequalities (39) and (40), we obtain

$$\begin{aligned}\nu\big\|\vec{W}_x(x,T)\big\|_{2,\Omega}^2&\leq\left(1+\frac{3}{2}T^2\right)\|\vec{F}_1\|_{2,Q_T}^2+\frac{1}{T-\varepsilon}\left[2\left\|\nu\Delta\vec{v}_0-\nu_{0k}\vec{v}_{0x_k}+\frac{\mu}{\rho}H_{0k}\vec{H}_{0x_k}\right\|_{2,\Omega}^2\right.\\&\quad+6\big\|\vec{f}(x)g(x,0)\big\|_{2,\Omega}^2+6\left(c(\Omega)\sup_{[0,T]}\|\vec{v}\|_{2,\Omega}^{(2)}\right)^2\|\vec{v}_{0x}\|_{2,\Omega}^2\end{aligned}$$

$$+ 6\Big(c(\Omega)\sup_{[0,T]}\|\vec{H}\|^{(2)}_{2,\Omega}\Big)^2\|\operatorname{rot}\vec{H}_0\|^2_{2,\Omega}\Bigg],$$

$$\begin{aligned}\frac{1}{\sigma\mu}\big\|\operatorname{rot}\vec{h}(x,T)\big\|^2_{2,\Omega} \le{}& \Big(1+\frac{3}{2}T^2\Big)\|\vec{G}_1\|^2_{2,Q_T}\\ &+ \frac{1}{T-\varepsilon}\Bigg[2\bigg\|\frac{1}{\sigma\mu}\operatorname{rot}\operatorname{rot}\vec{H}_0 + \operatorname{rot}[\vec{v}_0\times\vec{H}_0]\bigg\|^2_{2,\Omega}\\ &+ 6\bigg\|\frac{\xi(x,0)}{\sigma\mu}\vec{r}\bigg\|^2_{2,\Omega} + 6\Big(c(\Omega)\sup_{[0,T]}\|\vec{H}\|^{(2)}_{2,\Omega}\Big)^2\|\vec{v}_{0x}\|^2_{2,\Omega}\\ &+ 6\Big(c(\Omega)\sup_{[0,T]}\|\vec{v}\|^{(2)}_{2,\Omega}\Big)^2\|\operatorname{rot}\vec{H}_0\|^2_{2,\Omega}\Bigg].\end{aligned} \tag{42}$$

Moreover, the equality (41) implies that

$$\begin{aligned}\big\|\vec{F}_1(x,t)\big\|^2_{2,Q_T} \le{}& 3\big\|\vec{f}(x)g_t(x,t)\big\|^2_{2,Q_T}\\ &+ 3\Big(c_1(\Omega)\sup_{[0,T]}\|\vec{v}\|^2_{2,\Omega}\Big)^2\cdot\big\|\vec{v}_{tx}(x,t)\big\|^2_{2,Q_T}\\ &+ 3\Big(c_1(\Omega)\sup_{[0,T]}\big\|\vec{H}(x,t)\big\|^{(2)}_{2,\Omega}\Big)^2\cdot\big\|\operatorname{rot}\vec{H}_t(x,t)\big\|^2_{2,Q_T},\\ \big\|\vec{G}_1(x,t)\big\|^2_{2,Q_T} \le{}& \bigg\|\frac{\xi_t(x,t)}{\sigma\mu}\vec{r}(x)\bigg\|^2_{2,Q_T}\\ &+ 3\Big(c_1(\Omega)\sup_{[0,T]}\big(\big\|\vec{v}(x,t)\big\|^{(2)}_{2,\Omega}\big)^2\Big)^2\cdot\big\|\operatorname{rot}\vec{H}_t(x,t)\big\|^2_{2,Q_T}\\ &+ 3\Big(c_1(\Omega)\sup_{[0,T]}\big(\big\|\vec{H}(x,t)\big\|^{(2)}_{2,\Omega}\big)^2\Big)\cdot\big\|\vec{v}_{tx}(x,t)\big\|^2_{2,Q_T}.\end{aligned} \tag{43}$$

Here $\|\vec{F}_1\|^2_{2,Q_T}$ and $\|\vec{G}_1\|^2_{2,Q_T}$ are bounded, since one can estimate $\vec{v}_t(x,t)$, $\vec{v}_{tx}(x,t)$, $\vec{H}_t(x,t)$ and $\operatorname{rot}\vec{H}_t$ in $L_2(Q_T)$ via the data of the problem and $\|\vec{f}\|_{2,\Omega}$, $\|\vec{r}\|_{2,\Omega}$. By combining (40) and (41), we have

$$\big\|\vec{v}_{tx}(x,T)\big\|^2_{2,\Omega} + \big\|\operatorname{rot}\vec{H}_t(x,T)\big\|^2_{2,\Omega} \le M_0. \tag{44}$$

Thus, the operators T_g and S_ξ map any bounded set D from $L_2(\Omega)$, which is the domain of definition of T_g and S_ξ into a bounded set $\tilde{D}$ from $W^1_2(\Omega)$. Then the set $\tilde{D}$ is compact in $L_2(\Omega)$ by the Rellich theorem. Thus the operators T_g and S_ξ are continuous and map every bounded set into a compact set. Consequently, the operators T_g and S_ξ are completely continuous. The operators A and B are also completely continuous as a composition of linear bounded and completely continuous operators. □

Theorem 4 *If* $g, g_t \in C(\bar{Q}_T)$, $\xi, \xi_t \in C(\bar{Q}_T)$, $|g(x,t)| \ge g_T > 0$, $|\xi(x,t)| \ge \xi_T > 0$ *as* $x \in \Omega$, $\vec{U}(x) \in W^2_2(\Omega)\cap \overset{0}{J}_1(\Omega)$, $\vec{H}_0(x) \in \hat{J}(\Omega)$, $\vec{\Psi}(x) \in \hat{J}(\Omega)$, $\vec{v}_0(x) \in W^2_2(\Omega)\cap\overset{0}{J}_1(\Omega)$, $\nabla\pi(x) \in G(\Omega)$. *Let the inequalities*

$$\frac{1}{\sigma} > \mu c_1\big(\|\vec{U}\|_{4,\Omega} + \|\vec{\Psi}\|_{4,\Omega}\big), \qquad \nu > c_1\bigg(\|\vec{U}\|_{4,\Omega} + \frac{\mu}{\rho}\|\vec{\Psi}\|_{4,\Omega}\bigg) \tag{45}$$

hold. Then the problem (1)-(6) *is solvable if and only if* (11) *is solvable in* $L_2(\Omega)$.

Proof We suppose that the operator equations (11) are solvable. We introduce the notation $\vec{f}_1(x)$ and $\vec{r}_1 = \operatorname{rot}\vec{j}_1(x)$. Then by [9–15], we can find $\vec{v}_1$, ∇p_1, and $\vec{H}_1$ in the necessary classes of functions, satisfying (1)-(5) with vectors $\vec{f}_1(x)g(x,t)$ and $\xi(x,t)\operatorname{rot}\vec{j}_1(x)$. We show that these functions satisfy the overdetermination condition (6), also. Let $\vec{v}(x,T) = \vec{U}_1(x)$, $\vec{H}(x,T) = \vec{\Psi}_1(x)$, $\nabla p(x,T) = \nabla\pi(x)$, then for the function $\vec{w} = \vec{U} - \vec{U}_1$, $\vec{z} = \vec{\Psi} - \vec{\Psi}_1$, we have the following problem:

$$-\nu\Delta\vec{w} + w_k\vec{U}_{x_k} + U_{1k}\vec{w}_{x_k} - \frac{\mu}{\rho}z_k\vec{\Psi}_{x_k} - \frac{\mu}{\rho}\Psi_{1k}\vec{z}_{x_k} = -\nabla q, \tag{46}$$

$$\frac{1}{\sigma\mu}\operatorname{rot}\operatorname{rot}\vec{z} + w_k\vec{\Psi}_{x_k} + U_{1k}\vec{z}_{x_k} - z_k\vec{U}_{x_k} - \Psi_{1k}\vec{w}_{x_k} = 0, \tag{47}$$

$$\operatorname{div}\vec{w} = 0, \qquad \operatorname{div}\mu\vec{z} = 0, \tag{48}$$

$$\vec{w}|_\Gamma = 0, \qquad z_n|_\Gamma = 0, \qquad \left.\frac{\partial z_2}{\partial x_1} - \frac{\partial z_1}{\partial x_2}\right|_\Gamma = 0. \tag{49}$$

Now we take the dot product of (46) with the function $\vec{w}$ and of (47) with $\vec{z}$, respectively, in $L_2(\Omega)$,

$$\begin{aligned}
&\nu\rho\|\vec{w}_x\|^2 + \rho\int_\Omega w_k\vec{U}_{x_k}\vec{w}\,dx + \rho\int_\Omega w_{x_k}U_{1k}\vec{w}\,dx \\
&\quad - \mu\int_\Omega z_k\vec{\Psi}_{x_k}\vec{w}\,dx - \mu\int_\Omega \Psi_{1k}\vec{z}_{x_k}\vec{w}\,dx = 0, \\
&\frac{1}{\sigma}\|\operatorname{rot}\vec{z}\|^2 - \mu\int_\Omega w_k\vec{\Psi}_{x_k}\vec{z}\,dx + \mu\int_\Omega U_{1k}\vec{z}_{x_k}\vec{z}\,dx \\
&\quad - \mu\int_\Omega z_k\vec{U}_{x_k}\vec{z}\,dx - \mu\int_\Omega \vec{w}_{x_k}\Psi_{1k}\vec{z}\,dx = 0.
\end{aligned}$$

We transform the previous equalities into the following form:

$$\nu\rho\|\vec{w}_x\|^2 - \rho\int_\Omega w_k\vec{U}\vec{w}_{x_k}\,dx + \mu\int_\Omega z_k\vec{\Psi}\vec{w}_{x_k}\,dx - \mu\int_\Omega \Psi_{1k}\vec{z}_{x_k}\vec{w}\,dx = 0, \tag{50}$$

$$\frac{1}{\sigma}\|\operatorname{rot}\vec{z}\|^2 - \mu\int_\Omega w_k\vec{\Psi}\vec{z}_{x_k}\,dx + \mu\int_\Omega \vec{U}z_k\vec{z}_{x_k}\,dx + \mu\int_\Omega \vec{w}\Psi_{1k}\vec{z}_{x_k}\,dx = 0. \tag{51}$$

Now we add (50) and (51),

$$\begin{aligned}
&\nu\rho\|\vec{w}_x\|^2 + \frac{1}{\sigma}\|\operatorname{rot}\vec{z}\|^2 \\
&\quad = \rho\int_\Omega w_k\vec{U}\vec{w}_{x_k}\,dx - \mu\int_\Omega z_k\vec{\Psi}\vec{w}_{x_k}\,dx \\
&\qquad + \mu\int_\Omega w_k\vec{\Psi}\vec{z}_{x_k}\,dx - \mu\int_\Omega \vec{U}z_k\vec{z}_{x_k}\,dx,
\end{aligned}$$

the right-hand side of previous equality implies that

$$\begin{aligned}
&\nu\rho\|\vec{w}_x\|^2 + \frac{1}{\sigma}\|\operatorname{rot}\vec{z}\|^2 \\
&\quad \le \rho c_1\|\vec{U}\|_{4,\Omega}\|\vec{w}_x\|_{2,\Omega}^2 + \mu\|\vec{w}\|_{4,\Omega}\|\vec{\Psi}\|_{4,\Omega}\|\vec{z}_x\|_{2,\Omega}
\end{aligned}$$

$$
\begin{aligned}
&+ \mu\|\vec{z}\|_{4,\Omega}\|\vec{\Psi}\|_{4,\Omega}\|\vec{w}_x\|_{2,\Omega} + \mu c_1\|\vec{U}\|_{4,\Omega}\|\vec{z}_x\|_{2,\Omega}^2 \\
\le{}& \rho c_1\|\vec{U}\|_{4,\Omega}\|\vec{w}_x\|_{2,\Omega}^2 + \mu c_1\|\vec{U}\|_{4,\Omega}\|\vec{z}_x\|_{2,\Omega}^2 \\
&+ \mu c_1\|\vec{w}_x\|_{2,\Omega}\|\vec{\Psi}\|_{4,\Omega}\|\vec{z}_x\|_{2,\Omega} + \mu c_1\|\vec{z}_x\|_{2,\Omega}\|\vec{\Psi}\|_{4,\Omega}\|\vec{w}_x\|_{2,\Omega} \\
\le{}& \rho c_1\|\vec{U}\|_{4,\Omega}\|\vec{w}_x\|_{2,\Omega}^2 + \mu c_1\|\vec{U}\|_{4,\Omega}\|\vec{z}_x\|_{2,\Omega}^2 \\
&+ \mu c_1\|\vec{\Psi}\|_{4,\Omega}\big(\|\vec{w}_x\|_{2,\Omega}^2 + \|\vec{z}_x\|_{2,\Omega}^2\big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\big(\nu\rho - c_1\big(\rho\|\vec{U}\|_{4,\Omega} + \mu\|\vec{\Psi}\|_{4,\Omega}\big)\big)\|\vec{w}_x\|^2 \\
&\quad + \left(\frac{1}{\sigma} - \mu c_1\big(\|\vec{U}\|_{4,\Omega} + \|\vec{\Psi}\|_{4,\Omega}\big)\right)\|\operatorname{rot}\vec{z}\|^2 \le 0.
\end{aligned}
\tag{52}
$$

By conditions (45), from inequality (52) we obtain $\vec{U} = \vec{U}_1$, $\vec{\Psi} = \vec{\Psi}_1$, then $\nabla\pi(x) = \nabla\pi_1(x)$. Thus, $\vec{v}_1$, ∇p_1, $\vec{H}_1$, $\vec{f}_1(x)$, and $\vec{r}_1 = \operatorname{rot}\vec{j}_1(x)$ satisfy all conditions (1)-(6). Consequently, the problem (1)-(6) is solvable.

Necessity. Assume that the problem (1)-(6) is solvable. Let us denote this solution by $\{\vec{v}, \nabla p, \vec{H}, \operatorname{rot}\vec{j}\}$. Hence, we obtain the operator equations (11). Furthermore, we see that $\vec{f}(x)$ and $\vec{r} = \operatorname{rot}\vec{j}(x)$ are solutions of this equation. □

Theorem 5 *Let the condition of Theorem* 4 *hold. Let the following inequality hold*:

$$
M_1 + \|\vec{\aleph}\|_{2,\Omega} + \|\vec{\lambda}\|_{2,\Omega} < 1,
\tag{53}
$$

where

$$
\begin{aligned}
&\|\Phi\|_{2,Q_T}^2 = \nu\|\vec{v}_x\|_{2,Q_T}^2 + \frac{1}{\sigma}\|\operatorname{rot}\vec{H}\|_{2,Q_T}^2, \qquad \beta = \frac{2-\varepsilon}{2d^2}, \quad \varepsilon \in (0,2], \\
&\vec{\aleph} = \frac{1}{g(x,T)}\left[-\nu\Delta\vec{U} + U_k\vec{U}_{x_k} - \frac{\mu}{\rho}\Psi_k\vec{\Psi}_{x_k} + \frac{1}{\rho}\nabla\left(\pi + \frac{\mu\vec{\Psi}^2}{2}\right)\right], \\
&\vec{\lambda} = \frac{\sigma\mu}{\xi(x,T)}\left[\frac{1}{\sigma\mu}\operatorname{rot}\operatorname{rot}\vec{h} - \operatorname{rot}(\vec{U}\times\vec{\Psi})\right], \\
&M_1 = \exp\left\{\frac{c^2}{2d}\|\Phi\|_{2,Q_T}^2\right\}\left(\frac{1}{\inf_\Omega|g(x,T)|} + \frac{1}{\inf_\Omega|\xi(x,T)|}\right) \\
&\quad\times\left[\left(\left\|\nu\Delta\vec{v}_0 - v_{0k}\vec{v}_{0x_k} + \frac{\mu}{\rho}H_{0k}\vec{H}_{0x_k}\right\|_{2,\Omega}\right.\right. \\
&\quad + \sup_\Omega\left(\big|g(x,0)\big| + \frac{1}{\sigma\mu}\big|\xi(x,0)\big|\right) \\
&\quad \left. + \left\|\frac{1}{\sigma\mu}\operatorname{rot}\operatorname{rot}\vec{H}_0 + \operatorname{rot}[\vec{v}_0\times\vec{H}_0]\right\|\right)\exp\{-\beta T\} \\
&\quad \left. + \int_0^T \exp\{-\beta(T-t)\}\sup_\Omega\left(\big|g_t(x,t)\big| + \frac{1}{\sqrt{\sigma}}\big|\xi_t(x,t)\big|\right)dt\right].
\end{aligned}
$$

Then there exists a solution of the inverse problem (1)-(6).

Proof It is known that for the direct problem (1)-(5), one can obtain the following inequality:

$$\begin{aligned}
&\chi^2(t) + \int_0^t \Phi^2(\tau)\,d\tau \\
&\quad\le \rho\|\vec{v}_0\|^2_{2,\Omega} + \mu\|\vec{H}_0\|^2_{2,\Omega} \\
&\qquad + \frac{3}{2}\left\{\rho\left(\int_0^T \sup_\Omega\bigl|g(x,t)\bigr|\,dt\right)^2 + \frac{1}{\sigma^2\mu}\left(\int_0^T \sup_\Omega\bigl|\xi(x,t)\bigr|\,dt\right)^2\right\}. \qquad (54)
\end{aligned}$$

We return to the problem (32)-(34). We rewrite (32), (33). By differentiating (1) and (2) with respect to t, we have

$$\begin{aligned}
&\vec{v}_{tt} + (v_{kt}\vec{v}_{x_k} + v_k\vec{v}_{x_kt}) - \frac{\mu}{\rho}(H_{kt}\vec{H}_{x_k} + H_k\vec{H}_{x_kt}) - \nu\Delta\vec{v}_t \\
&\quad = -\frac{1}{\rho}\operatorname{grad}\left(p + \frac{\mu\vec{H}^2}{2}\right)_t + \vec{f}g_t, \qquad (55)
\end{aligned}$$

$$\vec{H}_{tt} + \frac{1}{\sigma\mu}\operatorname{rot}\operatorname{rot}\vec{H}_t - \bigl(\operatorname{rot}[\vec{v}\times\vec{H}]\bigr)_t = \frac{1}{\sigma\mu}\xi_t\operatorname{rot}\vec{j}, \qquad (56)$$

$$\bigl(\operatorname{rot}[\vec{v}\times\vec{H}]\bigr)_t = H_{kt}\vec{v}_{x_k} + H_k\vec{v}_{x_kt} - v_{kt}\vec{H}_{x_k} - v_k\vec{H}_{x_kt}.$$

Taking the dot product of (55) and (56) with $\rho\vec{v}_t$ and $\mu\vec{H}_t$ in $L_2(\Omega)$, respectively, we obtain

$$\begin{aligned}
&\frac{\rho}{2}\frac{d}{dt}\|\vec{v}_t\|^2 + \nu\|\vec{v}_{tx}\|^2 + \rho\int_\Omega v_{kt}\vec{v}_{x_k}\vec{v}_t\,dx \\
&\quad - \mu\int_\Omega (H_{kt}\vec{H}_{x_k} + H_k\vec{H}_{x_kt})\vec{v}_t\,dx = \rho\int_\Omega \vec{f}g_t\cdot\vec{v}_t\,dx, \qquad (57)
\end{aligned}$$

$$\frac{\mu}{2}\frac{d}{dt}\|\vec{H}_t\|^2 + \frac{1}{\sigma}\|\operatorname{rot}\vec{H}_t\|^2 - \mu\int_\Omega \vec{H}_t\cdot\bigl(\operatorname{rot}[\vec{v}\times\vec{H}]\bigr)_t\,dx = \frac{1}{\sigma}\int_\Omega \vec{H}_t\cdot\xi_t\operatorname{rot}\vec{j}\,dx. \qquad (58)$$

Here

$$\begin{aligned}
&\int_\Omega \vec{H}_t\cdot\bigl(\operatorname{rot}[\vec{v}\times\vec{H}]\bigr)_t\,dx \\
&\quad = \int_\Omega \vec{H}_t(H_{kt}\vec{v}_{x_k} + H_k\vec{v}_{x_kt} - v_{kt}\vec{H}_{x_k} - v_k\vec{H}_{x_kt})\,dx \\
&\quad = \int_\Omega (H_{kt}\vec{v}_{x_k}\vec{H}_t - H_k\vec{v}_t\vec{H}_{x_kt} - \vec{H}_{x_k}v_{kt}\vec{H}_t)\,dx.
\end{aligned}$$

By combining (57) and (58), we obtain

$$\frac{1}{2}\frac{d}{dt}\omega^2(t) + F^2(t) = I(t) + \rho\int_\Omega \vec{f}g_t\cdot\vec{v}_t\,dx + \frac{1}{\sigma}\int_\Omega \vec{H}_t\cdot\xi_t\operatorname{rot}\vec{j}\,dx. \qquad (59)$$

Here we introduced the following notation:

$$\chi^2(t) = \rho\|\vec{v}\|^2 + \mu\|\vec{H}\|^2, \qquad \Phi^2(t) = \rho\nu\|\vec{v}_x\|^2 + \frac{1}{\sigma}\|\operatorname{rot}\vec{H}\|^2,$$

$$\omega^2(t) = \rho\|\vec{v}_t\|^2 + \mu\|\vec{H}_t\|^2, \qquad F^2(t) = \rho\nu\|\vec{v}_{tx}\|^2 + \frac{1}{\sigma}\|\operatorname{rot}\vec{H}_t\|^2,$$

$$z(t) = \sqrt{\rho}\|\vec{f}g_t\| + \frac{1}{\sqrt{\sigma}}\|\xi_t \operatorname{rot}\vec{j}\|,$$

$$I(t) = \int_\Omega \bigl[\vec{v}_{x_k}(\mu H_{kt}\vec{H}_t - \rho v_{kt}\vec{v}_t) + \mu\vec{H}_{x_k}(H_{kt}\vec{v}_t - v_{kt}\vec{H}_t)\bigr]\,dx.$$

By applying the Young inequality to the right-hand side of (59), we obtain

$$\begin{aligned}
\bigl|I(t)\bigr| &\le \left(\int_\Omega (\mu H_{kt}\vec{H}_t + \rho v_{kt}\vec{v}_t)^2\right)^{\frac{1}{2}}\|\vec{v}_x\| + \mu\int_\Omega 2\vec{H}_{x_k}\vec{H}_t\vec{v}_t\,dx \\
&\le c_3\left(\int_\Omega \left(\left(\vec{H}_t^2\right)^2 + \left(\vec{v}_t^2\right)^2\right)dx\right)^{\frac{1}{2}}\|\vec{v}_x\| + 2\mu\left(\int_\Omega \vec{H}_t^2\cdot\vec{v}_t^2\,dx\right)^{\frac{1}{2}}\|\vec{H}_x\| \\
&\le c_3\Phi(t)\omega(t)F(t),
\end{aligned}$$

$$\begin{aligned}
&\left|\rho\int_\Omega \vec{f}g_t\cdot\vec{v}_t\,dx + \frac{1}{\sigma}\int_\Omega \vec{H}_t\cdot\xi_t \operatorname{rot}\vec{j}\,dx\right| \\
&\quad\le \rho\|\vec{f}g_t\|\|\vec{v}_t\| + \frac{1}{\sigma}\|\xi_t \operatorname{rot}\vec{j}\|\|\vec{H}_t\| \\
&\quad\le \omega(t)\left(\sqrt{\rho}\|\vec{f}g_t\| + \frac{1}{\sqrt{\sigma}}\|\xi_t \operatorname{rot}\vec{j}\|\right).
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{1}{2}\frac{d\omega^2(t)}{dt} + F^2(t) &\le c\Phi(t)\omega(t)F(t) + \omega(t)z(t) \\
&\le \frac{\delta}{2}F^2(t) + \frac{c^2}{2\delta}\Phi^2(t)\omega^2(t) + \omega(t)z(t),
\end{aligned}$$

$$\frac{d\omega(t)}{dt} + \frac{2-\delta}{2d^2}\omega(t) \le \frac{c^2}{2\delta}\Phi^2(t)\omega(t) + z(t).$$

Multiplying the last inequality by $\exp\{-\beta(T-t)\}$, $\beta = \frac{2-\delta}{2d^2}$, we obtain

$$e^{-\beta(T-t)}\frac{d\omega(t)}{dt} + \beta\omega(t)e^{-\beta(T-t)} \le \frac{c^2}{2\delta}\Phi^2(t)\omega(t)e^{-\beta(T-t)} + z(t)e^{-\beta(T-t)},$$

here $y(t) = \omega(t)e^{-\beta(T-t)}$, $\alpha_1(t) = \frac{c^2}{2\delta}\Phi^2(t)$, $\alpha_2(t) = z(t)e^{-\beta(T-t)}$. Then the last inequality can be rewritten as

$$\frac{dy(t)}{dt} \le \alpha_1(t)y(t) + \alpha_2(t). \tag{60}$$

By the Gronwall lemma, we have

$$\begin{aligned}
&\exp\left\{-\int_0^t \alpha_1(\tau)\,d\tau\right\}\frac{dy(t)}{dt} \\
&\quad\le \alpha_1(t)y(t)\exp\left\{-\int_0^t \alpha_1(\tau)\,d\tau\right\} + \alpha_2(t)\exp\left\{-\int_0^t \alpha_1(\tau)\,d\tau\right\},
\end{aligned}$$

$$\frac{d}{dt}\bigl[y(t)e^{-\int_0^t \alpha_1(\tau)\,d\tau}\bigr] \le \alpha_2(t)e^{-\int_0^t \alpha_1(\tau)\,d\tau}.$$

Integrating this inequality with respect to t from 0 to T, we obtain

$$y(T)e^{-\int_0^T \alpha_1(\tau)\,d\tau} \le y(0) + \int_0^T \alpha_2(t)e^{-\int_0^t \alpha_1(\tau)\,d\tau}\,dt,$$

$$\omega(T) \le \exp\left\{\frac{c^2}{2\delta}\int_0^T \Phi^2(t)\,dt\right\}\left[\omega(0)e^{-\beta T} + \int_0^T z(t)e^{-\beta(T-t)}\,dt\right].$$

Returning to the original notation, we obtain the following inequality:

$$\begin{aligned}
&\rho\left\|\vec{v}_t(x,T)\right\|^2 + \mu\left\|\vec{H}_t(x,T)\right\|^2 \\
&\quad\le \exp\left\{\frac{c^2}{2\delta}\int_0^T \Phi^2(t)\,dt\right\}\left[\left(\rho\left\|\vec{v}_t(x,0)\right\|^2 + \mu\left\|\vec{H}_t(x,0)\right\|^2\right)e^{-\frac{2-\delta}{2d^2}T}\right. \\
&\qquad\left. + \int_0^T\left(\sqrt{\rho}\left\|g_t(x,t)\vec{f}(x)\right\| + \frac{1}{\sqrt{\sigma}}\left\|\xi_t(x,t)\operatorname{rot}\vec{j}(x)\right\|\right)e^{-\frac{2-\delta}{2d^2}(T-t)}\,dt\right],
\end{aligned}$$

where $\vec{v}_t(x,0) = \nu\Delta\vec{v}_0 + \vec{f}(x)g(x,0) - v_{0k}\vec{v}_{0x_k} + \frac{\mu}{\rho}H_{0k}\vec{H}_{0x_k}$,

$$\vec{H}_t(x,0) = \frac{1}{\sigma\mu}\operatorname{rot}\operatorname{rot}\vec{H}_0 + \frac{\xi(x,0)}{\sigma\mu}\operatorname{rot}\vec{j}(x) + \operatorname{rot}[\vec{v}_0\times\vec{H}_0].$$

We consider the bounded, convex, closed set

$$D = \left\{\vec{f}\in L_2(\Omega), \operatorname{rot}\vec{j}\in L_2(\Omega), \|\vec{f}\|\le 1, \|\vec{r}\|\le 1\right\}.$$

Since $(A\vec{f})(x) = \frac{1}{g(x,T)}(T_g\vec{f})(x)$, $(B\vec{r})(x) = \frac{\sigma\mu}{\xi(x,T)}(S_\xi\vec{r})(x)$, the following inequality holds:

$$\|A\vec{f}\|^2 + \|B\vec{r}\|^2 \le M_1^2 \tag{61}$$

for the operators A and B in D.

We define in D the nonlinear operators A_1 and B_1 in the following form:

$$A_1\vec{f} = A\vec{f} + \vec{\aleph}, \qquad B_1\vec{r} = B\vec{r} + \vec{\lambda}. \tag{62}$$

By the condition (53) the operators A_1 and B_1 map D onto itself. By virtue of this and by Theorem 3, the operators A and B are completely continuous, and the combined Schauder principle implies the solvability of the operator equation

$$A_1\vec{f} = \vec{f}, \qquad B_1\vec{r} = \vec{r}$$

in D. Thus, by Theorem 4 the inverse problem is solvable (1)-(6). The proof is complete. □

4 Conclusion

The inverse problem with final overdetermination for a non-stationary magnetic hydrodynamics system has been reduced to an operator equation. By skillfully using the method proposed in [21] and [30], the compactness of the operator is proved and Schauder's theorem for the operator equation is used. The important thing is the previously unexamined inverse problem with final overdetermination for a non-stationary flat system of magnetic hydrodynamics. The results were formulated in the form of theorems and were proved rigorously. The search for new methods for facilitating the solution of the problem of the existence of a global solution of inverse problems for the Navier-Stokes equations, free convection, magnetohydrodynamics, and other nonlinear evolution equations is relevant. Therefore the proposed method by [21] and [30] is definitely applicable, also to the investigation of many other inverse problems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to the writing of this article equally. All authors read and approved the final manuscript.

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