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Nonconstant periodic solutions created by impulses for singular differential equations

Naima Daoudi-Merzagui[1] and Abdelkader Boucherif[2*]

*Correspondence: aboucher@kfupm.edu.sa
[2]Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, P.O. Box 5046, Dhahran, 31261, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
In this work we discuss the existence of nonconstant periodic solutions for nonautonomous singular second order differential equations in the presence of impulses. Our approach is variational.

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1 Introduction

This paper is devoted to the study of the existence of nonconstant periodic solutions for non-autonomous singular second order differential equations,

$$u''(t) + f(t,u) = e(t), \quad \text{for a.e. } t \in [0,T], T > 0, \tag{1.1}$$

under impulse conditions

$$\Delta u'(t_j) = I_j\bigl(u(t_j)\bigr), \quad j = 1, 2, \ldots, p-1, \tag{1.2}$$

where f is a singular negative function and $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$, with $u'(t_j^\pm) = \lim_{t\to t_j^\pm} u'(t)$; t_j for $j = 1, 2, \ldots, p-1$, are the instants where the impulses occur with $0 = t_0 < t_1 < \cdots < t_{p-1} < t_p = T$, $t_{j+p} = t_j + T$. The functions $I_j : \mathbb{R} \to \mathbb{R}$; $j = 1, 2, \ldots, p-1$, are continuous and represent the jump discontinuities of u' at the impulse moments, and $I_{j+p} \equiv I_j$. Applications of impulsive differential equations with or without delays occur in medicine, population dynamics, and chaos theory; see [1, 2]. For the general aspects of impulsive differential equations, we refer the reader to the classical monographs [3, 4]. Due to its significance, a great deal of work has been done in the theory of impulsive differential equations; see for example [5–7]. It was pointed out in [8] that singular differential equations of the form (1.1) appear in the description of many phenomena in the applied sciences, such as nonlinear elasticity. Singular problems without impulse effects have been investigated extensively in the literature (see [2, 9–12] and the references therein). Some classical tools have been used to study such problems. These classical techniques include the coincidence degree theory of Mawhin and Willem [13], the method of upper and lower solutions [14], some fixed point theorems [15], and variational methods [16, 17]. For example, the authors in [18] obtained multiple periodic solutions for second-order perturbed Hamiltonian systems with impulse effects via variational methods. We believe that singular problems with impulsive effects have not been sufficiently studied; for some work on the subject, see [19, 20]. Inspired by the above facts, and the following important result (see [12]): *if e is an integrable T-periodic function, then* (1.1) *has a positive T-periodic weak solution if and only if* $\int_0^T e(t)\,dt < 0$, the aim of this paper is to prove a new existence result on a weak nonconstant T-periodic solutions *generated by impulses* (1.2) for the singular equation (1.1). Here, we say that *a solution is generated by impulses if this solution exists when $I_j \neq 0$, for some $1 < j < p-1$, and if it disappears when $I_j \equiv 0$ for all $1 < j < p-1$.*

The paper is organized as follows. Section 2 contains the basic preliminaries. An existence result of periodic solutions is given in Section 3. We conclude with an example.

2 Preliminaries

In this section we introduce some basic notions that will be used in the rest of the paper. P_T denotes the set of T-periodic functions $u : \mathbb{R} \to \mathbb{R}$ satisfying $u(t+T) = u(t)$ for all $t \in \mathbb{R}$; $C_T = \{u \in P_T; u \text{ is continuous}\}$. For $u \in C_T$ we denote its norm by $\|u\|_\infty = \sup\{|u(t)|; t \in [0,T]\}$. Then $(C_T, \|\cdot\|_\infty)$ is a Banach space. For $p \geq 1$, $L^p := L^p(0,T;\mathbb{R})$ is the classical Lebesgue space of measurable functions $u : [0;T] \to \mathbb{R}$ such that $|u(\cdot)|^p$ is integrable, and for $u \in L^p$ we define its norm by

$$\|u\|_{L^p} = \left(\int_0^T \left|u(t)\right|^p dt\right)^{\frac{1}{p}}.$$

We consider the Sobolev space $H_T^1 = \{u : \mathbb{R} \to \mathbb{R}; u \text{ is absolutely continuous}, u' \in L^2, \text{ and } u(t) = u(t+T) \text{ for } t \in \mathbb{R}\}$. H_T^1, equipped with the inner product

$$(u,v) = \int_0^T u'(t)v'(t)\,dt + \int_0^T u(t)v(t)\,dt$$

and the norm

$$\|u\|_{H_T^1} := \left(\|u\|_{L^2}^2 + \left\|u'\right\|_{L^2}^2\right)^{\frac{1}{2}}$$

is a reflexive Banach space. Also, H_T^1 admits the orthogonal decomposition, $H_T^1 = E + F$, where F is the subspace of constant functions in H_T^1 and E denotes the subspace of functions in H_T^1 with zero mean value. E is a weakly closed subspace of H_T^1. If $u \in E$, then the Wirtinger inequality

$$\int_0^T \left|u(t)\right|^2 dt \leq \frac{T^2}{4\pi^2}\int_0^T \left|u'(t)\right|^2 dt \tag{2.1}$$

implies that, on E, we can obtain the equivalent norm

$$\|u\| := \left\|u'\right\|_{L^2}.$$

Also, for $u \in E$ we have

$$\|u\|_\infty \leq \sqrt{T}\|u\|. \tag{2.2}$$

It is easy to see that a T-periodic solution of (1.1), (1.2) with zero mean value must be a nonconstant T-periodic solution of (1.1), (1.2).

Definition 1 $u \in H^1_T$ is solution of (1.1), (1.2) if $u \in C_T$ such that for every j, $u_j = u|_{[t_j,t_{j+1}]} \in H^2(t_j,t_{j+1})$, and it satisfies (1.1) for a.e. $t \in [0,T]$, $t \neq t_j$, the limits $u'(t_j^-)$, $u'(t_j^+)$ exist and the impulsive conditions (1.2) are satisfied.

3 Main result

We consider the impulsive second-order periodic boundary value problem,

$$\begin{cases} u''(t) + f(t,u) = e(t), & \text{for } t \in (0,T), t \neq t_j, \\ \Delta u'(t_j) = I_j(u(t_j)), & j = 1,2,\ldots,p-1, \\ u(0) - u(T) = 0, & u'(0) - u'(T) = 0, \end{cases} \tag{3.1}$$

under the following assumptions:

(H1) (i) $f : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$, is a negative Carathéodory function which is T-periodic in its first argument,
(ii) $\lim_{u\to 0^+} f(t,u) = -\infty$, for a.e. $t \in [0,T]$,
(iii) $\lim_{u\to +\infty} f(t,u) = 0$, for a.e. $t \in [0,T]$,

(H2) (i) e is a locally integrable T-periodic function and $\bar{e} := \frac{1}{T}\int_0^T e(t)\,dt > 0$,
(ii) $I_j : \mathbb{R} \to \mathbb{R}$, is a continuous bounded function for all $j = 1,\ldots,p-1$, such that $m = \inf I_j < \sup I_j = M < -\frac{T}{p-1}\bar{e}$.

Remark 1 (H1)(iii) implies that $\lim_{u\to+\infty} \frac{F(t,u)}{u^2} = 0$, for a.e. $t \in [0,T]$ where $F(t,u) := \int_1^u f(t,s)\,ds$.

Remark 2 Consider (1.1) and suppose that $I_j \equiv 0$ for all $1 < j < p-1$. In this case f verifies the conditions of the second result in [12]. Then (1.1) has a positive T-periodic solution if and only if $\bar{e} < 0$. This means that (1.1) under (H1) and (H2)(i) does not have a T-periodic weak solution. However, if the impulses happen, *i.e.* if (H2)(ii) is fulfilled for this singular equation (1.1), there may exist a positive T-periodic weak solution. Such a solution is called a periodic solution generated by impulses as pointed out in [20].

Theorem 1 *Suppose* (H1) *and* (H2) *hold. Then* (3.1) *admits at least one weak nonconstant* T-*periodic solution*.

Proof To prove this result, we rely on a variational method. In order to study problem (3.1), we consider the following modified problem:

$$\begin{cases} u''(t) + f_r(t,u(t)) = e(t), & \text{for a.e. } t \in (0,T), t \neq t_j, \\ \Delta u'(t_j) = I_j(u(t_j)), & j = 1,2,\ldots,p-1, \\ u(0) - u(T) = 0, & u'(0) - u'(T) = 0, \end{cases} \tag{3.2}$$

where $f_r : [0,T] \times \mathbb{R} \to \mathbb{R}$ is the truncation function defined for $r \in (0,1]$ by

$$f_r(t,u) = \begin{cases} f(t,r), & u \leq r, \\ f(t,u), & u > r. \end{cases}$$

f_r is a negative, continuous, and T-periodic function in t, which satisfies (H1)(iii). Let $F_r(t,u) := \int_1^u f_r(t,s)\,ds$.

Take $v \in H_T^1$ and multiply the two sides of the equality $-u'' - f_r(t;u) + e(t) = 0$ by v and integrate from 0 to T

$$\int_0^T \bigl[-u''(t) - f_r\bigl(t;u(t)\bigr) + e(t)\bigr]v(t)\,dt = 0. \tag{3.3}$$

Due to the jump discontinuities of u' at each t_j, $j = 1,2,\ldots,p-1$, and since v is T-periodic and $u'(0) - u'(T) = 0$, the first term of (3.3) becomes

$$\begin{aligned}\int_0^T u''(t)v(t)\,dt &= \sum_{j=0}^{p} \int_{t_j}^{t_{j+1}} u''(t)v(t)\,dt \\ &= u'(T)v(T) - u'(0)v(0) - \sum_{j=1}^{p-1} \Delta u'(t_j)v(t_j) - \int_0^T u'(t)v'(t)\,dt \\ &= -\sum_{j=1}^{p-1} \Delta u'(t_j)v(t_j) - \int_0^T u'(t)v'(t)\,dt.\end{aligned}$$

Combining the above with (3.3) we obtain

$$\sum_{j=1}^{p-1} \Delta u'(t_j)v(t_j) + \int_0^T u'(t)v'(t)\,dt - \int_0^T f_r\bigl(t;u(t)\bigr)\,dt + \int_0^T e(t)]v(t)\,dt = 0. \tag{3.4}$$

As a result, we introduce the concept of a weak solution for problem (3.2). We say that a function u is a weak solution of problem (3.2) if (3.4) holds for any $v \in H_T^1$.

Hence, we define the energy functional Φ_r: $H_T^1 \to \mathbb{R}$, associated to (3.2) by

$$\Phi_r(u) := \frac{1}{2}\int_0^T \bigl|u'(t)\bigr|^2\,dt + \sum_{j=1}^{p-1} \int_0^{u(t_j)} I_j(s)\,ds - \int_0^T F_r\bigl(t,u(t)\bigr)\,dt + \int_0^T e(t)u(t)\,dt. \tag{3.5}$$

Clearly, Φ_r is well defined on H_T^1. Combining the weak lower semicontinuity of the L^2-norm and Fatou's lemma we infer that Φ_r is weakly lower semi continuous, by means of the assumptions (H1)(i), (H2). Also, it is a differentiable functional whose derivative is the functional $\Phi_r'(u)$, given by

$$\Phi_r'(u)v = \int_0^T u'(t)v'(t)\,dt + \sum_{j=1}^{p-1} I_j\bigl(u(t_j)\bigr)v(t_j) - \int_0^T f_r\bigl(t,u(t)\bigr)v(t)\,dt + \int_0^T e(t)v(t)\,dt.$$

Obviously, from (3.4), if $u \in H_T^1$ is a critical point of the functional Φ_r, then u is a weak solution of problem (3.2). So, to obtain nonconstant weak solutions, it is sufficient to prove the existence of critical points of Φ_r, on the weakly closed subspace E of H_T^1.

Now, we claim that Φ_r is coercive on E. Indeed, the assumption (H1)(iii), implies that, for all $\varepsilon \in (0, \min(1, \frac{\pi^2}{T^2}))$, there exists $\delta_\varepsilon > 0$ such that, for almost every $t \in [0,T]$, we have

$$\bigl|f_r(t,u)\bigr| \le 2\varepsilon u \tag{3.6}$$

whenever $|u| > \delta_\varepsilon$. Using (3.2), (3.6), and (H1)(iii) we obtain for all $u \in \mathbb{R}$ and a.e $t \in [0, T]$

$$\left|F_r(t,u)\right| \leq \varepsilon u^2 + \max_{|u|\leq\delta_\varepsilon}\left|F_r(t;u)\right| - \varepsilon, \tag{3.7}$$

so that

$$\int_0^T F_r\bigl(t,u(t)\bigr)\,dt \leq \varepsilon \int_0^T \left|u(t)\right|^2 dt + C_\varepsilon, \tag{3.8}$$

where $C_\varepsilon = \int_0^T \max_{|u|\leq\delta}|F_r(t;u)|\,dt - T\varepsilon < +\infty$. Also, one can easily see that

$$\int_0^T e(t)u(t)\,dt + \sum_{j=1}^{p-1}\int_0^{u(t_j)} I_j(s)\,ds \geq \bigl(m(p-1) - \|e\|_{L^1}\bigr)\|u\|_\infty.$$

Thus, for $u \in E$, by (H2) and the previous inequalities we obtain

$$\begin{aligned}
\Phi_r(u) &= \frac{1}{2}\int_0^T \left|u'(t)\right|^2 dt - \int_0^T F_r\bigl(t,u(t)\bigr)\,dt + \int_0^T e(t)u(t)\,dt + \sum_{j=1}^{p-1}\int_0^{u(t_j)} I_j(s)\,ds \\
&\geq \frac{1}{2}\int_0^T \left|u'(t)\right|^2 dt - \varepsilon\int_0^T u(t)^2\,dt - C_\varepsilon T + \bigl(m(p-1) - \|e\|_{L^1}\bigr)\|u\|_\infty \\
&\geq \frac{1}{2}\left(1 - \frac{T^2}{2\pi^2}\varepsilon\right)\|u\|^2 + \sqrt{T}\bigl(m(p-1) - \|e\|_{L^1}\bigr)\|u\| - C\varepsilon T.
\end{aligned}$$

So, $\Phi_r(u) \to +\infty$ as $\|u\| \to +\infty$, which shows that Φ_r is coercive on E. Since E is a weakly closed subspace of H^1_T, using the direct method of the calculus of variations, we see that there exists $u^* \in E$ such that

$$\Phi_r\bigl(u^*\bigr) = \inf_E \Phi_r.$$

Notice that by (H1)(i) and (H2)(i), we have $\int_0^T f_r(t,0)\,dt = \int_0^T f(t,r)\,dt \leq 0 < \int_0^T e(t)\,dt$, so that the function $u \equiv 0$ cannot be a solution of (3.2). Hence u^* is a nontrivial solution of (3.2). □

In the following, we shall show that u^* is a solution of (3.1). For this purpose we introduce the following auxiliary result.

Lemma 1 *There exist $r_0 \in (0,1)$ and a constant $\beta_0 > 0$ such that each solution u of* (3.2) *satisfies $r_0 \leq u(t) \leq \beta_0$, for all t. In particular, any T-periodic solution of* (3.2) *is a solution of* (3.1).

Proof Here, we shall use some ideas from [10].

We proceed by contradiction. Suppose, on the contrary, that, for each $r \in (0,1)$ and for each constant $\beta > 0$, there exists a T-periodic solution u of (3.2) which satisfies

$$u(t) < r \quad \text{or} \quad u(t) > \beta \quad \text{for some } t \in [0, T]. \tag{3.9}$$

In particular, if for each integer $n > 1$ we consider $r_n = \frac{1}{n}$ and $\beta = n$, the above assumption implies that there exists a solution u_n of (3.2) for $r = r_n$ such that

$$\{u_n(t); t \in \mathbb{R}\} \nsubseteqq [r_n, n]. \tag{3.10}$$

We will show that this assumption leads to a contradiction.

First, we claim that for every $n > 1$ there must exist $\tau_n \in [0, T]$ such that

$$u_n(\tau_n) \in \left[\frac{1}{n}, n\right].$$

Indeed, suppose on the contrary that there exists a subsequence of $(u_n)_n$, which we label the same, for which $\min u_n(t) > n$. It follows from (H1)(iii) and the Fatou lemma that

$$\begin{aligned}(p-1)M &\geq \liminf_{n\to+\infty} \sum_{j=1}^{p-1} I_j\bigl(u_n(t_j)\bigr) = \liminf_{n\to+\infty} \int_0^T \bigl(f_{r_n}\bigl(t, u_n(t)\bigr) - e(t)\bigr)\, dt \\ &\geq \int_0^T \liminf_{n\to+\infty} \bigl(f_{r_n}\bigl(t, u_n(t)\bigr) - e(t)\bigr)\, dt\end{aligned}$$

and

$$\begin{aligned}\int_0^T \liminf_{n\to+\infty} \bigl(f_{r_n}\bigl(t, u_n(t)\bigr) - e(t)\bigr)\, dt &= \int_0^T \liminf_{x\to+\infty} \bigl(f(t,x) - e(t)\bigr)\, dt \\ &= \int_0^T \lim_{x\to+\infty} \bigl(f(t,x) - e(t)\bigr)\, dt = -T\bar{e},\end{aligned}$$

which leads to

$$(p-1)M \geq -T\bar{e}.$$

This is a contradiction to (H2)(ii). Similarly, we will arrive at a contradiction with (H2), if we assume that $\max u_n < \frac{1}{n}$. In fact, by the Fatou lemma we have

$$\begin{aligned}\limsup_{n\to+\infty} \sum_{j=1}^{p-1} I_j\bigl(u_n(t_j)\bigr) &= \limsup_{n\to+\infty} \int_0^T \bigl(f_{r_n}\bigl(t, u_n(t)\bigr) - e(t)\bigr)\, dt \\ &\leq \int_0^T \limsup_{n\to+\infty} \bigl(f_{r_n}\bigl(t, u_n(t)\bigr) - e(t)\bigr)\, dt \\ &\leq \int_0^T \limsup_{x\to0^+} \bigl[f(t,x) - e(t)\bigr]\, dt.\end{aligned}$$

Hence

$$\limsup_{n\to+\infty} \sum_{j=1}^{p-1} I_j\bigl(u_n(t_j)\bigr) \leq \int_0^T \lim_{x\to0^+} \bigl[f(t,x) - e(t)\bigr]\, dt = -\infty.$$

This contradicts the assumption that I_j is bounded.

Next, we show that u_n is bounded from above. Since for all $n > 1$, u_n is a T-periodic solution of (3.2), $\Phi_r'(u_n) = 0$. Hence for all $v \in H_T^1$ and for all $n > 1$ we have, for all ε, $0 < \varepsilon < 1$,

$$\left| \int_0^T \left[u_n'(t)v'(t) - f_{r_n}\big(t, u_n(t)\big)v(t) + e(t)v(t) \right] + \sum_{j=1}^{p-1} I_j\big(u_n(t_{j+sp})\big)v(t_{j+sp}) \right| \le \varepsilon \|v\|. \tag{3.11}$$

Taking $v(t) \equiv -1$ in the above inequality, we obtain

$$\begin{aligned} &\left| \int_0^T \left[f_{r_n}\big(t, u_n(t)\big) - e(t) \right] dt \right| - \left| \sum_{j=1}^{p-1} I_j(u_n(t_j)) \right| \\ &\quad \le \left| \int_0^T \left[f_{r_n}\big(t, u_n(t)\big) - e(t) \right] dt - \sum_{j=1}^{p-1} I_j(u_n(t_j)) \right| \le \varepsilon\sqrt{T}. \end{aligned}$$

Then

$$\begin{aligned} \left| \int_0^T \left[f_{r_n}\big(t, u_n(t)\big) - e(t) \right] dt \right| &\le \left| \sum_{j=1}^{p-1} I_j(u_n(t_j)) \right| + \varepsilon\sqrt{T} \\ &\le (p-1)|m| + \varepsilon\sqrt{T}. \end{aligned}$$

Now, from the above inequality, we get for all $n > 1$

$$\begin{aligned} \int_0^T \left| f_{r_n}\big(t, u_n(t)\big) \right| dt &\le \int_0^T \left| f_{r_n}\big(t, u_n(t)\big) - e(t) \right| dt + \int_0^T e(t)\, dt \\ &= \left| \int_0^T \left[f_{r_n}\big(t, u_n(t)\big) - e(t) \right] dt \right| + \bar{e}T \\ &\le (p-1)|m| + \varepsilon\sqrt{T} + T\bar{e}. \end{aligned} \tag{3.12}$$

Also, taking $v = u_n$ in (3.11), we obtain

$$\varepsilon \|u_n\| \ge \left\| u' \right\|_{L_2}^2 - \int_0^T \left[f_{r_n}\big(t, u_n(t)\big) - e(t) \right] u_n(t)\, dt + \sum_{j=1}^{p-1} I_j\big(u_n(t_j)\big) u_n(t_j). \tag{3.13}$$

Using (3.12) we get for all $n > 1$

$$\begin{aligned} \left| \int_0^T \left[f_{r_n}\big(t, u_n(t)\big) - e(t) \right] u_n(t)\, dt \right| &\le \|u_n\|_\infty \left(\int_0^T \left| f_{r_n}\big(t, u_n(t)\big) \right| dt + \int_0^T \left| e(t) \right| dt \right) \\ &\le \|u_n\|_\infty \big(\varepsilon\sqrt{T} + T\bar{e} + (p-1)|m| + \|e\|_{L^1} \big). \end{aligned}$$

Thus (3.13) implies that

$$\varepsilon \|u_n\| \ge \|u_n\|^2 - \|u_n\|_\infty \big(\varepsilon\sqrt{T} + T\bar{e} + \|e\|_{L^1} + (p-1)|m| \big). \tag{3.14}$$

Wirtinger's inequality (2.1) combined with (3.14) gives, for all $n > 1$,

$$\varepsilon \|u_n\| \ge \|u_n\|^2 - \sqrt{\frac{T}{12}} \|u_n\| \big(\varepsilon\sqrt{T} + T\bar{e} + \|e\|_{L^1} + (p-1)|m| \big).$$

We deduce that, for $n > 1$,

$$\|u_n\| \le \beta_1,$$

where

$$\beta_1 = \left(1 + \frac{T}{2\sqrt{3}}\right) + \sqrt{\frac{T}{12}}\big(\|e\|_{L^1} + T\bar{e} + (p-1)|m|\big).$$

Notice that β_1 is independent of n. Hence $(u_n)_n$ is bounded in H^1_T. Since $\|u_n\|_\infty \le \sqrt{T}\|u_n\|$, we deduce that

$$u_n(t) \le \beta := \beta_1\sqrt{T}.$$

Consequently, for n sufficiently large ($n > \beta$), for all $t \in [0, T]$, we have $u_n(t) \le n$. Furthermore, we cannot have $u_n(t) \ge \frac{1}{n}$ for all $t \in [0, T]$; otherwise we would get $\frac{1}{n} \le u_n(t) \le n$ for all $t \in [0, T]$ and this contradicts the assumption (3.10). Therefore, for n sufficiently large ($n > \beta$), there must exist a $t_n^* \in [0, T]$ such that $u_n(t_n^*) < \frac{1}{n}$. This means that $t_n^* \in I_{\frac{1}{n}}$, where $I_{\frac{1}{n}}$ is the set defined by

$$I_{\frac{1}{n}} = \big\{t \in [0, T]; u_n(t) < r_n\big\}. \tag{3.15}$$

Hence the set $I_{\frac{1}{n}}$ is not empty. The continuity of the solution u_n at $t = t_n^*$ implies that $\operatorname{meas}(I_{\frac{1}{n}}) > 0$, which implies

$$\int_{I_{\frac{1}{n}}} \big[f_{r_n}\big(t, u_n(t)\big) - e(t)\big]\,dt \ne 0.$$

Now, consider the sets

$$I_{1,\beta} = \big\{t \in [0, T]; 1 \le u_n(t) \le \beta\big\}, \tag{3.16}$$

$$I_{\frac{1}{n},1} = \big\{t \in [0, T]; r_n \le u_n(t) < 1\big\}, \tag{3.17}$$

so that we can write

$$[0, T] = I_{\frac{1}{n}} \cup I_{\frac{1}{n},1} \cup I_{1,\beta}.$$

Then integrating the differential equation in (3.2) from 0 to T we obtain

$$\begin{aligned}\Upsilon_n &:= \int_0^T -u_n''(t)\,dt = \int_0^T \big(f_{r_n}\big(t, u_n(t)\big) - e(t)\big)\,dt \\ &= \int_{I_{\frac{1}{n}}} \big[f_{r_n}\big(t, u_n(t)\big) - e(t)\big]\,dt \\ &\quad + \int_{I_{\frac{1}{n},1}} \big[f_{r_n}\big(t, u_n(t)\big) - e(t)\big]\,dt + \int_{I_{1,\beta}} \big[f_{r_n}\big(t, u_n(t)\big) - e(t)\big]\,dt.\end{aligned} \tag{3.18}$$

(1) Assume we are integrating positively on all subintervals of $[0, T]$.

If $t \in I_{\frac{1}{n}}$ then $u_n(t) < r_n$. It follows from (3.15) and (H1)(ii) that

$$\int_{I_{\frac{1}{n}}} \left[f_{r_n}\left(t, u_n(t)\right) - e(t)\right] dt = \int_{I_{\frac{1}{n}}} \left[f(t, r_n) - e(t)\right] dt < 0,$$

which yields

$$\Upsilon_n < \int_{I_{\frac{1}{n},1}} \left[f_{r_n}\left(t, u_n(t)\right) - e(t)\right] dt + \int_{I_{1,\beta}} \left[f_{r_n}\left(t, u_n(t)\right) - e(t)\right] dt. \tag{3.19}$$

If $t \in I_{1,\beta}$ then $u_n(t) \in [1, \beta]$. This means that $u_n(t)$ is bounded on $I_{1,\beta}$, since f_{r_n} is continuous in u, then f_{r_n} is bounded almost everywhere in $I_{1,\beta}$. Let

$$C = C(\beta) = \max\left\{\left|f_{r_n}(t,x)\right|; t \in [0, T], 1 \leq x \leq \beta\right\}. \tag{3.20}$$

Then

$$\left|\int_{I_{1,\beta}} \left[f_{r_n}\left(t, u_n(t)\right) - e(t)\right] dt\right| \leq \int_{I_{1,\beta}} \left|f_{r_n}\left(t, u_n(t)\right)\right| + \left|e(t)\right| dt \leq T\left(C + \|e\|_{L^1}\right), \tag{3.21}$$

and (3.19) leads to

$$\Upsilon_n \leq \int_{I_{\frac{1}{n},1}} \left[f_{r_n}\left(t, u_n(t)\right) - \bar{e}\right] dt + T(C + \bar{e}). \tag{3.22}$$

By (H1)(ii), we see that, for every $\sigma > 0$, there exists $\gamma_\sigma > 0$ such that $f(t,x) - \bar{e} < -\sigma$, for all $x \in I_{\gamma_\sigma} := (0, \gamma_\sigma)$ and for every $t \in [0, T]$. Then, for n large enough ($n > \beta$), we have $J := I_{\frac{1}{n},1} \cap I_{\gamma_\sigma} \neq \emptyset$. Hence, (H1)(i) implies

$$\int_{I_{\frac{1}{n},1}} \left[f_{r_n}\left(t, u_n(t)\right) - \bar{e}\right] dt < \int_J \left[f_{r_n}\left(t, u_n(t)\right) - \bar{e}\right] dt < -\sigma \operatorname{meas}(J). \tag{3.23}$$

Thus, for $\sigma = \frac{1}{\operatorname{meas}(J)} n^2 T(C + \bar{e})$, we obtain

$$\begin{aligned} \Upsilon_n &< \int_{I_{\frac{1}{n},1}} \left[f_{r_n}\left(t, u_n(t)\right) - \bar{e}\right] dt + T(C + \bar{e}) \\ &< T(C + \bar{e})\left(1 - n^2\right) \underset{n \to +\infty}{\to} -\infty. \end{aligned} \tag{3.24}$$

Then Υ_n is not bounded.

(2) If we integrate negatively on all subintervals of $[0, T]$ then, instead of (3.23), we get

$$\int_{I_{\frac{1}{n},1}} \left[f\left(t, u_0 + \frac{1}{n}\right) - \bar{e}\right] dt > \sigma \operatorname{meas}(I_J).$$

This, together with (3.20), leads to

$$\Upsilon_n \to +\infty, \quad \text{as } n \to +\infty. \tag{3.25}$$

On the other hand, integrating the differential equation in (3.2) from 0 to T and using T-periodicity of u_n', we obtain

$$\begin{aligned}\Upsilon_n &= -\int_0^T u_n''(t)\,dt = -\sum_{j=0}^{p}\int_{t_j^+}^{t_{j+1}^-} u_n''(t)\,dt \\ &= \sum_{j=1}^{p-1}\Delta u_n'(t_j) = \sum_{j=1}^{p-1} I_j\bigl(u_n(t_j)\bigr) \le (p-1)M.\end{aligned}$$

Thus by (H2)

$$\text{for each } n\in\mathbb{N}^*,\ \Upsilon_n < 0 \text{ and } \Upsilon_n \text{ is bounded.} \tag{3.26}$$

We see that (3.26) contradicts (3.24) and (3.25). This contradiction shows that Lemma 1 is proved. In particular, Lemma 1 shows that there exists $r\in(0,1)$ such that every T-periodic solution u of (3.2) is a solution of (3.1), since it satisfies $u(t)\ge r$ for all $t\in\mathbb{R}$ and $f_r(t,u(t)) = f(t,u(t))$, if $u(t)\ge r$. Therefore u^* is a nonconstant T-periodic solution of (1.1), (1.2). This completes the proof of our main result. □

4 Example

Consider the impulsive singular problem

$$\begin{cases} u''(t) - \frac{e^t}{u^\alpha} = e(t), & \text{for } t\in(0,T), t\neq t_j,\\ \Delta u'(t_j) = I_j(u(t_j)), & j=1,\\ u(0)-u(T)=0,\end{cases} \tag{4.1}$$

where $\alpha > 1$ and $T > 0$. Take $I_j(s) = \cos s - 2$, and $e\in L^2([0;T],\mathbb{R})$ such that $\bar{e} < \frac{1}{T}$. In this case $m = -3$ and $M = -1$. Then (H1)-(H2) hold. Therefore, by Theorem 1, problem (4.1) has at least one nonconstant T-periodic solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, University of Tlemcen, Tlemcen, 13000, Algeria. [2]Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, P.O. Box 5046, Dhahran, 31261, Saudi Arabia.

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