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On global behavior of weak solutions to the Navier-Stokes equations of compressible fluid for $\gamma = 5/3$

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Abstract
In this article, we consider the global behavior of weak solutions of the Navier-Stokes equations of a compressible fluid in a bounded domain driven by bounded forces for the adiabatic constant $\gamma = 5/3$. Under the condition of a small mass depending on the given forces, we prove the existence of bounded absorbing sets of weak solutions, and thus we further get global bounded trajectories and global attractors to the weak solutions.

MSC: Primary 35B40; 35B41; secondary 76N99

Keywords: weak solutions; bounded absorbing set; bounded trajectories; attractors

1 Introduction

In this article, we investigate the global behavior of finite energy weak solutions to the Navier-Stokes equations of a viscous compressible isentropic fluid:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = \mu \Delta \mathbf{u} + (\mu + \lambda)\nabla \operatorname{div} \mathbf{u} + \rho \mathbf{f}, \tag{2}$$

in $\Omega \times I$, and with a non-slip boundary condition:

$$\mathbf{u}(t,x)|_{\partial\Omega} = \mathbf{0}, \quad t \in I \subset \mathbb{R}. \tag{3}$$

In this article, we always assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary, and I an open time interval. The unknown functions $\varrho = \rho(t,x)$ and $\mathbf{u} = \mathbf{u}(t,x) = (u^1(t,x), u^2(t,x), u^3(t,x))$ represent the density and velocity of fluid, respectively. The external force $\mathbf{f} = (f^1(t,x), f^2(t,x), f^3(t,x))$ is given. The pressure takes the form $P = a\rho^\gamma$, where a is a positive constant, and γ the adiabatic constant. $\mu > 0$ and λ are the viscosity constants, satisfying $3\lambda + 2\mu \geq 0$.

Next we give the standard definition of finite energy weak solutions to the problem (1)-(3) as in [1, 2].

Definition 1.1 Let $\gamma > 3/2$. we call the couple $(\rho, \mathbf{u})$ a finite energy weak solution to the problem (1)-(3), if it satisfies the following properties:

- ρ, $\mathbf{u}$ enjoy the regularity

$$\rho \in L^{\infty}_{\text{loc}}\big(I; L^{\gamma}(\Omega)\big) \cap L^{s(\gamma)}_{\text{loc}}\big(I; L^{s(\gamma)}(\Omega)\big), \qquad u^i \in L^2_{\text{loc}}\big(I; W^{1,2}_0(\Omega)\big) \tag{4}$$

for $i = 1, 2, 3$ and $s(\gamma) = (5\gamma - 3)/3$.

- Let the energy E be defined as follows:

$$E[\rho, \mathbf{u}](t) = \int_{\Omega} \left[\frac{1}{2}\rho(t,x)\big|\mathbf{u}(t,x)\big|^2 + \frac{a}{\gamma - 1}\rho^{\gamma}(t,x)\right] \mathrm{d}x, \tag{5}$$

then $E \in L^1_{\text{loc}}(I)$ satisfies the following energy inequality in $\mathcal{D}'(I)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}E[\rho, \mathbf{u}](t) + \int_{\Omega} \big[\mu\big|\nabla \mathbf{u}(t)\big|^2 + (\lambda + \mu)\big|\operatorname{div} \mathbf{u}(t)\big|^2\big] \,\mathrm{d}x \le \int_{\Omega} \rho(t)\mathbf{f}(t) \cdot \mathbf{u}(t) \,\mathrm{d}x. \tag{6}$$

- Equations (1) and (2) hold in $D'(I \times \Omega)$; moreover, (1) is satisfied in $D'(I \times \mathbb{R}^3)$ provided we prolong $\rho, \mathbf{u}$ to be zero on $\mathbb{R}^3/\Omega$.
- Equation (1) is satisfied in the sense of renormalized solutions, *i.e.*,

$$b(\rho)_t + \operatorname{div}\big(b(\rho)\mathbf{u}\big) + \big(b'(\rho)\rho - b(\rho)\big) \operatorname{div} \mathbf{u} = 0 \tag{7}$$

holds in $D'(I \times \Omega)$ for any b satisfying

$$b \in C^0\big([0, \infty]\big) \cap C^1\big((0, \infty)\big), \qquad \big|b'(t)\big| \le Ct^{-\lambda_0}, \quad t \in (0,1), \lambda_0 < 1, \tag{8}$$

and

$$\big|b'(t)\big| \le Ct^{\lambda_1}, \quad t \ge 1, \textit{ where } C > 0, -1 < \lambda_1 \le \frac{s(\gamma)}{2} - 1. \tag{9}$$

The existence of globally defined weak solutions for $\Omega \subset \mathbb{R}^3$ was proved by Lions [3] under the hypothesis that $\gamma > 9/5$. Then, by using the curl-div lemma to subtly derive a certain compactness, and applying Lions' idea and a technique from [4], Feireisl *et al.* [1] extended Lions' existence result to the case $\gamma > 3/2$. For any $1 \le \gamma \le 3/2$, a global weak solution still exists when the initial data have a certain symmetry (*e.g.*, spherical, or axisymmetric symmetry); see [4, 5]. The theory of weak solutions is also applied to other models of fluid mechanics; see [6–10] for examples. In [2, 11–13] Feireisl and Petzeltová investigated the global behavior of weak solutions of the problem (1)-(3), and showed the existence of bounded absorbing sets, global bounded trajectories and global attractors to weak solutions of compressible flows for $\gamma > 5/3$. Jiang *et al.* [14–16] and Wang [17] further generalized their results to the Navier-Stokes-Poisson equations and nematic liquid crystals, respectively. However, it is still an open problem for the case $\gamma > 3/2$. In this article, under the proof-frame of [2, 11], we investigate the global behavior of weak solutions of the problem (1)-(3) for $\gamma = 5/3$ under the assumption of small mass depending on the given forces. Finally, we mention that, from the definition of renormalized solutions, the total mass m is conserved, *i.e.*

$$m = \int_{\Omega} \rho(x,t) \,\mathrm{d}x \text{ is independent of } t \in I. \tag{10}$$

Next we start to state our main results. The first result concerns the existence of bounded absorbing sets of weak solutions to the problem (1)-(3).

Theorem 1.1 *Let $\gamma = 5/3$, $a_0 > -\infty$, $I = (a_0, \infty) \subset \mathbb{R}$ be an open interval, and the bounded measurable function $\mathbf{f} = (f^1(t,x), f^2(t,x), f^3(t,x))$ satisfy*

$$\max_{i=1,2,3}\left\{\operatorname{ess}\sup_{t\in I, x\in\Omega}\left|f^i(t,x)\right|\right\} \leq K. \tag{11}$$

Then there exist constants $m_0 := m_0(K) \in (0,1)$ and $E_\infty := E(K)$ satisfying the following property:

For any positive constant E_0 and any finite energy weak solution $(\rho, \mathbf{u})$ of the problem (1)-(3), *if*

$$\operatorname{ess}\limsup_{t\to a_0} E(t) \leq E_0 \quad \textit{and} \quad m \leq m_0, \tag{12}$$

then there exists a time point $T = T(E_0, a_0)$ such that

$$E(t) := E[\rho, \mathbf{u}](t) \leq E_\infty \quad \textit{for a.e. } t > T. \tag{13}$$

Here we explain why our arguments work only for $\gamma = 5/3$. In Feireisl and Petzeltová's article [2], they deduced the following key estimate:

$$\begin{aligned}
&\sup_{t\in[T,T+1]} E(t+) \\
&\quad \leq c(K,m)\Big(1 + \sup_{t\in[T,T+1]} \sqrt{E(t+)} + \tilde{c}(m) \sup_{t\in[T,T+1]} \|\varrho(t)\|_{L^\gamma(\Omega)}^{(4\gamma-3)(3(\gamma+\theta-1))}\Big),
\end{aligned} \tag{14}$$

where $c(K,m)$ and $\tilde{c}(m)$ are two positive constants. Under the condition $\gamma > 5/3$, $(4\gamma - 3)/(3(\gamma + \theta - 1)) < \gamma$, and thus one can apply the Young inequality to the estimate above to deduce

$$\sup_{t\in[T,T+1]} E(t+) \leq L \tag{15}$$

for some constant $L > 0$. The local-time boundedness (15) is very important to further deduce the existence of a bounded absorbing set. However, if $\gamma \leq 5/3$, then $(4\gamma - 3)/(3(\gamma + \theta - 1)) \geq \gamma$, and thus the above idea to deduce (15) obviously fails. However, when $\gamma = 5/3$, (14) implies

$$\sup_{t\in[T,T+1]} E(t+) \leq c(K,m)\Big(1 + \sup_{t\in[T,T+1]} \sqrt{E(t+)} + \tilde{c}(m) \sup_{t\in[T,T+1]} E(t+)\Big). \tag{16}$$

By careful analyzing the derivation of (15), we observe that $c(K,m)$ and m converge to zero as $m \to 0$, and thus (15) can be still deduced from (16) provided that the mass is sufficiently small.

Based on Theorem 1.1, we can further get global bounded trajectories of weak solutions to the problem (1)-(3) as in [12], since the family of trajectories generated by the finite energy weak solutions of (1)-(3) defined on I possesses a bounded absorbing set in the energy 'norm'. To this purpose, we define

$$\begin{aligned} U^s[E_0,\mathcal{F}](t_0,t) = \Big\{ &\big(\rho(\tau),\mathbf{q}(\tau)\big), \tau \in [0,1] | \rho(\tau) = \rho(t+\tau),\ \mathbf{q}(\tau) = (\rho\mathbf{u})(t+\tau), \\ &\text{where } (\rho,\mathbf{u}) \text{ is a finite energy weak solution to the} \\ &\text{problem (1)-(3) on an open interval } I, \text{ such that } (t_0,t_0+1] \subset I, \\ &\mathbf{f} \in \mathcal{F}, \operatorname{ess}\limsup_{t\to t_0} E(t) \le E_0 \text{ and } m \text{ satisfy (12)}\Big\}. \end{aligned} \tag{17}$$

Then we have the second result concerning the large-time behavior of the short trajectories defined in (17).

Theorem 1.2 *Let* $\gamma = 5/3$, $J_1 = (0,1)$,

$$\mathcal{F} \textit{ be bounded subset of the } \big(L^\infty(\mathbb{R}\times\Omega)\big)^3 \tag{18}$$

and

$$\begin{aligned} \mathcal{F}^+ = \Big\{ \mathbf{f} \Big| \mathbf{f} = &\lim_{\tau_n\to\infty} \mathbf{h}_n(t+\tau_n,x) \textit{ weak star in } L^\infty(\mathbb{R}\times\Omega) \\ &\textit{for a certain } \mathbf{h}_n \in \mathcal{F} \textit{ and } \tau_n \to \infty \Big\}. \end{aligned} \tag{19}$$

Assume that there exists a certain sequence $t_n \to \infty$ *satisfying*

$$\big(\rho_n(t_n+t,x),\mathbf{q}_n(t_n+t,x)\big) \in U^s[E_0,\mathcal{F}](a_0,t_n) \quad (a_0 \in \mathbb{R}),$$

then we can extract a subsequence (*not relabeled*) *such that*

$$\rho_n(t_n+t,x) \to \bar\rho(t,x) \quad \textit{in } L^{5/3}(J_1\times\Omega) \textit{ and in } C\big(\bar J_1; L^\alpha(\Omega)\big) \textit{ for } 1 \le \alpha < 5/3, \tag{20}$$

$$\mathbf{q}_n(t_n+t,x) \to (\bar\rho\bar{\mathbf{u}})(t,x) \quad \textit{in } L^p(J_1\times\Omega) \cap C\big(\bar J_1; \big(L^{\frac{5}{4}}_{\text{weak}}(\Omega)\big)^3\big), \tag{21}$$

and

$$E\big[\rho_n(t_n+t,x),\mathbf{u}_n(t_n+t,x)\big] \to E\big[\bar\rho(t,x),\bar{\mathbf{u}}(t,x)\big] \quad \textit{in } L^1(J_1) \tag{22}$$

for any $p \in [1,\frac{5}{4})$, *where* $(\bar\rho, \bar{\mathbf{u}})$ *is a finite energy weak solution of the problem* (1)-(3) *defined on the whole real line* $I = \mathbb{R}$ *such that* $E \in L^\infty(\mathbb{R})$, $\int_\Omega \bar\rho \,\mathrm{d}x = m$, *and* $\mathbf{f} \in \mathcal{F}^+$.

The theorem above presents that the energy E of finite energy weak solutions defined on $I = \mathbb{R}$ is uniformly bounded on $\mathbb{R}$, and thus we can further construct a set of short trajectories to which any finite energy weak solution is asymptotically attracted by Theorem 1.2. To this end, we define

$$\begin{aligned} \mathcal{A}^s[\mathcal{F}] = \big\{ &\big(\rho(\tau),\mathbf{q}(\tau)\big)_{\tau\in[0,1]} | \big(\rho,\mathbf{q} = (\rho\mathbf{u})\big) \text{ is a finite energy weak solution} \\ &\text{of the problem (1)-(3) on } I = \mathbb{R}, \text{with } \mathbf{f} \in \mathcal{F}^+, E[\rho,\mathbf{u}] \in L^\infty(\mathbb{R}) \\ &\text{and } m \text{ satisfy (12)}\big\}. \end{aligned} \tag{23}$$

Thus, we have the third conclusion as regards a global attractor to the short trajectories of the set $\mathcal{A}^s(\mathcal{F})$ as in [12].

Theorem 1.3 *Assume* $\gamma = 5/3$ *and* $\mathcal{F}$ *satisfies* (18). *Then the set* $\mathcal{A}^s[\mathcal{F}]$ *is compact in* $L^{5/3}(J_1 \times \Omega) \times (L^p(J_1 \times \Omega))^3$. *Moreover, for any* $p \in [1,5)$,

$$\sup_{(\rho,\mathbf{q})\in U^s[E_0,\mathcal{F}](t_0,t)} \left[\inf_{(\bar{\rho},\bar{\mathbf{q}})\in\mathcal{A}^s[\mathcal{F}]} \left(\|\rho-\bar{\rho}\|_{L^{5/3}(J_1\times\Omega)}\right) + \|\mathbf{q}-\bar{\mathbf{q}}\|_{L^p(J_1\times\Omega)}\right] \to 0, \tag{24}$$

as $t \to \infty$.

The theorem above shows that the set $\mathcal{A}^s(\mathcal{F})$ is a global attractor to the space of short trajectories; moreover, the set $\mathcal{A}^s(\mathcal{F})$ is nonempty and compact, if $\mathcal{F}$ is nonempty. Similar to [18], we can further build a set of global trajectories. To this end, we define

$$\begin{aligned}\mathcal{A}[\mathcal{F}] = \bigl\{(\rho,\mathbf{q})|\rho = \rho(0), \mathbf{q} = (\rho\mathbf{u})(0),\ &\text{where } \rho,\mathbf{u} \text{ is a finite energy} \\ &\text{weak solution of the problem (1)-(3) on } I = \mathbb{R} \\ &\text{with } \mathbf{f} \in \mathcal{F}^+ \text{ and } E \in L^\infty(\mathbb{R}),\ \text{and } m \text{ satisfy (12)}\bigr\},\end{aligned} \tag{25}$$

and

$$\begin{aligned}&U[E_0,\mathcal{F}](t_0,t) \\ &\quad = \Bigl\{(\rho,\mathbf{q})(t)\big|(\rho,\mathbf{u}) \text{ is a finite energy weak solution of the problem} \\ &\qquad \text{(1)-(3) on } I \text{ such that } (t_0,t] \subset I, \mathbf{f} \in \mathcal{F} \text{ and } \operatorname{ess}\limsup_{t\to t_0} E(t) \le E_0, \\ &\qquad \text{and } m \text{ satisfy (12)}\Bigr\},\end{aligned} \tag{26}$$

thus we get the fourth result on attractors as in [12].

Theorem 1.4 *We redefine the energy E by*

$$E[\rho,\mathbf{u}](t) = \int_{\rho(x,t)>0} \left[\frac{1}{2}\frac{|\rho\mathbf{u}|^2}{\rho}(t,x) + \frac{3a}{2}\rho^{5/3}(t,x)\right] \mathrm{d}x. \tag{27}$$

Assume that $\gamma = 5/3$ *and* $\mathcal{F}$ *satisfies* (18), *then* $\mathcal{A}[\mathcal{F}]$ *is compact in* $L^\alpha(\Omega) \times (L^{\frac{5}{4}}_{\mathrm{weak}}(\Omega))^3$, *i.e., for any* $1 \le \alpha < 5/3$ *and any* $\phi \in (L^5(\Omega))^3$,

$$\sup_{(\rho,\mathbf{q})\in U[E_0,\mathcal{F}](t_0,t)} \left[\inf_{(\bar{\rho},\bar{\mathbf{q}})\in\mathcal{A}[\mathcal{F}]} \left(\|\rho-\bar{\rho}\|_{L^\alpha(\Omega)} + \left|\int_\Omega (\mathbf{q}-\bar{\mathbf{q}})\cdot\phi\,\mathrm{d}x\right|\right)\right] \to 0 \quad \textit{as } t \to \infty. \tag{28}$$

Remark 1.1 It should be noted that the energy $E(t)$ defined by (27) is equal to (5) a.e. in I (see [18], Lemma 7.18) and (27) is lower semicontinuous (see [18], Proposition 7.21). Then the two conditions '$\operatorname{ess}\limsup_{t\to a} E(t) \le E_0$' and '$\limsup_{t\to a} E(t) \le E_0$' are equivalent, and, thus, the conclusions in Theorems 1.1-1.2 with $E(t)$ defined by (27) still hold, in particular, we have $E(t) := E[\rho,\mathbf{u}](t) \le E_\infty$ for $t > T$ in Theorem 1.1.

In next section, we use the proof-frame of [2] to prove Theorem 1.1 under the condition of small mass. Once we establish Theorem 1.1, the conclusions in Theorems 1.2-1.4 obviously hold by the standard compactness method as in [1, 12]; hence we omit the proof.

2 Proof of Theorem 1.1

Similar to [2], to get Theorem 1.1, it suffices to obtain the following two results.

Proposition 2.1 *Under the hypotheses of Theorem* 1.1, *let* $m \in (0,1)$ *and* $(\rho, \mathbf{u})$ *be a renormalized solution of* (1)-(3), *then the energy E is locally bounded variation on I* (*being redefined on a set of measure zero if necessary*), *and*

$$E(t+) = \lim_{s\to t+} E(s) \leq \lim_{s\to t-} E(s) = E(t-) \quad \textit{for any } t \in I. \tag{29}$$

Moreover, there exists a constant $c(K)$, only depending on K and independent of m, such that

$$E(t_2-) \leq \big(1 + E(t_1+)\big)e^{c(K)(t_2-t_1)} - 1 \quad \textit{for all } 0 < t_1 < t_2. \tag{30}$$

Proposition 2.2 *Under the assumptions of Theorem* 1.1, *there exists a constant* $m_0 \in (0,1)$ *such that for any* $m \in (0, m_0)$ *there exists a constant* $L := L(K)$ *enjoying the following property*:

If

$$E\big((T+1)-\big) > E(T+) - 1 \quad \textit{for some } T \in I, \tag{31}$$

then

$$\sup_{t\in(T,T+1)} E(t+) \leq L. \tag{32}$$

For completeness of this article, we provide the proof of Theorem 1.1 in detail, based on Propositions 2.1-2.2. It is easy to see that there exists $T = T(E_0, a_0) > a_0$ satisfying $E(T-) > E((T-1)+) - 1$. Indeed if it fails, then, when the t is sufficiently large, the energy would be negative. This contradicts the fact that the energy is non-negative. Therefore $E(t_0) \leq L$ for some $t_0 < T$, where L is defined as in Proposition 2.2.

Next we claim that

$$E\big((t_0+n)+\big) \leq L \quad \text{for any } n \geq 0, \tag{33}$$

By induction, we assume $E((t_0+n)+) \leq L$. Making use of (29) and Proposition 2.2, either

$$\sup_{t\in(t_0+n,t_0+n+1)} E(t+) \leq L,$$

which implies $E((t_0+n+1)-) \leq L$, or

$$E\big((t_0+n+1)+\big) \leq E\big((t_0+n+1)-\big) \leq E\big((t_0+n)+\big) - 1 \leq L - 1.$$

Consequently, in view of (33) and Lemma 2.1, we take the value

$$E_\infty = (1+L)e^{c(K)} - 1,$$

to obtain Theorem 1.1. This completes the proof of Theorem 1.1.

Next we turn to strictly show the two propositions above. We mention that all the estimate constants appearing in this section is independent of m.

2.1 Proof of Proposition 2.1

Let $E_1(t)$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1(t) + \int_\Omega \mu|\nabla \mathbf{u}|^2 + (\lambda+\mu)|\operatorname{div}\mathbf{u}|^2\,\mathrm{d}x = \int_\Omega \rho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x \quad \text{a.e. for } t\in I, \tag{34}$$

then $E_2 := (E - E_1) \in L^1_{\mathrm{loc}}(I)$. In view of (6), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2(t) \le 0 \quad \text{in } \mathcal{D}'(I). \tag{35}$$

Hence E is the sum of 'an absolutely function' and 'a nonincreasing function', and thus, E is a continuous function except a countable set of points in which (29) holds. In addition, using the condition (11), we can control the right-hand side of (6) as follows:

$$\begin{aligned}\int_\Omega \rho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x &\le K\left(\int_\Omega \rho\,\mathrm{d}x\right)^{\frac12}\left(\int_\Omega \rho|\mathbf{u}|^2\,\mathrm{d}x\right)^{\frac12} \le \sqrt{2m}K\left(1+\int_\Omega \rho|\mathbf{u}|^2\,\mathrm{d}x\right)\\ &\le \sqrt{2m}K\bigl(1+2E(t)\bigr) \le 2\sqrt{2}K\bigl(1+E(t)\bigr) := c(K)\bigl(1+E(t)\bigr),\end{aligned} \tag{36}$$

where we have used the condition $m\in(0,1)$. Thus, using the Gronwall lemma, we immediately get (30), and we complete the proof of Proposition 2.1.

2.2 Proof of Proposition 2.2

Before further providing the proof of Proposition 2.2, we shall establish the following four auxiliary lemmas.

Lemma 2.1 *Under the hypotheses of Theorem* 1.1 *and* (31), *let* $m\in(0,1)$, *then*

$$\int_T^{T+1} \|\mathbf{u}(t)\|^2_{W_0^{1,2}(\Omega)}\,\mathrm{d}t \le c_1\left(1+\int_T^{T+1}\|\rho(t)\|_{L^{\frac32}(\Omega)}\,\mathrm{d}t\right) \tag{37}$$

holds for a constant $c_1 = c_1(K)$.

Proof Exploiting (31), the energy inequality (6), the embedding theorem $W^{1,2}(\Omega)\subset L^6(\Omega)$, and the Poincaré inequality, we can estimate

$$\int_\Omega \mu\bigl|\nabla\mathbf{u}(t)\bigr|^2\,\mathrm{d}x \le c_{1,1}\left(1+\int_T^{T+1}\int_\Omega \rho|\mathbf{u}|\,\mathrm{d}x\,\mathrm{d}t\right).$$

On the other hand, we can use the Hölder inequality and the condition $m \in (0,1)$ to estimate

$$\int_\Omega \rho|\mathbf{u}|\,\mathrm{d}x \le \sqrt{m}\left(\int_\Omega \rho|\mathbf{u}|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \le \|\rho\|_{L^{3/2}(\Omega)}^{1/2}\|\mathbf{u}\|_{L^6(\Omega)}.$$

Consequently, we immediately get the desired result by using the embedding theorem again. □

Lemma 2.2 *Under the assumptions of Theorem* 1.1 *and* (31), *there exists a constant* $m_0 \in (0,1)$ *depending on* K *such that, for any*

$$m \in (0, m_0], \tag{38}$$

we have

$$E(t+) \le c_2\left(1 + \int_T^{T+1} \|\rho(s)\|_{L^{5/3}(\Omega)}^{5/3}\,\mathrm{d}s\right) \quad \textit{for any } t \in [T, T+1] \tag{39}$$

for some constant $c_2 = c_2(K)$.

Proof We integrate (30) for the choice $t_2 = T+1$ with respect to t_1 to obtain

$$E\big((T+1)-\big) \le c_{2,1}\left(1 + \int_T^{T+1} E(s)\,\mathrm{d}s\right).$$

In addition,

$$E(T+) < E\big((T+1)-\big) + 1 \le c_{2,2}\left(1 + \int_T^{T+1} E(s)\,\mathrm{d}s\right). \tag{40}$$

Thus, we can take $t_1 = T$ in (30) and use (40) to obtain

$$E(T+) \le c_{2,3}\left(1 + \int_T^{T+1} E(s)\,\mathrm{d}s\right) \quad \text{for any } t \in [T, T+1).$$

Now, exploiting the Hölder inequality and Lemma 2.1, we can infer that

$$\begin{aligned}\int_T^{T+1}\int_\Omega \rho|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t &\le \sup_{t\in[T,T+1)} \|\rho(t)\|_{L^{3/2}(\Omega)} \int_T^{T+1} \|\mathbf{u}\|_{W_0^{1,2}(\Omega)}^2\,\mathrm{d}s \\ &\le c_{2,4} \sup_{t\in[T,T+1]} \|\rho(t)\|_{L^{3/2}(\Omega)}\left(1 + \int_T^{T+1} \|\rho\|_{L^{3/2}(\Omega)}\,\mathrm{d}t\right).\end{aligned}$$

We can use the interpolation inequality to get

$$\|\rho\|_{L^{3/2}(\Omega)} \le \|\rho\|_{L^1(\Omega)}^{1/6}\|\rho\|_{L^5/3(\Omega)}^{5/6},$$

and thus

$$\int_T^{T+1}\int_\Omega \rho|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t \le c_{2,5} \sup_{t\in[T,T+1]} E(t+)^{1/2}\left(1 + m^{1/6}\int_T^{T+1} \|\rho(s)\|_{L^{5/3}(\Omega)}^{5/6}\,\mathrm{d}s\right).$$

Hence we further have

$$\sup_{t\in[T,T+1]} E(t+) \le c_{2,6}\Bigg[1 + \int_T^{T+1} \big\|\rho(t)\big\|_{L^{5/3}(\Omega)}^{5/3}\,\mathrm{d}s$$
$$+ \sup_{t\in[T,T+1]} E(t+)^{1/2}\left(1 + m^{1/6}\int_T^{T+1} \|\rho\|_{L^{5/3}(\Omega)}^{5/6}\,\mathrm{d}t\right)\Bigg].$$

Consequently, there exists a sufficiently small constant $m_0 \in (0,1)$ dependent on K such that, for any $m \in (0, m_0]$, (39) holds. □

Lemma 2.3 *Let* $(\rho, \mathbf{u})$ *be a finite energy weak solutions to the problem* (1)-(3) *and*

$$S_\varepsilon[v] = \vartheta_\varepsilon * v, \quad \text{where } \vartheta_\varepsilon = \vartheta_\varepsilon(x) \text{ is a regularizing sequence.}$$

Then

$$\partial_t S_\varepsilon\big[b(\rho)\big] + \operatorname{div}\big(S_\varepsilon\big[b(\rho)\big]\mathbf{u}\big) + S_\varepsilon\big[\big(b'(\rho)\rho - b(\rho)\big)\operatorname{div}\mathbf{u}\big] = r_\varepsilon \tag{41}$$

a.e. in $I \times \mathbb{R}^3$. *Moreover, if*

$$b(\rho) \text{ is in } L^\infty_{\mathrm{loc}}\big(\mathbb{R}^+, L^\beta(\Omega)\big), \quad \beta \ge 2,$$

then

$$r_\varepsilon \to 0 \text{ in } L^2_{\mathrm{loc}}\big(\mathbb{R}^+; L^\alpha(\Omega)\big) \text{ for } \varepsilon \to 0 \text{ with } \alpha = \frac{2\beta}{\beta+2}. \tag{42}$$

Proof Please, refer to [2], Lemma 2.1 or [18], Lemmas 6.7-6.9. □

Lemma 2.4 *Let* $p, r \in (1, \infty)$ *be given numbers, then there exists a bounded linear operator* $\mathcal{B}$,

$$\mathcal{B} = [\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3] : \left\{f \in L^p(\Omega) \,\Big|\, \int_\Omega f\,\mathrm{d}x = 0\right\} \mapsto \big[W_0^{1,p}(\Omega)\big]^3,$$
$$\big\|\mathcal{B}\{f\}\big\|_{W_0^{1,p}(\Omega)} \le c_3(p, \Omega)\|f\|_{L^p(\Omega)} \tag{43}$$

such that $\mathbf{v} := \mathcal{B}\{f\}$ *satisfies*

$$\operatorname{div}\mathbf{v} = f \quad \text{a.e. in } \Omega, \mathbf{v}|_{\partial\Omega} = 0. \tag{44}$$

In addition, if $f \in L^r(\Omega)$ *can be written by*

$$f = \operatorname{div}\mathbf{h} \quad \text{for a certain } \mathbf{h} \in \big[L^r(\Omega)\big]^3, \mathbf{h}\cdot\mathbf{n}|_{\partial\Omega} = 0,$$

then

$$\big\|\mathcal{B}\{f\}\big\|_{L^r(\Omega)} \le c_4(p, \gamma, \Omega)\|\mathbf{h}\|_{L^r(\Omega)}. \tag{45}$$

Proof The bounded linear operator $\mathcal{B}$ was first considered by Bogovskii [19], please refer to [20], Proposition 2.1, for a detailed proof. □

We are now in the position to prove Proposition 2.2. Let $0 \le \psi \le 1$, $\psi \in \mathcal{D}(T, T+1)$, and S_ε are the smoothing operators given by Lemma 2.3. We consider test functions

$$\varphi_i(t,x) = \psi(t)\mathcal{B}_i\left\{S_\epsilon\big[b(\rho)\big] - \frac{1}{|\Omega|}\int_\Omega S_\epsilon\big[b(\rho)\big]\,dx\right\}, \quad i = 1,2,3,$$

where

$$b \in C^1(R), \qquad b(z) = z^{1/15} \quad \text{for } z \ge 1. \tag{46}$$

Taking the φ_i as test functions for (2) and exploiting Lemmas 2.3 and 2.4, we can obtain the following identity:

$$\begin{aligned}
& a\int_T^{T+1}\int_\Omega \psi\rho^5/3S_\epsilon\big[b(\rho)\big]\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_T^{T+1}\psi\left(\int_\Omega a\rho^5/3\,\mathrm{d}x\right)\frac{1}{|\Omega|}\int_\Omega S_\epsilon\big[b(\rho)\big]\,\mathrm{d}x)\,\mathrm{d}t \\
&\qquad + (\lambda+\mu)\int_T^{T+1}\int_\Omega \psi S_\epsilon\big[b(\rho)\big]\operatorname{div}\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_T^{T+1}\int_\Omega \psi_t\rho u^i\mathcal{B}_i\left\{S_\epsilon\big[b(\rho)\big] - \frac{1}{|\Omega|}\int_\Omega S_\epsilon\big[b(\rho)\big]\,\mathrm{d}x\right\}\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \mu\int_T^{T+1}\int_\Omega \psi\,\partial_{x_j}u^i\partial_{x_j}\mathcal{B}_i\left\{S_\epsilon\big[b(\rho)\big] - \frac{1}{|\Omega|}\int_\Omega S_\epsilon\big[b(\rho)\big]\,\mathrm{d}x\right\}\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_T^{T+1}\int_\Omega \psi\rho u^iu^j\partial_{x_j}\mathcal{B}_i\left\{S_\epsilon\big[b(\rho)\big] - \frac{1}{|\Omega|}\int_\Omega S_\epsilon\big[b(\rho)\big]\,\mathrm{d}x\right\}\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_T^{T+1}\int_\Omega \psi\rho u^i\mathcal{B}_i\bigg\{S_\epsilon\big[\big(b(\rho)-b'(\rho)\rho\big)\operatorname{div}\mathbf{u}\big] \\
&\qquad - \frac{1}{|\Omega|}\int_\Omega S_\epsilon\big[\big(b(\rho)-b'(\rho)\rho\big)\operatorname{div}\mathbf{u}\big]\,\mathrm{d}x\bigg\}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_T^{T+1}\int_\Omega \psi\rho u^i\mathcal{B}_i\left\{r_\epsilon - \frac{1}{|\Omega|}\int_\Omega r_\epsilon\,\mathrm{d}x\right\}\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_T^{T+1}\int_\Omega \psi\rho u^i\mathcal{B}_i\big\{\operatorname{div}\big(S_\epsilon\big[b(\rho)\big]\big)\mathbf{u}\big\}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_T^{T+1}\int_\Omega \psi\rho f_i\mathcal{B}_i\left\{S_\epsilon\big[b(\rho)\big] - \frac{1}{|\Omega|}\int_\Omega S_\epsilon\big[b(\rho)\big]\,\mathrm{d}x\right\}\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{47}$$

Using the condition (38), we can get the following estimates; please, refer to [2] for the omitted details:

$$\left|\int_T^{T+1}\psi\left(\int_\Omega a\rho^{5/3}\,\mathrm{d}x\right)\left(\frac{1}{|\Omega|}\int_\Omega S_\varepsilon\big[b(\rho)\big]\,\mathrm{d}x\right)\mathrm{d}t\right| \le c_5\int_T^{T+1}\int_\Omega \rho^{5/3}\,\mathrm{d}x\,\mathrm{d}t, \tag{48}$$

$$\left|\int_T^{T+1}\int_\Omega \psi S_\varepsilon\big[b(\rho)\big]\operatorname{div}\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\right| \le c_6\int_T^{T+1}\big\|\mathbf{u}(t)\big\|_{W_0^{1,2}(\Omega)}\,\mathrm{d}t, \tag{49}$$

$$\begin{aligned}
&\left|\int_T^{T+1}\int_\Omega \psi_t \rho u^i \mathcal{B}_i\left\{S_\varepsilon\big[b(\rho)\big]-\frac{1}{|\Omega|}\int_\Omega S_\varepsilon\big[b(\rho)\big]\,\mathrm{d}x\right\}\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\le c_7\int_T^{T+1}|\psi_t|\|\sqrt{\rho}\mathbf{u}\|_{L^2(\Omega)}\,\mathrm{d}t, \qquad (50)\\
&\left|\int_T^{T+1}\int_\Omega \psi\,\partial_{x_j}u^i\partial_{x_j}\mathcal{B}_i\left\{S_\varepsilon\big[b(\rho)\big]-\frac{1}{|\Omega|}\int_\Omega S_\varepsilon\big[b(\rho)\big]\,\mathrm{d}x\right\}\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\le c_8\int_T^{T+1}\big\|\mathbf{u}(t)\big\|_{W_0^{1,2}(\Omega)}\,\mathrm{d}t, \qquad (51)\\
&\left|\int_T^{T+1}\int_\Omega \psi\rho u^i u^j\partial_{x_j}\mathcal{B}_i\left\{S_\varepsilon\big[b(\rho)\big]-\frac{1}{|\Omega|}\int_\Omega S_\varepsilon\big[b(\rho\big]\,\mathrm{d}x\right\}\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\le c_9\sup_{t\in[T,T+1]}\big\|\rho(t)\big\|_{L^{5/3}(\Omega)}\int_T^{T+1}\big\|\mathbf{u}(t)\big\|^2_{W_0^{1,2}(\Omega)}\,\mathrm{d}t, \qquad (52)\\
&\left|\int_T^{T+1}\int_\Omega \psi\rho u^i\mathcal{B}_i\left\{S_\varepsilon\big[\big(b(\rho)-b'(\rho)\rho\big)\operatorname{div}\mathbf{u}\big]\right.\right. \\
&\qquad\left.\left.-\frac{1}{|\Omega|}\int_\Omega S_\varepsilon\big[(b(\rho)-b'(\rho)\rho\operatorname{div}\mathbf{u}\big]\,\mathrm{d}x\right\}\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\le c_{10}\sup_{t\in[T,T+1]}\big\|\rho(t)\big\|_{L^{5/3}(\Omega)}\int_T^{T+1}\big\|\mathbf{u}(t)\big\|^2_{W_0^{1,2}(\Omega)}\,\mathrm{d}t, \qquad (53)\\
&\left|\int_T^{T+1}\int_\Omega \psi\rho u^i\mathcal{B}_i\left\{r_\varepsilon-\frac{1}{|\Omega|}\int r_\varepsilon\right\}\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\le c_{11}\int_T^{T+1}\|\rho\|_{L^{5/3}(\Omega)}\|\mathbf{u}\|_{W^{1,2}(\Omega)}\|r_\varepsilon\|_{L^{5/3}(\Omega)}\,\mathrm{d}t, \qquad (54)\\
&\left|\int_T^{T+1}\int_\Omega \psi\rho f_i\mathcal{B}_i\big\{\operatorname{div}\big(S_\varepsilon\big[b(\rho)\big]\mathbf{u}\big)\big\}\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\le c_{12}\sup_{t\in[T,T+1]}\big\|\rho(t)\big\|_{L^{5/3}(\Omega)}\int_T^{T+1}\big\|\mathbf{u}(t)\big\|^2_{W_0^{1,2}(\Omega)}\,\mathrm{d}t, \qquad (55)
\end{aligned}$$

and

$$\left|\int_T^{T+1}\int_\Omega \psi\rho f_i\mathcal{B}_i\left\{S_\varepsilon\big[b(\rho)\big]-\frac{1}{|\Omega|}\int_\Omega S_\varepsilon\big[b(\rho)\big]\,\mathrm{d}x\right\}\mathrm{d}x\,\mathrm{d}t\right|\le c_{13}(K). \qquad (56)$$

In addition, we can use (46) to see that

$$b(\rho)\text{ is in }L^\infty_{\mathrm{loc}}\big(R^+,L^{10}(\Omega)\big),$$

thus, exploiting (42) and (54), we further get

$$\lim_{\varepsilon\to 0+}\left|\int_T^{T+1}\int_\Omega \psi\rho u^i\mathcal{B}_i\left\{r_\varepsilon-\frac{1}{|\Omega|}\int r_\varepsilon\right\}\mathrm{d}x\,\mathrm{d}t\right|=0. \qquad (57)$$

Noting that there exists a sequence ψ_ε approximating the characteristic function of the interval $[T,T+1]$, thus, letting $\varepsilon\to 0$ in (48)-(56), we can obtain

$$\int_T^{T+1}\int_\Omega \rho^{26/15}\,\mathrm{d}x\,\mathrm{d}t \le c_{15}(K)\Big[1+\sup_{t\in[T,T+1]}\big\|\sqrt{\rho}\mathbf{u}(t)\big\|_{L^2(\Omega)} + \Big(1+\sup_{t\in[T,T+1]}\big\|\rho(t)\big\|_{L^{5/3}(\Omega)}\Big)\int_T^{T+1}\big\|\mathbf{u}(t)\big\|^2_{W_0^{1,2}(\Omega)}\,\mathrm{d}t\Big]. \tag{58}$$

Recalling the interpolating the spaces L^1 and $L^{26/15}$, we have

$$\int_T^{T+1}\|\rho\|^{5/3}_{L^{5/3}(\Omega)}\,\mathrm{d}t \le c_{16}m^{3/55}\Big[\int_T^{T+1}\int_\Omega \rho^{26/15}\,\mathrm{d}x\,\mathrm{d}t\Big]^{10/11}. \tag{59}$$

Then, exploiting Lemma 2.1, one has

$$\begin{aligned}&\Big|\sup_{t\in[T,T+1]}\big\|\rho(t)\big\|_{L^{5/3}(\Omega)}\int_T^{T+1}\big\|\mathbf{u}(t)\big\|^2_{W_0^{1,2}(\Omega)}\,\mathrm{d}t\Big|^{\frac{10}{11}}\\ &\quad\le c_{17}(K)\Big[1+\sup_{t\in[T,T+1]}\big\|\rho(t)\big\|_{L^{5/3}(\Omega)}\sup_{t\in[T,T+1]}\big\|\rho(t)\big\|_{L^{\frac{3}{2}}(\Omega)}\Big]^{\frac{10}{11}}\\ &\quad\le c_{18}(K)\Big[1+\sup_{t\in[T,T+1]}\big\|\rho(t)\big\|_{L^{5/3}(\Omega)}\Big]^{5/3}.\end{aligned} \tag{60}$$

In addition, thanks to (5), we have

$$\operatorname{ess}\sup_{t\in[T,T+1]}\big\|\sqrt{\rho}\mathbf{u}(t)\big\|_{L^2(\Omega)} \le \sup_{t\in[T,T+1]} 2\sqrt{E(t+)}. \tag{61}$$

Finally, making use of Lemma 2.2 and the estimates (58)-(61), we conclude

$$\begin{aligned}&\sup_{t\in[T,T+1]} E(t+)\\ &\quad\le c_{19}(K)\Big(1+\sup_{t\in[T,T+1]}2\sqrt{E(t+)}+m^{3/55}\sup_{t\in[T,T+1]}\big\|\rho(t)\big\|^{5/3}_{L^{5/3}(\Omega)}\Big).\end{aligned} \tag{62}$$

Consequently, (62) implies the existence of the constant L which has the property stated in Proposition 2.2 provided that

$$m \le \min\Big\{\Big(\frac{a}{2c_{19}(K)(\gamma-1)}\Big)^{\frac{55}{3}}, m_0\Big\}.$$

This completes the proof of Proposition 2.2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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